UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sc. and M.Sci. (Level M)

ANALYTIC NUMBER THEORY

MATH M0007

(Paper Code MATH-0007)

May 2015, 2 hours and 30 minutes

This paper contains **five** *questions*
A candidate's **FOUR** *best answers will be used for assessment.*
Calculators are **not** *permitted in this examination.*

Do not turn over until instructed.

1. Let $\zeta(s)$ denote the Riemann zeta function.

   (a) i. (**2 marks**) Define the *divisor function* $\tau(n)$.

   ii. (**4 marks**) Show that

   $$\sum_{n\le x}\frac{1}{n}=\log x+\gamma+O(1/x)\qquad (x\ge 1)$$

   for some $\gamma\in\mathbb{R}$.

   iii. (**6 marks**) Show that

   $$\sum_{n\le x}\tau(n)=x\log x+(2\gamma-1)x+O(x^{1/2}),$$

   for some $\gamma\in\mathbb{R}$.

   (b) Recall that the *Möbius function* is given by

   $$\mu(n)=\begin{cases}1, & \text{if } n=1,\\ (-1)^k, & \text{if } n \text{ is a product of } k \text{ distinct primes}\\ 0, & \text{otherwise.}\end{cases}$$

   i. (**3 marks**) Show that

   $$\sum_{d|n}\mu(d)=\begin{cases}1, & \text{if } n=1\\ 0, & \text{if } n>1.\end{cases}$$

   ii. (**2 marks**) Ignoring issues of convergence, show that

   $$\zeta(s)^{-1}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}.$$

   (c) i. (**2 marks**) Define *Euler's totient function* $\varphi(n)$.

   ii. (**3 marks**) Show that

   $$\varphi(n)=\sum_{d|n}\mu(d)\frac{n}{d}.$$

   iii. (**3 marks**) Ignoring issues of convergence, show that

   $$\sum_{n=1}^{\infty}\frac{\varphi(n)}{n^s}=\frac{\zeta(s-1)}{\zeta(s)}.$$

2. (a) The *Gamma function*, denoted by $\Gamma(s)$, is defined for $\mathrm{Re}(s) > 0$ by

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \mathrm{d}x.$$

i. (**2 marks**) Using the identity

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \mathrm{d}y = 1,$$

or otherwise, show that $\Gamma(1/2) = \sqrt{\pi}$.

ii. (**4 marks**) Explain why $\Gamma(s)$ is analytic in the region $\{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$.

iii. (**2 marks**) Show that

$$\Gamma(s) = \Gamma(s+1)/s \qquad (\mathrm{Re}(s) > 0).$$

(b) i. (**4 marks**) Show that $\Gamma(s)$ can be extended to a meromorphic function on $\mathbb{C}$. Where are its poles?

ii. (**2 marks**) Show that $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1.

(c) One can define an entire function $\xi(s)$ by

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

It satisfies the functional equation

$$\xi(1-s) = \xi(s).$$

i. (**6 marks**) Show that

$$\zeta(0) = -1/2.$$

ii. (**5 marks**) Using the functional equation, determine (with proof) all zeros of $\zeta(s)$ that lie *outside* of the region

$$\{s \in \mathbb{C} : 0 \leq \mathrm{Re}(s) \leq 1\}.$$

Continued...

3. (a) i. (**4 marks**) Let $a, b \in \mathbb{Z}$ with $a < b$, and let $f(x)$ be a function with a continuous derivative in the interval $[a, b]$. Recall that the *Euler–Maclaurin summation formula* states that

$$\sum_{a<n\leq b} f(n) = \int_a^b f(x)\mathrm{d}x + \int_a^b (\{x\} - \tfrac{1}{2})f'(x)\mathrm{d}x + \frac{1}{2}(f(b) - f(a)),$$

where $\{x\}$ denotes the fractional part of x. Use this to prove that

$$\zeta(s) = \frac{1}{s-1} - s\int_1^\infty \frac{\{x\} - \frac{1}{2}}{x^{s+1}}\mathrm{d}x + \frac{1}{2},$$

for $\mathrm{Re}(s) > 1$.

ii. (**6 marks**) Let $D = \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$. Prove that the equation in (i) gives an analytic continuation of $(s-1)\zeta(s)$ to the region D.

(b) Let $\pi(x)$ be the number of primes $p \leq x$. *Chebyshev's functions* are given by

$$\theta(x) = \sum_{p\leq x} \log p \quad \text{and} \quad \psi(x) = \sum_{n\leq x} \Lambda(n),$$

where the summation in $\theta(x)$ is over primes $p \leq x$ and $\Lambda(n)$ is the *von Mangoldt function.*

i. (**5 marks**) Show that $\psi(x) = \theta(x) + O(x^{1/2}\log x)$.

ii. (**5 marks**) Use partial summation to prove that

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t(\log t)^2}\mathrm{d}t.$$

iii. (**5 marks**) Let $E(x)$ be an increasing function of x with $E(x) \geq x^{1/2}\log x$, such that $\psi(x) = x + O(E(x))$. Use (i) and (ii) to show that

$$\pi(x) = \mathrm{L}(x) + O(E(x)),$$

where

$$\mathrm{L}(x) = \frac{x}{\log x} + \int_2^x \frac{\mathrm{d}t}{(\log t)^2}.$$

Continued...

4. Suppose that q is a positive integer and let $\varphi(q)$ be the *Euler totient function*.

   (a) (**2 marks**) Define what it means for a function $\chi : \mathbb{Z} \to \mathbb{C}$ to be a *Dirichlet character* modulo q.

   (b) (**5 marks**) Let a be an integer which is coprime to q and let n be an integer such that $n \not\equiv a \bmod q$. Show that

   $$\sum_{\chi \bmod q} \overline{\chi(a)}\chi(n) = 0,$$

   where the summation is over all Dirichlet characters modulo q.

   (c) i. (**8 marks**) State the Pólya–Vinogradov inequality. For any odd prime p use it to prove that the interval $[1, x]$ contains

   $$\frac{p-1}{2p}x + O\left(p^{1/2}\log p\right)$$

   integers coprime to p which are quadratic residues modulo p.

   ii. (**4 marks**) Let χ be a non-trivial Dirichlet character modulo q. For any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and any $x \geq 1$, show that

   $$L(s, \chi) = \sum_{n \leq x} \frac{\chi(n)}{n^s} + O\left(\frac{|s| q^{1/2}\log q}{\sigma x^{\sigma}}\right),$$

   where $\sigma = \mathrm{Re}(s)$.

   iii. (**6 marks**) Prove that for any $\sigma > 1/2$,

   $$\sum_{\chi \neq \chi_0} L(\sigma, \chi) = \varphi(q) + O\left(q^{1/2} + q^{3/2-\sigma}\log q\right),$$

   where the summation is over all non-trivial Dirichlet characters modulo q.

5. The *Ramanujan sum* is defined to be

$$c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e(an/q),$$

for positive integers n and q, where $e(x) = \exp(2\pi i x)$.

(a) Let

$$g(q) = \sum_{b=1}^{q} e(bn/q)$$

for positive integers n and q.

i. (**4 marks**) Show that

$$g(q) = \begin{cases} q, & \text{if } q \mid n, \\ 0, & \text{otherwise.} \end{cases}$$

ii. (**4 marks**) By extracting common factors between b and q, show that

$$g(q) = \sum_{d \mid q} c_{q/d}(n).$$

iii. (**5 marks**) Use Möbius inversion to deduce from (i) and (ii) that

$$c_q(n) = \sum_{d \mid (n,q)} d\mu(q/d).$$

iv. (**2 marks**) Show that

$$\mu(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e(a/q),$$

for any positive integer q.

(b) i. (**5 marks**) Let $[q_1, q_2] = q_1 q_2 / (q_1, q_2)$ denote the least common multiple of positive integers q_1, q_2. Show that if $q_1 \neq q_2$ then

$$\sum_{n=1}^{[q_1,q_2]} c_{q_1}(n) c_{q_2}(n) = 0.$$

ii. (**5 marks**) For any positive integer q show that

$$\sum_{n=1}^{q} c_q(n)^2 = q\varphi(q).$$

End of examination.