10. Average orders

Now that we have defined some arithmetic functions it is natural to ask how large they are on average. The main tools for studying this question are partial summation and Perron's formula, together with the identities from the preceding section.

Definition 10.1. Let $f \in \mathcal{A}$ and let $g(x)$ be a monotonic increasing function of x. We say that $g(n)$ is the *average order* of $f(n)$ if

$$\sum_{n\le x} f(n) = xg(x) + o(xg(x)).$$

The prime number theorem says that the average size of the von Mangoldt function $\Lambda(n)$ is 1. We saw in the example after Theorem 2.3 that the average order of $\log n$ is $\log n$.

Lemma 10.2. *The average order of $\varphi(n)/n$ is $6/\pi^2$.*

Proof. We have

$$\sum_{n\le x} \frac{\varphi(n)}{n} = \sum_{n\le x}\prod_{p|n}\left(1-\frac{1}{p}\right) = \sum_{n\le x}\sum_{d|n}\frac{\mu(d)}{d} = \sum_{d\le x}\frac{\mu(d)}{d}\sum_{\substack{n\le x\\ d|n}} 1.$$

The inner sum is $[x/d] = x/d + O(1)$, whence

$$\sum_{n\le x}\frac{\varphi(n)}{n} = x\sum_{d\le x}\frac{\mu(d)}{d^2} + O\left(\sum_{d\le x}\frac{1}{d}\right).$$

Now

$$\sum_{d\le x}\frac{1}{d} \ll \int_1^x \frac{dt}{t} \ll \log x$$

and

$$\sum_{d>x}\frac{1}{d^2} \ll \int_x^\infty \frac{dt}{t^2} \ll \frac{1}{x}.$$

Hence

$$\sum_{d\le x}\frac{\mu(d)}{d^2} = \sum_{d=1}^\infty \frac{\mu(d)}{d^2} - \sum_{d>x}\frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} + O(1/x).$$

Since $\zeta(2) = \pi^2/6$, we have therefore shown that

$$\sum_{n\le x}\frac{\varphi(n)}{n} = \frac{6}{\pi^2}x + O(1+\log x) = \frac{6}{\pi^2}x + O(\log x),$$

for $x \ge 2$. □

Using this result and partial summation (Theorem 2.3) one can show that the average order of $\varphi(n)$ is $3n/\pi^2$ (exercise).

The average order of $1/n$ is $(\log n)/n$. This can be seen through an application of partial summation. Taking $a_n = 1$ and $f(n) = 1/n$ in Theorem 2.3, we find that

$$\sum_{n\le x}\frac{1}{n} = \frac{[x]}{x} + \int_1^x \frac{[t]dt}{t^2} = 1 + O\left(\frac{1}{x}\right) + \int_1^x \frac{dt}{t} - \int_1^x \frac{\{t\}dt}{t^2},$$

since $[t] = t - \{t\} = t + O(1)$. Thus

$$\sum_{n \le x} \frac{1}{n} = 1 + O\left(\frac{1}{x}\right) + \int_1^x \frac{dt}{t} - \int_1^x \frac{\{t\}}{t^2} dt$$
$$= \log x + \gamma + O\left(\frac{1}{x}\right),$$

where

$$\gamma = 1 - \int_1^\infty \frac{\{t\}}{t^2} dt = 0.5772\ldots$$

is the *Euler–Mascheroni constant*. Note that we must have $\gamma = \lim_{x\to\infty}(\sum_{n\le x} 1/n - \log x)$.

We now have everything in place to calculate the average order of the divisor function.

Theorem 10.3 (Dirichlet)**.** *We have*

$$\sum_{n\le x} d(n) = x \log x + (2\gamma - 1)x + O(x^{1/2}).$$

In particular the average order of $d(n)$ is $\log n$.

Proof. We have

$$\sum_{n\le x} d(n) = \sum_{n\le x}\sum_{d|n} 1 = \sum_{de\le x} 1 = \sum_{\substack{de\le x\\ d\le\sqrt{x}}} 1 + \sum_{\substack{de\le x\\ e\le\sqrt{x}}} 1 - \sum_{\substack{de\le x\\ d,e\le\sqrt{x}}} 1.$$

The final term is $[\sqrt{x}]^2 = (\sqrt{x} + O(1))^2 = x + O(\sqrt{x})$. The remaining terms give

$$\sum_{\substack{de\le x\\ d\le\sqrt{x}}} 1 + \sum_{\substack{de\le x\\ e\le\sqrt{x}}} 1 = 2\sum_{\substack{de\le x\\ d\le\sqrt{x}}} 1 = 2\sum_{d\le\sqrt{x}} \left[\frac{x}{d}\right] = 2x\sum_{d\le\sqrt{x}}\frac{1}{d} + O(\sqrt{x})$$
$$= 2x\left(\frac{\log x}{2} + \gamma + O(1/x)\right) + O(\sqrt{x}).$$

The statement of the theorem now follows. ☐

The method of proof in this argument is usually referred to as *Dirichlet's hyperbola method*. The *Dirichlet divisor problem* is to bound the error term

$$\Delta(x) = \sum_{n\le x} d(n) - x\log x - (2\gamma - 1)x.$$

The current record is due to Huxley (*Exponential Sums and Lattice Points III*, Proc. London Math. Soc. Vol. 87 (2003), 591609): $\Delta(x) = O_\varepsilon(x^{131/416+\varepsilon})$. Note that $\frac{131}{416} = 0.3149\cdots < \frac{1}{2}$. It is conjectured that $\frac{131}{416}$ can be replaced by $\frac{1}{4}$.

An alternative means of estimating the average order of arithmetic functions comes from studying their associated Dirichlet series and applying the truncated Perron formula (Theorem 4.2). This approach was at the heart of our proof of the prime number theorem in §5. It will be instructive to see what this approach gives for the divisor function. We have seen in §9 that the divisor function has associated Dirichlet series $\sum_{n=1}^{\infty} d(n)n^{-s} = \zeta(s)^2$ for $\Re(s) > 1$. Hence Theorem 4.2 implies that

$$\sum_{n\le x} d(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta(s)^2 x^s \frac{ds}{s} + O\left(x^c \sum_{n=1}^{\infty} \frac{d(n)}{n^c \max\{1, T|\log \frac{x}{n}|\}}\right).$$

for any $c > 1$ and any $x, T \ge 1$. For any $\varepsilon > 0$ we have $d(n) = O_\varepsilon(n^\varepsilon)$, by Lemma 9.13. To estimate the error term we reapply the method from the proof of Theorem 4.3, which involves splitting the sum into three parts: $n \le x/e$, $x/e < n < ex$ and $n \ge ex$. For the first and last parts we have $|\log \frac{x}{n}| \ge 1$ so that

$$x^c \sum_{n\le x/e} \frac{d(n)}{n^c|\log\frac{x}{n}|} \ll_\varepsilon x^{c+\varepsilon} \quad \text{and} \quad x^c \sum_{n\ge ex} \frac{d(n)}{n^c|\log\frac{x}{n}|} \ll_\varepsilon x^c \sum_{n\ge ex} \frac{1}{n^{c-\varepsilon}} \ll_\varepsilon x^{1+\varepsilon}.$$

Finally, for the middle part we use the following from the proof of Theorem 4.3:

$$\sum_{\frac{x}{e}<n<ex} \frac{1}{\max(1, T|\log\frac{x}{n}|)} \ll \frac{x\log x}{T}.$$

Hence

$$\sum_{n\le x} d(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta(s)^2 x^s \frac{ds}{s} + O_\varepsilon\left(\frac{x^{c+\varepsilon}}{T}\right),$$

for any $\varepsilon > 0$. To handle the main term we let $\mathcal{C}$ be the contour joining the four corners $c - iT, c + iT, \frac{1}{2} + iT, \frac{1}{2} - iT$. Cauchy's theorem gives

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \zeta(s)^2 x^s \frac{ds}{s} = \mathrm{Res}_{s=1} \frac{\zeta(s)^2 x^s}{s} = x\log x + (2\gamma - 1)x,$$

since $\zeta(s)^2$ has a unique pole of order 2 in $\mathcal{C}$ at $s = 1$. (Note that one can use Theorem 5.1 to calculate the above residue at $s = 1$.)

Putting everything together we have shown that

$$\begin{aligned}\Delta(x) &\ll_\varepsilon \left|\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \zeta(s)^2 x^s \frac{ds}{s}\right| + \left|\int_{\frac{1}{2}-iT}^{c-iT} \zeta(s)^2 x^s \frac{ds}{s}\right| + \left|\int_{\frac{1}{2}+iT}^{c+iT} \zeta(s)^2 x^s \frac{ds}{s}\right| + \frac{x^{c+\varepsilon}}{T} \\ &\ll_\varepsilon x^{1/2}\int_{-T}^{T} \frac{|\zeta(\frac{1}{2}+it)|^2}{|\frac{1}{2}+it|} dt + \int_{\frac{1}{2}}^{c} \frac{|\zeta(\sigma - iT)|^2 x^\sigma}{T} d\sigma + \int_{\frac{1}{2}}^{c} \frac{|\zeta(\sigma + iT)|^2 x^\sigma}{T} d\sigma + \frac{x^{c+\varepsilon}}{T}.\end{aligned}$$

since $|x^{\sigma+it}| = x^\sigma$ and $|\sigma + it| \ge |t|$. In order to proceed we need to know about the size of $|\zeta(\sigma + it)|$ for $\sigma \in [\frac{1}{2}, c]$. This is an area of intensive study, even today. In order to simplify things we will assume the following:

Conjecture 10.4 (The Lindelöf hypothesis)**.** *Let $\sigma \ge \frac{1}{2}$ and let $\varepsilon > 0$. Then $|\zeta(\sigma + it)| = O_\varepsilon(1 + |t|^\varepsilon)$.*

Remark. In fact one has RH $\Rightarrow$ LH. The current record is $|\zeta(\sigma + it)| = O_\varepsilon(1 + |t|^{53/342+\varepsilon})$ for $\sigma \ge \frac{1}{2}$, which is due to Bourgain (*Decoupling, exponential sums and the Riemann zeta function*, 2014, arXiv:1408.5794). Note that $\frac{53}{342} = 0.1549\ldots$.

Assuming the Lindelöf hypothesis, we easily obtain

$$\Delta(x) \ll_\varepsilon x^{1/2}T^\varepsilon + \frac{x^c}{T^{1-\varepsilon}\log x} + \frac{x^{c+\varepsilon}}{T}.$$

Choosing $T = x$ and $c = 1 + 1/\log x$ we conclude that $\Delta(x) = O_\varepsilon(x^{1/2+\varepsilon})$, under the assumption of the Lindelöf hypothesis. This (essentially) recovers Theorem 10.3 — albeit only conditionally! In fact, by working a bit harder, one can adapt this method to show that $\Delta(x) = O_\varepsilon(x^{1/3+\varepsilon})$ without any unproved hypothesis on the size of the Riemann zeta function. This is discussed in detail by Titchmarsh (see Chapter XII in *The theory of the Riemann zeta function*, 2nd ed., Oxford Univ. Press, 1986).