Recall the Riemman zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ defined for $\Re(s) > 1$. We saw in Theorem 6.1 that it has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$.

Definition 12.1. Let $q \in \mathbb{N}$ and let χ be a Dirichlet character mod q. Then the *Dirichlet L-function* associated to χ is defined to be

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

for $\Re(s) > 1$.

It is clear that $L(s,\chi)$ is absolutely convergent for $\Re(s) > 1$. Since χ is completely multiplicative, moreover, we have

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

for $\Re(s) > 1$. If $\chi = \chi_0$ then

$$L(s,\chi_0) = \prod_{p \nmid q} \left(1 - p^{-s}\right)^{-1} = \prod_p \left(1 - p^{-s}\right)^{-1} \prod_{p|q} \left(1 - p^{-s}\right) = \zeta(s) \prod_{p|q} \left(1 - p^{-s}\right).$$

Hence $L(s,\chi_0)$ behaves like $\zeta(s)$.

For $\chi \neq \chi_0$, $L(s,\chi)$ behaves very differently.

Theorem 12.2. *Let $q \in \mathbb{N}$ and let χ be a Dirichlet character mod q such that $\chi \neq \chi_0$. Then $L(s,\chi)$ converges for $\Re(s) > 0$.*

Corollary 12.3. *$L(s,\chi)$ is holomorphic for $\Re(s) > 0$ if $\chi \neq \chi_0$.*

Proof of Theorem 12.2. Since $\chi \neq \chi_0$ we have

$$\sum_{n=1}^{q} \chi(n) = \sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi(n) = 0,$$

by Corollary 11.8. Hence

$$\left| \sum_{n=M}^{N} \chi(n) \right| = O_q(1),$$

where the implied constant depends only on q (and not on M, N). An application of partial summation therefore yields

$$\begin{aligned} \sum_{M \le n \le N} \chi(n) n^{-s} &= N^{-s} \sum_{n \le N} \chi(n) - M^{-s} \sum_{n \le M} \chi(n) + s \int_M^N x^{-s-1} \left(\sum_{M \le n \le x} \chi(n) \right) dx \\ &= O_q(M^{-\Re(s)}) + O\left(\frac{|s|}{|\Re(s)|} \left(M^{-\Re(s)} + N^{-\Re(s)}\right) \right). \end{aligned}$$

This tends to 0 as $M, N \to \infty$ for $\Re(s) > 0$. Hence $\sum_{M \le n \le N} \chi(n) n^{-s}$ is a Cauchy sequence. Thus, since the partial sums of $\sum_{n=1}^{\infty} \chi(n) n^{-s}$ form a Cauchy sequence for $\Re(s) > 0$, it follows from the Cauchy convergence criterion that $\sum_{n=1}^{\infty} \chi(n) n^{-s}$ converges for $\Re(s) > 0$. $\square$

Remark. Although $L(s,\chi)$ converges for $\Re(s) > 0$, the Euler product is only valid for $\Re(s) > 1$ (since we need absolute convergence). In particular we can't use the Euler product to say anything about $L(1,\chi)$ for $\chi \neq \chi_0$.

Theorem 12.4. *Let $q \in \mathbb{N}$ and let χ be a Dirichlet character mod q. Then $L(1,\chi) \neq 0$ if $\chi \neq \chi_0$.*

Proof of Theorem 12.4. We first claim that

$$\prod_{\chi \bmod q} L(s,\chi) \geq 1, \tag{*}$$

for any real number $s > 1$. We begin by taking logs to get

$$\log \prod_{\chi \bmod q} L(s,\chi) = \log \prod_{\chi \bmod q} \prod_p (1-\chi(p)p^{-s})^{-1} = -\sum_{\chi \bmod q} \sum_p \log(1-\chi(p)p^{-s}),$$

where each factor $(1-\chi(p)p^{-s})$ is non-zero since $s > 1$. We now invoke the well-known series expansion

$$\log(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$$

for $|x| < 1$. (This remains true for $x \in \mathbb{C}$ by taking the standard branch of the logarithm.) Hence it follows that

$$\begin{aligned} \log \prod_{\chi \bmod q} L(s,\chi) &= \sum_{\chi \bmod q} \sum_p \sum_{k=1}^{\infty} \chi(p)^k \frac{p^{-ks}}{k} \\ &= \sum_p \sum_{k=1}^{\infty} \frac{p^{-ks}}{k} \sum_{\chi \bmod q} \chi(p^k). \end{aligned}$$

An application of Corollary 11.8 shows that the inner sum is $\varphi(q)$ if $p^k \equiv 1 \bmod q$ and 0 otherwise. Hence

$$\log \prod_{\chi \bmod q} L(s,\chi) = \varphi(q) \sum_p \sum_{\substack{k=1 \\ p^k \equiv 1 \bmod q}}^{\infty} \frac{p^{-ks}}{k} \geq 0,$$

as claimed in (*).

Now suppose that there exist Dirichlet characters $\chi_1 \neq \chi_2 \bmod q$ such that $L(1,\chi_1) = L(1,\chi_2) = 0$. Recall that

$$L(s,\chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right)$$

has a pole of order 1 at $s = 1$. On the other hand, $L(s,\chi_1)L(s,\chi_2)$ has a zero of multiplicity 2 at $s = 1$. Hence $\prod_\chi L(s,\chi)$ has a zero at $s = 1$, which contradicts (*).

If χ is not a real character then $\bar{\chi} \neq \chi$. Hence if $L(1,\chi) = 0$ then

$$L(1,\bar{\chi}) = \sum_{n=1}^{\infty} \bar{\chi}(n)n^{-1} = \overline{\sum_{n=1}^{\infty} \chi(n)n^{-1}} = \overline{L(1,\chi)} = 0.$$

Thus we have distinct characters $\chi, \bar{\chi}$ such that $L(1,\chi) = L(1,\bar{\chi}) = 0$, which we've already seen is impossible.

Hence there is at most 1 character χ mod q such that $L(1,\chi)=0$ and it must be real. In this case $L(s,\chi)L(s,\chi_0)$ is holomorphic in $\Re(s)>0$. Hence

$$\psi(s)=\frac{L(s,\chi)L(s,\chi_0)}{L(2s,\chi_0)}$$

is holomorphic for $\Re(s)>\frac{1}{2}$. Clearly $\psi(s)\to 0$ as $s\to\frac{1}{2}+$, since

$$L(2s,\chi_0)=\zeta(2s)\prod_{p|q}\left(1-\frac{1}{p^{2s}}\right)\to\infty$$

as $s\to\frac{1}{2}+$. The Euler expansion for $\psi(s)$ gives

$$\begin{aligned}\psi(s)&=\prod_p\frac{(1-\chi(p)p^{-s})^{-1}(1-\chi_0(p)p^{-s})^{-1}}{(1-\chi_0(p)p^{-2s})^{-1}}\\&=\prod_{p|q}1\prod_{\substack{p\nmid q\\ \chi(p)=1}}\frac{(1-p^{-s})^{-2}}{(1-p^{-2s})^{-1}}\prod_{\substack{p\nmid q\\ \chi(p)=-1}}\frac{(1+p^{-s})^{-1}(1-p^{-s})^{-1}}{(1-p^{-2s})^{-1}}\\&=\prod_{\substack{p\\ \chi(p)=1}}\frac{1+p^{-s}}{1-p^{-s}},\end{aligned}$$

for $\Re(s)>1$. Note that the product is non-empty, since otherwise $\psi(s)=1$ for $\Re(s)>1$, hence for $\Re(s)>\frac{1}{2}$ by analytic continuation, which contradicts $\psi(s)\to 0$ as $s\to\frac{1}{2}+$.

By expanding $(1-p^{-s})^{-1}$ into a geometric series, we obtain

$$\psi(s)=\sum_{n=1}^{\infty}a_n n^{-s},$$

for $\Re(s)>1$, where all $a_n\geq 0$ and $a_1=1$. Since $\psi(s)$ is holomorphic for $\Re(s)>\frac{1}{2}$, Taylor expansion at 2 yields

$$\psi(s)=\sum_{m=0}^{\infty}\frac{1}{m!}\psi^{(m)}(2)(s-2)^m.$$

Comparing these two formulae, and noting that $n^{-s}=e^{-s\log n}$, we get

$$\psi^{(m)}(2)=(-1)^m\sum_{n=1}^{\infty}a_n(\log n)^m n^{-2}=(-1)^m b_m,$$

say, where $b_m\geq 0$ since $a_n\geq 0$. Hence

$$\psi(s)=\sum_{m=0}^{\infty}\frac{1}{m!}b_m(2-s)^m$$

for $|2-s|<\frac{3}{2}$. So for $\frac{1}{2}<s<2$ we obtain

$$\psi(s)\geq\psi(2)\geq a_1=1,$$

since $b_m\geq 0$. This contradicts the fact that $\psi(s)\to 0$ as $s\to\frac{1}{2}+$. Hence there is no real character χ such that $L(1,\chi)=0$. □

One of the most important consequences of using Dirichlet L-functions is to say something about primes in a fixed congruence class.

Exercise. Adapt Euclid's proof to show that there are infinitely many primes $p \equiv -1 \bmod 4$. (Hint: consider $4p_1 \dots p_r - 1$.)

Theorem 12.5 (Dirichlet)**.** *Let $q \in \mathbb{N}$ and let $a \in \mathbb{Z}$ such that $(a, q) = 1$. Then*

$$\sum_{p \equiv a \bmod q} \frac{1}{p}$$

is divergent. In particular, there are infinitely many primes $p \equiv a \bmod q$.

Note that the condition $(a, q) = 1$ is clearly necessary in the statement of the theorem. The conclusion can be rephrased as saying that the linear polynomial $a + qX$ is prime infinitely often. No analogue is known for any $f \in \mathbb{Z}[X]$ of degree > 1. (The case $f(X) = X^2 + 1$ is a famous conjecture.)

Proof of Theorem 12.5. The proof of Theorem 12.4 shows that

$$\log L(s, \chi) = \sum_p \sum_{k=1}^{\infty} \chi(p^k) \frac{p^{-ks}}{k},$$

for $\Re(s) > 1$. Hence Corollary 11.9 implies that

$$\begin{aligned} \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \log L(s, \chi) &= \frac{1}{\varphi(q)} \sum_p \sum_{k=1}^{\infty} \frac{p^{-ks}}{k} \sum_{\chi \bmod q} \bar{\chi}(a)\chi(p^k) \\ &= \sum_p \sum_{\substack{k=1 \\ p^k \equiv a \bmod q}}^{\infty} \frac{p^{-ks}}{k} \\ &= \sum_{p \equiv a \bmod q} \frac{1}{p^s} + O(1), \end{aligned}$$

as $s \to 1$. The left hand side is

$$\frac{1}{\varphi(q)} \left(\bar{\chi}_0(a) \log L(s, \chi_0) + \sum_{\chi \neq \chi_0} \bar{\chi}(a) \log L(s, \chi) \right)$$

The second term is bounded as $s \to 1$ by Theorem 12.4, while the first term tends to infinity as $s \to 1$, since $\log L(s, \chi_0) = \log \left(\zeta(s) \prod_{p|q} (1 - p^{-s}) \right)$ This shows that

$$\sum_{p \equiv a \bmod q} \frac{1}{p^s} \to \infty$$

as $s \to 1$, as required. □

Although it is beyond the scope of this course, it is also possible to establish a version of the prime number theorem for primes in arithmetic progression. For $(a, q) = 1$, let

$$\psi(s; q, a) = \sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \Lambda(n) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \sum_{n \leq x} \chi(n)\Lambda(n).$$

The contribution of the trivial character χ_0 gives the main term. Dealing with the non-trivial characters ultimately leads to the following important result.

Theorem 12.6 (Siegel–Walfiscz)**.** *Let* $(a, q) = 1$ *such that* $q \leq (\log x)^{1-\delta}$ *for some fixed* $\delta > 0$*. Then there exists* $c > 0$ *such that*

$$\psi(x; q, a) = \frac{x}{\varphi(q)} + O\left(x \exp(-c\sqrt{\log x})\right),$$

where all the constants are computable and don't depend on q*.*