We take $x \geq T \geq 1$ and we wish to bound

$$\sum_{\frac{x}{e}<n<ex} \frac{1}{\max(1, T|\log\frac{x}{n}|)}.$$

We split $\sum_{\frac{x}{e}<n<ex}$ as $\sum_{\frac{x}{e}<n<xe^{-\frac{1}{T}}} + \sum_{xe^{-\frac{1}{T}} \leq n \leq xe^{\frac{1}{T}}} + \sum_{xe^{\frac{1}{T}}<n<ex}$. On the first range, $\frac{1}{T\log\frac{x}{n}}$ is increasing and each term is bounded by 1, so the sum is at most

$$1 + \int_{x/e}^{xe^{-\frac{1}{T}}} \frac{dt}{T\log\frac{x}{t}} = 1 + \int_{1/T}^{1} \frac{xe^{-u}}{Tu}\, du \leq 1 + \frac{x}{T}\int_{1/T}^{1} \frac{du}{u} = 1 + \frac{x\log T}{T} \ll \frac{x\log x}{T},$$

and similarly for the third range. For the middle range we use the trivial estimate

$$\sum_{xe^{-\frac{1}{T}} \leq n \leq xe^{\frac{1}{T}}} 1 \leq x\left(e^{\frac{1}{T}} - e^{-\frac{1}{T}}\right) + 1 = 2x\sinh\left(T^{-1}\right) + 1 \ll \frac{x}{T}.$$