Forest fires, explosions, and random trees

Edward Crane

HIMR, UoB

13th January 2014

Acknowledgements

This is joint work with Nic Freeman (Heilbronn Institute) and Balint Tóth (UoB / Budapest).

We have benefitted from a Heilbronn Institute collaboration grant to work with Christina Goldschmidt and James Martin at the University of Oxford.

Overview

This talk is about the *mean field forest fire model* introduced by Balázs Ráth and Bálint Tóth in a 2009 paper.

It is a coagulation-fragmentation process that exhibits *self-organized criticality*.

- Definition of the finite model.
- The global limit as $n \to \infty$ – work of Ráth and Toth.
- Our latest result - the Benjamini-Schramm (tagged particle) limit.
- The stationary cluster growth process and the Brownian CRT.

The mean field forest fire model

The mean field forest fire model is a continuous-time Markov chain evolving in the space of graphs on n (labelled) vertices.

- Simplest case: start with no edges at time $t = 0$.
- Each pair of non-adjacent vertices is joined by an edge at rate $1/n$, independently.
- Each vertex is hit by lightning at rate $\lambda(n)$, independently.
- When v is hit by lightning, all the edges in the connected component (cluster) of v disappear instantaneously.

The global limit

We write $v_k^n(t)$ for the proportion of vertices at time t that are contained in clusters of size k.

$$\sum_{k=1}^{n} v_k^n(t) = 1$$

The vector $v_k^n(t)$ is a Markov chain.

What is the limiting behaviour of $v_k^n(t)$ as $n \to \infty$?

Regime 1: $n\lambda_n \to 0$ as $n \to \infty$

Over any fixed time interval $[0, T]$, the probability of seeing any lightning tends to zero as $n \to \infty$.

So the asymptotic behaviour is the same as for the dynamical formulation of the Erdős-Rényi random graph model.

The limiting process $(v_k)(t)$ is deterministic and satisfies a system of ODEs called the *Smoluchowski coagulation equations*:

$$\dot{v}_k \;=\; \frac{k}{2} \sum_{i=1}^{k-1} v_i(t) v_{k-i}(t) \;-\; k\, v_k(t), \quad k \geq 1\,.$$

Regime I: $n\lambda_n \to 0$ as $n \to \infty$

For the *monodisperse* initial condition, an induction shows

$$v_k(t) = \frac{k^{k-1}}{k!} e^{-kt} t^{k-1} \,.$$

Regime I: $n\lambda_n \to 0$ as $n \to \infty$

For the *monodisperse* initial condition, an induction shows

$$v_k(t) = \frac{k^{k-1}}{k!} e^{-kt} t^{k-1} .$$

Gelation: the appearance of a unique giant component at $t = 1$.

For $t < 1$, $\sum_{k=1}^{\infty} v_k(t) = 1$, exponential decay – *subcritical*.

For $t = 1$, $\sum_{k=1}^{\infty} v_k(1) = 1$, and $v_k(1) \sim \frac{1}{k^{3/2}\sqrt{2\pi}}$ – *critical*.

For $t > 1$, $\sum_{k=1}^{\infty} v_k(t) < 1$, exponential decay – *supercritical*.

Regime 2: $n\lambda_n = \lambda \in (0, \infty)$

Lightning strikes occur as a Poisson process of rate λ.

In between lightning strikes, the limiting system evolves according to the Smoluchowski coagulation equations.

If lightning strikes at a time t when the system is subcritical, the lightning has no effect.

Otherwise, it causes $v_1(t)$ to jump so that $\sum_{k=1}^{\infty} v_k(t^+) = 1$.

Regime 3: $\lambda_n \to 0$ but $n\lambda_n \to \infty$ as $n \to \infty$

The most interesting case!

Balázs Ráth and Bálint Tóth [1] proved that the process $(v_k)^n(t)$ converges in probability (with respect to a suitable topology).

The limit is the (unique) solution of the *critical forest fire equations*

$$\dot{v}_k = \frac{k}{2} \sum_{i=1}^{k-1} v_i(t) v_{k-i}(t) - k\, v_k(t), \quad k \geq 2,$$

$$\sum_{k=1}^{\infty} v_k(t) = 1$$

[1] B. Ráth and B. Tóth, *Erdős-Rényi random graphs + forest fires = self-organized criticality*, Elec. J. Prob. **14** (2009), 1290–1327.

Regime 4: $\lambda_n = \lambda \in (0, \infty)$

Each vertex is hit by lightning at times of a Poisson process of rate λ.

In the limit the system satisfies a system of ODEs:

$$\dot{v}_k = \frac{k}{2} \sum_{i=1}^{k-1} v_i(t) v_{k-i}(t) - (1+\lambda)\, k\, v_k(t), \quad k \geq 2,$$

$$\dot{v}_1 = \lambda \sum_{k=2}^{\infty} k v_k(t) - v_1(t)$$

The methods of [1] show that this has a unique solution, which stays subcritical for all time.

The local limit in regime 3

We define the *cluster growth process* $X(t)$ to be the unique càdlàg Markov process $X(t)$ satisfying

- $X(0) = 1$,
- $X(t)$ has holding time exponential with mean $1/k$ in state k,
- When $X(t)$ jumps from k at time t, its increment is independent of the jump time. The increment is a sample from the law $(v_k)(t)$, the solution of the critical forest fire equations at time t.
- When $X(t)$ explodes, it returns instantaneously to 1.

The local limit in regime 3 (new result)

Let $X_n(t)$ be the cluster size of vertex 1 at time t in the n-node mean field forest fire model.

Let E be $\mathbb{N}$ obtained by making 1 the limit of n as $n \to \infty$.

Theorem (C, Freeman and Tóth, 2014)

For each $T > 0$, the process $X_n(t)$ restricted to the time interval $[0, T]$ converges as $n \to \infty$ to the cluster growth process $X(t)$. The convergence is in probability with respect to the Skorohod topology on the space of càdlàg paths in E.

An idea of the proof

We prove that $\mathbb{P}(X_n(t) = k) = v_k(t)$ for all t, by showing that $X_n(t)$ satisfies a linearized version of the critical forest fire equations which has a unique solution. The difficult part is to show that any solution must have the same rate of return of mass to state 1 as the critical forest fire solution.

The convergence proof works by coupling the cluster of vertex 1 in the n-node model to the cluster growth process. We show that for any $\epsilon > 0$, the probability that they get more than ϵ apart in E before time T is less than ϵ for n sufficiently large.

The coupling relies on many details from [1].

The key step is to show that once the watched cluster gets large, it burns quickly with high probability, uniformly in n.

The fixed point of the critical forest fire equations

The critical forest fire equations have a unique fixed point, given by

$$v_k = \frac{2}{k}\binom{2k-2}{k-1}4^{-k},$$

This Catalan-type distribution has probability generating function

$$\sum_{k=1}^{\infty} v_k z^k \;=\; 1 - \sqrt{1-z}\,.$$

In fact there is a unique probability distribution P on isomorphism classes of finite rooted trees such that the cluster growth process in the environment P almost surely explodes, with finite expected time to explosion, and has stationary distribution equal to P.

This distribution can be constructed in terms of the genealogical trees of the clusters, which have the law of the critical binary Galton-Watson tree.

The cluster growth process in the fixed-point environment

The cluster growth model in the fixed point environment can be defined as a continuous-time Markov chain in the space of finite rooted trees. It has a unique stationary distribution. We can compute many of its properties exactly using generating functions:

- The degree of the root has pgf $\left(4^s - 2\right)/(4s - 2)$.
- The joint distribution of root degree and cluster size has pgf

$$\frac{2}{2s-1}\left(\left(\frac{1+\sqrt{1-z}}{2}\right)^{2-2s} - \left(\frac{1+\sqrt{1-z}}{2}\right)\right).$$

The cluster growth process in the fixed-point environment

- The joint distribution of age (time since last explosion) and watched cluster size has generating function

$$\sum_{k=1}^{\infty} z^k \mathbb{P}(X(t) = k \text{ and age} > t) \; = \; \frac{(1-\sqrt{1-z})(1-\tanh(t/2))}{1+\tanh(t/2)\sqrt{1-z}} .$$

The cluster growth process in the fixed-point environment

- The distribution of the excess time to the next explosion is

$$\mathbb{P}(t_\infty - t < x \,|\, c_t = k) \;=\; 1 - \cosh(x)^{-2k}\,,$$

which has mean $\dfrac{4^k}{k\binom{2k}{k}}$.

The Brownian CRT

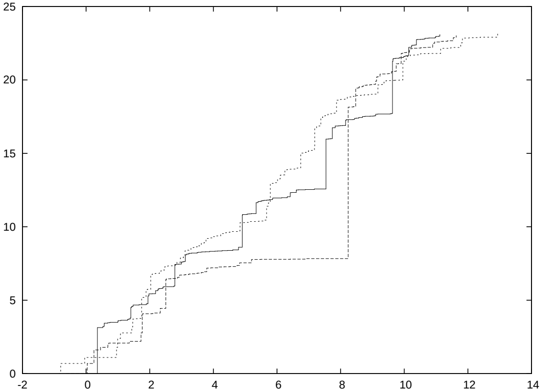
The Brownian continuum random tree is a random pointed measured metric space that is known to occur as a scaling limit in a number of different probabilistic models, and has several simple and not obviously related constructions. It occurs here too:

Theorem

Let Z_k be the stationary cluster growth process $X(t)$ in the fixed-point environment, conditioned to have size k. Make Z_k a random pointed metric space by giving each edge length $3/(2\sqrt{2k})$. Then Z_k converges in distribution with respect to the Gromov-Hausdorff topology to the Brownian continuum random tree.

The proof is analytic, using generating functions, singularity analysis, the moment method, and the Crámer-Wold device.

Infinitely many doublings before each explosion



log(cluster size) versus -log(time until explosion)
(three sample explosions)

Some questions

- Do the stationary states of the mean field forest fire models converge in probability to the constant system whose value is the fixed point of the critical forest fire equation?
- How do the largest clusters in the mean field forest fire model evolve?
- Is there a way to construct an exchangeable infinite limit of the mean field forest fire models?
- What happens in the case where edges are oriented and the forest fire spreads only in the direction of the oriented edges?