UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sc. and M.Sci. (Level M)

ANALYTIC NUMBER THEORY

MATH M0007

(Paper Code MATH-M0007)

May 2016

2 hours and 30 minutes

This paper contains **four** *questions*
All **four** *answers will be used for assessment.*
Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

1. Recall that the arithmetic functions μ, φ, Λ and $\mathbf{1}$ are defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^k, & \text{if } n \text{ is a product of } k \text{ distinct primes,} \\ 0, & \text{otherwise,} \end{cases}$$
$$\varphi(n) = \#\{a \bmod n : (a, n) = 1\},$$
$$\Lambda(n) = \begin{cases} 0, & \text{if } n = 1, \\ \log p, & \text{if } n \text{ is a positive power of the prime } p, \\ 0, & \text{otherwise,} \end{cases}$$
$$\mathbf{1}(n) = 1.$$

(a) i. (**3 marks**) Show that

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1. \end{cases}$$

ii. (**5 marks**) For a fixed positive integer k, show that

$$\sum_{\substack{n \le x \\ (n,k)=1}} \frac{1}{n} = \frac{\varphi(k)}{k} \log x + O(1).$$

You may assume that

$$\sum_{m \le y} \frac{1}{m} = \log y + O(1).$$

(b) i. (**2 marks**) Show that $\Lambda * \mathbf{1} = \log$.

ii. (**5 marks**) Prove that

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d.$$

iii. (**5 marks**) Ignoring issues of convergence, show that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}.$$

(c) The *Liouville function* is defined to be $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ denotes the total number of prime factors of n (counted with multiplicity).

i. (**2 marks**) Show that $\lambda(n)$ is a completely multiplicative function.

ii. (**3 marks**) Ignoring issues of convergence, show that

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}.$$

Continued...

2. (a) i. (**4 marks**) Prove that for $\mathrm{Re}(s) > 1$ we have

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\mathrm{d}x,$$

where $\{x\}$ denotes the fractional part of x.

ii. (**6 marks**) Let $D = \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$. Prove that the equation in (i) gives an analytic continuation of $(s-1)\zeta(s)$ to the region D.

(b) i. (**4 marks**) Prove that

$$\sum_{n\le x} \log n = x\log x + O(x).$$

ii. (**5 marks**) Using Q1 part (b)(i), or otherwise, show that

$$\sum_{n\le x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

You may assume that $\pi(x) = O(x/\log x)$.

iii. (**2 marks**) Show that

$$\sum_{n\le x} \frac{\Lambda(n)}{n} = \sum_{p\le x} \frac{\log p}{p} + O(1).$$

iv. (**4 marks**) Hence prove that

$$\sum_{p\le x} \frac{1}{p} = \log\log x + O(1).$$

Continued...

3. (a) i. (**3 marks**) Prove that for any $\theta \in \mathbb{R}$

$$3 + 4\cos(\theta) + \cos(2\theta) \geq 0.$$

ii. (**5 marks**) Use the previous part to prove that for any $\sigma > 1$ and any $t \in \mathbb{R}$

$$3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma + it)| + \log|\zeta(\sigma + 2it)| \geq 0.$$

iii. (**5 marks**) Deduce that $\zeta(1 + it) \neq 0$ for any $t \in \mathbb{R}$ with $t \neq 0$. You may assume that

$$\lim_{\sigma \to 1}(\sigma - 1)\zeta(\sigma) = 1.$$

(b) i. (**4 marks**) Recall that the *Gamma function* is given by the analytic function

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\mathrm{d}x,$$

for $\mathrm{Re}(s) > 0$. Show that $\Gamma(s)$ can be extended to a meromorphic function on $\mathbb{C}$.

ii. (**3 marks**) Give a complete list of all zeros and poles (with multiplicities) of $\Gamma(s)$.

iii. (**5 marks**) Using the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

determine (with proof) all zeros of $\zeta(s)$ that lie *outside* of the region

$$\{s \in \mathbb{C} : 0 \leq \mathrm{Re}(s) \leq 1\}.$$

You may assume that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$.

Continued...

4. Suppose that q is a positive integer.

   (a) i. (**2 marks**) Define what it means for a function $\chi : \mathbb{Z} \to \mathbb{C}$ to be a *Dirichlet character* modulo q.

   ii. (**2 marks**) Define the Gauss sum $\tau(\chi)$ associated to a Dirichlet character χ modulo q. Assuming that χ is primitive, what is the value of $|\tau(\chi)|$? (You do not need to include a proof.)

   iii. (**4 marks**) Prove that if χ is a Dirichlet character modulo q, which is not the trivial character, then

$$\sum_{n=1}^{q} \chi(n) = 0.$$

   (b) i. (**5 marks**) If χ is a Dirichlet character modulo q then define $\hat{\chi} : \mathbb{Z} \to \mathbb{C}$ by

$$\hat{\chi}(m) = \frac{1}{\sqrt{q}} \sum_{n=1}^{q} \chi(n) e\left(\frac{-mn}{q}\right),$$

   where $e(z) = \exp(2\pi i z)$. Prove that if χ is a Dirichlet character modulo q then

$$\chi(n) = \frac{1}{\sqrt{q}} \sum_{m=1}^{q} \hat{\chi}(m) e\left(\frac{mn}{q}\right).$$

   ii. (**4 marks**) Let χ be a Dirichlet character modulo q and let $m \in \mathbb{Z}$ satisfy $(m, q) = 1$. Prove that

$$\hat{\chi}(m) = \bar{\chi}(m)\hat{\chi}(1).$$

   (c) i. (**2 marks**) State the Pólya–Vinogradov inequality.

   ii. (**6 marks**) Let χ be a non-trivial Dirichlet character modulo q and let $f = \mathbf{1} * \chi$, where $\mathbf{1}(n) = 1$ for all positive integers n. Show that

$$\sum_{n \le x} f(n) = xL(1, \chi) + O(\sqrt{q} \log q).$$

End of examination.