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A BOUND FOR SMALE'S MEAN VALUE CONJECTURE FOR COMPLEX POLYNOMIALS

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Abstract

Smale's mean value conjecture is an inequality that relates the locations of critical points and critical values of a polynomial p to the value and derivative of p at some given non-critical point. Using known estimates for the logarithmic capacity of a connected set in the plane containing three given points, we give a new bound for the constant in Smale's inequality in terms of the degree d of p. The bound improves previous results when $d \geq 8$.

1. *Introduction*

Let p be a polynomial with coefficients in $\mathbb{C}$. We say $\zeta \in \mathbb{C}$ is a *critical point* of p when $p'(\zeta) = 0$. Its image $p(\zeta)$ is the corresponding *critical value*. In 1981 Smale proved the following inequality concerning the critical points and critical values of polynomials, in connection with root-finding algorithms.

THEOREM Smale, [**14**].
Let p be a polynomial of degree $N \geq 2$ over $\mathbb{C}$ and suppose that $x \in \mathbb{C}$ is not a critical point of p. Then there exists a critical point ζ of p such that

$$\left| \frac{p(\zeta) - p(x)}{\zeta - x} \right| \leq 4 \left| p'(x) \right| . \tag{1.1}$$

Thus the derivative of p at each point can be estimated in terms of the gradients of chords of the graph $\{(z, w) \in \mathbb{C}^2 \ : \ w = p(z)\}$, joining the point $(x, p(x))$ to a stationary point $(\zeta, p(\zeta))$, where $p'(\zeta) = 0$. In this way Smale's inequality is analogous to the mean value theorem.

We define

$$S(p, x) \ = \ \min \left(\left| \frac{p(\zeta) - p(x)}{(\zeta - x) p'(x)} \right| \ : \ p'(\zeta) = 0 \right)$$

Any critical point ζ which achieves this minimum will be called an *essential* critical point of p with respect to x. For each $d \geq 2$ we denote the best possible constant on the right-hand side of (1.1) by $K(d)$. That is to say,

$$K(d) \ = \ \sup \{ \, S(p, x) : \deg(p) = d, \ p'(x) \neq 0 \, \} .$$

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Smale conjectured that $K(d) = 1 - 1/d$, in view of the example

$$p(z) = \chi_d(z) = z^d + z\,, \quad S(\chi_d, 0) = 1 - 1/d\,. \tag{1.2}$$

In [**1**], Beardon, Minda and Ng observed that Smale's proof of $K(d) \leq 4$ could be sharpened using an estimate of the hyperbolic density in a certain plane domain. They showed

$$K(d) \leq 4^{1-1/(d-1)}\,.$$

More recently Conte, Fujikawa and Lakic [**2**] proved

$$K(d) \leq 4\left(\frac{d-1}{d+1}\right)$$

by an ingenious repeated use of the bound $|a_2| \leq 2$ for the second coefficient of a schlicht function. These last two ideas were combined by Fujikawa and Sugawa [**6**] to give the bound

$$K(d) \;\leq\; 4\left(\frac{1 + (d-2)4^{\frac{1}{1-d}}}{d+1}\right)\,.$$

In each of the above results, the upper bound on $K(d)$ is asymptotically of the form $4 - O(1/d)$ as $d \to \infty$. The main result of the present paper is the following.

THEOREM 1.1. *For $d \geq 8$ we have*

$$K(d) < 4 - \frac{2.263}{\sqrt{d}}\,.$$

This bound is also valid for $2 \leq d \leq 7$, since in fact for these degrees better bounds are known. Indeed, Schmeisser [**13**] gave an elementary proof that

$$K(d) \leq \frac{2^d - (d+1)}{d(d-1)}\,.$$

The full strength of Smale's conjecture has been established for degrees 2,3 and 4 using algebraic methods (see [**12**, §7.2]) and in a forthcoming paper [**4**] we settle the case of degree 5 by a computational method. This explains why we have restricted Theorem 1.1 to the cases where $d \geq 8$.

In [**3**] we showed that for each degree $d \geq 2$ there exists a polynomial p of degree d such that $p'(0) \neq 0$, $S(p, 0) = K(d)$, and all the critical points of p are essential with respect to 0. This result, together with the main ideas from [**1**, **2**, **6**], allows us to improve the bounds of the form $4 - O(1/d)$ to one of the form $4 - O(1/\sqrt{d})$.

In §2 of the present paper we give a simple proof of the following weaker result, which does not rely on the use of a computer for numerical approximation.

PROPOSITION 1.2. *There exists d_0 such that for all $d \geq d_0$,*

$$K(d) \;\leq\; 4 - 2/\sqrt{d}\,,$$

and hence there exists $C > 0$ such that for all $d \geq 2$,

$$K(d) \;\leq\; 4 - C/\sqrt{d}\,.$$

In §3 we work harder to make the bound effective. This involves making a careful choice of the parameters involved in the proof of Proposition 1.2, and allows us to prove Theorem 1.1, assuming an explicit lower bound for the logarithmic capacity of a certain subset of $\mathbb{C}$. In §4 we show how to compute this logarithmic capacity with rigorous error bounds.

In fact Theorem 1.1 is not the best possible bound that can be proved using the present method. For various values of d we used Maple to optimise two parameters that remain fixed in the proof of Theorem 1.1. Some upper bounds for $K(d)$ that we obtained in this way are given in the following table. They were obtained using the numerical methods described in §4, but we shall not say anything about the methods used to find optimal parameter values, since there were no interesting mathematical ideas involved.

d	Bound in Theorem 1	Optimised bound for $K(d)$
8	3.2	3.124
9	3.246	3.150
10	3.285	3.169
11	3.318	3.188
12	3.347	3.208
16	3.435	3.271
25	3.548	3.365
100	3.774	3.621
10000	3.978	3.955

Throughout the paper we use the notation $D(a, r)$ to denote the open disc of radius r about a complex number a. We use $\mathbb{D}$ to denote the unit disc $D(0, 1)$ and $\widehat{\mathbb{C}}$ to denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

The author would like to thank Professor Toshiyuki Sugawa for his assistance in verifying the numerical estimates of logarithmic capacity that appear in §4, using a different method. Professor Sugawa also independently suggested the use of Jenkins' result (Lemma 3.2) after reading an early draft of this paper.

2. *Short proof of Proposition 1.2*

For the rest of this paper we will deal with a normalized case of the problem. The reader can check that if $A(z) = ax + b$ and $B(x) = cx + d$ with $a, c \neq 0$ then

$$S(A \circ p \circ B, B^{-1}(x)) \;=\; S(p, x)\,.$$

This invariance of $S(\cdot,\cdot)$ under composition with affine maps implies that we need only consider the case where $x = 0$, $p(0) = 0$, and -1 is a critical point of minimal modulus, with $p(-1) = -1$. There is in general no further flexibility for normalization by composition with affine maps. In particular we do not assume either that p is monic or that $p'(0)$ is real.

We showed in [**3**] that in order to give an upper bound for $K(d)$ it suffices to consider only the case in which every critical point is essential, i.e. each of the critical points $\zeta_1, \ldots, \zeta_{d-1}$ satisfies

$$\left|\frac{p(\zeta_i)}{p'(0)\zeta_i}\right| = K(d)\,.$$

We can encapsulate the restrictions that we are now imposing on p in the following definition.

DEFINITION 1. p is a *standard extremal polynomial* when the following conditions are satisfied:

- $p(0) = 0$,
- -1 is a critical point of minimal modulus for p,
- $p(-1) = -1$,
- every critical point of p is essential (with respect to $x = 0$), and
- $S(p, 0) = K(d)$, where $d = \deg(p)$.

Let p be a standard extremal polynomial of degree d, given by

$$p(z) = a_1 z + a_2 z + \cdots + a_d z^d\,.$$

There are $d - 1$ critical points of p, counted by multiplicity. We label them in increasing order of modulus, repeated according to multiplicity, so that $\zeta_1 = -1$ and

$$1 = |\zeta_1| \le |\zeta_2| \le \cdots \le |\zeta_{d-1}|\,.$$

Since $a_1 = p'(0)$ and $p(\zeta_1) = \zeta_1$, we have

$$|a_1| = |p'(0)| = \frac{1}{S(p)} = \frac{1}{K(d)}\,.$$

We denote by f the branch of p^{-1} defined on the unit disc $\mathbb{D}$ such that $f(0) = 0$. We denote the Taylor expansion of f about 0 by

$$f(z) = b_1 z + b_2 z^2 + b_3 z^3 + \ldots\,.$$

We have $b_1 = 1/p'(0)$, and since f is univalent, we have $|b_2/b_1| \le 2$. The coefficients of p can all be recovered from those of f. In particular $a_1 = 1/b_1$ and $a_2 = -b_2/b_1^3$. The reciprocals of the critical points are the roots of the following polynomial:

$$z^{d-1}p'(1/z) = a_1 z^{d-1} + 2a_2 z^{d-2} + \cdots + d a_d\,.$$

Therefore

$$\sum_{i=1}^{d-1} \frac{1}{\zeta_i} = \frac{-2a_2}{a_1} = \frac{2b_2}{b_1^2} = \frac{2b_2 a_1}{b_1}\,,$$

$$\left|\sum_{i=1}^{d-1} \frac{1}{\zeta_i}\right| \le 2\left|\frac{b_2}{b_1}\right| . |a_1| \le \frac{4}{S(p)} = \frac{4}{K(d)} \le \frac{4d}{d-1}\,. \tag{2.1}$$

We now show that if p is a standard extremal polynomial and $S(p)$ is close to 4, then p must have many critical values of small modulus.

LEMMA 2.1. *Let p be a standard extremal polynomial. Suppose that $r > 1$ and $S(p) > 4\left(\frac{1+r^{-N}}{2}\right)^{2/N}$. Then p has at least $N+1$ distinct critical values with modulus at most r. In particular, if $N \geq 4$ and $S(p) > 4(1-1/N)$ then p has at least $N+1$ distinct critical values with modulus at most 2.*

Proof. Suppose that there are at most N distinct critical values with modulus at most r. We define a domain U as follows.

$$U \;=\; D(0,r) \setminus \bigcup_{i=1}^{d-1} \{\lambda p(\zeta_i) \;:\; \lambda \in [1,\infty)\}\,.$$

U is constructed by removing from $D(0,r)$ a radial slit emanating from each critical value of p in $D(0,r)$; there are at most N such slits. Note that U is simply connected and contains no critical values of p. We will now obtain an upper estimate the density at 0 of the complete hyperbolic metric on U, which we denote by $\lambda_U(0)$. Because $U \supset \mathbb{D}$, we find immediately from the Schwarz-Pick lemma that $\lambda_U(0) \leq 2$. However, we require a better estimate of the form $\lambda_U(0) \leq 2-c/N$ for some positive constant c. Such an estimate is provided by Dubinin's desymmetrization method [**5**, p. 270], and this is also the key ingredient in [**1**]. Dubinin proved the following:

THEOREM 2.2. *Suppose $r > 1$ and let $\alpha_1, \ldots, \alpha_k$ be distinct points on the unit circle. Let D denote the region obtained by removing from $D(0,r)$ the radial slits $\{t\alpha_j \;:\; 1 \leq t < r\}$ for $j = 1, \ldots, k$. Let D_0 be this region in the case where the α_j are the k-th roots of unity. Let f be the unique conformal map of D onto $\mathbb{D}$ with $f(0) = 0$ and $f'(0) > 0$, and let f_0 be the corresponding map for D_0. Then $f'(0) \leq f_0'(0)$.*

An equivalent conclusion in Dubinin's result is that $\lambda_D(0) \leq \lambda_{D_0}(0)$; it is also easy to check by a limiting argument that the result remains true if the α_i are not required to be distinct. In [**1**] this inequality was used in the limit $r \to \infty$. However, we apply it directly to the domain U, taking $k = N$ and taking the point α_i to be $p(\zeta_j)/|p(\zeta_j)|$ for $i = 1, \ldots, N$. Then the domain D in Dubinin's theorem is a subdomain of our domain U. Thus we have $\lambda_U(0) \leq \lambda_D(0) \leq \lambda_{D_0}(0)$.

On the other hand, we can compute the conformal map f_0^{-1} explicitly and hence determine $\lambda_{D_0}(0)$. Define

$$K_0(w) = \frac{w}{(1+w)^2}\,.$$

Then K_0 maps $\mathbb{D}$ univalently onto $\mathbb{C} \setminus [1/4, \infty)$. Let $t = 4K(r^{-N})$ and define

$$K_1(w) = r^N K_0^{-1}(tK_0(w))\,.$$

Then

$$K_1'(0) \;=\; r^N t \;=\; 4r^N K\left(r^{-N}\right) \;=\; \frac{4}{\left(1+r^{-N}\right)^2}\,.$$

K_1 maps $\mathbb{D}$ conformally onto $r^N\mathbb{D} \setminus [1, r^N)$. Now $(f_0(z))^N \;=\; K_1^{-1}\left(z^N\right)$, so by comparing coefficients of z^N we obtain

$$\left(f_0'(0)\right)^N \;=\; \left(K_1^{-1}\right)'(0) \;=\; \frac{\left(1+r^{-N}\right)^2}{4}\,.$$

Therefore

$$\lambda_U(0) \leq \lambda_{D_0}(0) \ = \ 2f_0'(0) \ = \ 2\left(\frac{1+r^{-N}}{2}\right)^{2/N} .$$

Now we apply Koebe's one-quarter theorem. The map f omits $\zeta_1 = -1$, so we have $\lambda_{f(U)}(0) \geq 1/2$. However,

$$\frac{|f'(0)|\,\lambda_{f(U)}(0)}{\lambda_U(0)} = 1\,,$$

so

$$S(p) = |f'(0)| \leq 2\lambda_U(0) \leq 4\left(\frac{1+r^{-N}}{2}\right)^{2/N} . \tag{2.2}$$

To prove the final statement of Lemma 2.1, we assume $N \geq 4$ and fix $r = 2$. Since $(1-1/N)^N$ is an increasing function of N, we have

$$\left(\frac{1+r^{-N}}{2}\right)^2 \ \leq \ \left(\frac{17}{32}\right)^2 < (1-1/4)^4 \leq (1-1/N)^N .$$

Therefore

$$S(p) \leq 4\left(\frac{1+r^{-N}}{2}\right)^{2/N} \ < \ 4\left(1-\frac{1}{N}\right),$$

as required. □

Lemma 2.3. *Let $r > 1$. For any $s > -1/r$ there exists a constant $\delta = \delta(s) > 0$ with the following property. If p is a standard extremal polynomial, $S(p) > 4-\delta$, and ζ_j is a critical point of p such that $|p(\zeta_j)| \leq r$, then* $\mathrm{Re}\,(1/\zeta_j) < s$.

We will give a quantitative version of this estimate in §3, but we include this version here since its proof is simple and it is all that is required for the proof of Proposition 1.2.

Proof. We claim that there exists $\delta > 0$ such that if $F : \mathbb{D} \to \mathbb{C}$ is univalent, omits -1 and has $|F'(0)| > 4-\delta$ then $F(\mathbb{D})$ contains the set E defined by

$$E = \{z \in \widehat{\mathbb{C}} \mid \mathrm{Re}(1/z) \geq s, |z| < r\}.$$

This suffices to prove the lemma, since we may take F to be f, the branch of p^{-1} defined above. Then F omits ζ_j, so $\zeta_j \notin E$. But $|\zeta_j| = |p(\zeta_j)| < r$, so we must have $\mathrm{Re}\,(1/\zeta_j) < s$, as required.

Suppose for a contradiction that the claim is false. Then take a sequence $F_n : \mathbb{D} \to \mathbb{C}$ of univalent functions omitting -1, such that $|F_n'(0)| \to 4$ as $n \to \infty$, yet each F_n omits some point $z_n \in E$. Since E is relatively compact and the space of schlicht functions is compact, we can pass to a subsequence so that z_n converges to $z \in \overline{E}$ and the functions $g_n = F_n/|F_n'(0)|$ converge locally uniformly on $\mathbb{D}$ to a schlicht function g. Now g omits $-1/4$ so it is the Koebe function $g(w) = \frac{w}{(1-w)^2}$. However, $|F_n'(0)|g_n$ omits z_n, so $4g$ omits z. Here we are using the Carathéodory kernel theorem. On the other hand, $\overline{E}$ is disjoint from the slit $(-\infty, -1]$, which is the set omitted by $4g$, so we have the desired contradiction. □

We are now in a position to prove Proposition 1.2. We suppose for a contradiction that $S(p) \geq 4(1 - 1/2\sqrt{d})$. Then Lemma 2.1 applies with $r = 2$ and any value of N satisfying $4 \leq N \leq 2\sqrt{d}$. For d sufficiently large, Lemma 2.3 applies with $r = 2$ and $s = -1/3$. We take $N = \lfloor 2\sqrt{d} \rfloor > 2\sqrt{d} - 1$. We then have $\mathrm{Re}(1/\zeta_1) = -1$ and $\mathrm{Re}(1/\zeta_i) < -1/3$ for $i = 2, \ldots, N+1$. There are $d - N - 2$ remaining critical points, counted by multiplicity. Since $S(p) \geq 2$, in equation (2.1) we have

$$\left| \sum_{i=1}^{d-1} \frac{1}{\zeta_i} \right| \leq \frac{4}{S(p)} \leq 2 .$$

It follows that for some $j > N$ we have

$$\begin{aligned} \mathrm{Re}(1/\zeta_j) &> \frac{1 + N/3 - 2}{d - N - 2} \\ = \frac{N-3}{3(d-N-2)} &> \frac{2\sqrt{d} - 4}{3(d - 2\sqrt{d})} > \frac{2}{3\sqrt{d}} . \end{aligned}$$

Now f omits the point -1 and also omits the point ζ_j, which lies in the disc $D\left(3\sqrt{d}/4, 3\sqrt{d}/4\right)$.

Lemma 2.4. *Let $r > 0$ and suppose that $g : \mathbb{D} \to \mathbb{C}$ is univalent with $g(0) = 0$ and that g omits the points -1 and w, where $w \in \overline{D(r,r)}$. Then*

$$|g'(0)| \leq \frac{8r}{2r+1} = 4 - \frac{4}{2r+1} .$$

Proof. The function $1/g(1/z)$ defined on $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ maps ∞ to ∞ and omits a compact set E' containing -1 and $1/w$. We have $\mathrm{Re}(1/w) > 1/2r$, so $\mathrm{diam}(E') \geq 1 + 1/2r$ and hence

$$\frac{1}{|g'(0)|} = \mathrm{cap}(E') \geq (1 + 1/2r)/4 ,$$

using Koebe's 1/4 theorem. □

We apply Lemma 2.4 to the map f and the point $w = \zeta_j$, to obtain

$$S(p) = |f'(0)| \leq 4 - \frac{4}{(3\sqrt{d}/2) + 1} \leq 4 - \frac{2}{\sqrt{d}} ,$$

where the last inequality holds when $d \geq 4$. This contradicts our assumption, completing the proof of Proposition 1.2.

3. *Proof of Theorem 1.1*

We begin with a quantitative version of Lemma 2.3:

Lemma 3.1. *Whenever p is a standard extremal polynomial with $S(p) > 3.1$, and ζ_j is a critical point of p such that $|p(\zeta_j)| \leq 1.4$, then $\mathrm{Re}(1/\zeta_j) < -0.28$.*

Proof. Suppose for a contradiction that $\mathrm{Re}(1/\zeta_j) \geq -0.28$.
Define the set

$$J = \{0\} \cup \{1/z : z \in \mathbb{C} \setminus f(\mathbb{D})\}.$$

As above, the bound $S(p) > 3.1$ implies that the logarithmic capacity of J is at most $1/3.1$. On the other hand, the points $0, -1$ and $1/\zeta_j$ all belong to J, and this implies a lower bound for $\mathrm{cap}(J)$ in terms of $1/\zeta_j$, which we can minimise over the allowed region for $1/\zeta_j$. The parameters 1.4 and -0.28 in Lemma 3.1 are chosen so that this minimal capacity exceeds $1/3.1$, and this will give the desired contradiction.

For any three distinct points $x_1, x_2, x_3 \in \mathbb{C}$, we define

$$c(x_1, x_2, x_3) = \inf\{\mathrm{cap}(E) \mid x_1, x_2, x_3 \in E,\ E \subset \mathbb{C} \text{ connected}\}.$$

The function $c(x_1, x_2, x_3)$ has been determined by Kuz'mina (see [**10**, Theorem 1] or [**11**]). However, she gave its value in terms of three unknown parameters related by three equations involving x_1, x_2, x_3 and Jacobi's elliptic functions. We could not see an easy way to use her result directly to obtain a numerical lower bound on $\mathrm{cap}(J)$ given only that $1/\zeta_j$ lies outside a certain region. Fortunately we have the following result due to Jenkins.

LEMMA 3.2. (Jenkins [**8**, Theorem 1 and Corollary 1])
Let z vary on the ellipse defined by $|z+1| + |z| = L$, where $L > 1$. Then $c(-1, 0, z)$ attains its minimum at the two points on the ellipse where $|z| = |z+1|$. Moreover, the minimal value is an increasing function of L.

We must have $\mathrm{Re}(1/\zeta_j) \leq 0.28$, for otherwise we would have

$$\mathrm{diam}(J)^2 \geq 1.28^2 + 0.7^2 - 0.28^2 > 1.43^2,$$

so we would have $\mathrm{cap}(J) \geq 1.43/4$ and $S(p) < 4/1.43 < 3.1$, contrary to hypothesis. So $|\mathrm{Re}(1/\zeta_j)| \leq 0.28$, and by hypothesis $|1/\zeta_j| \geq 1/1.4 > 0.7$. We deduce that $|\mathrm{Im}(1/\zeta_j)| \geq 0.65$, so $|1/\zeta_j + 1| \geq \left(0.65^2 + (1-0.28)^2\right)^{1/2} = 0.97$, and therefore

$$\left|\frac{1}{\zeta_j} - (-1)\right| + \left|\frac{1}{\zeta_j}\right| \geq 0.97 + 0.7 = 1.67.$$

We take $L = 1.67$ in Lemma 3.2, and using the monotonicity statement it now suffices to find a lower bound for $c(-1, 0, -0.5 + it)$ where t is any positive real satisfying $t^2 + (1/2)^2 \leq (1.67/2)^2$. We take $t = 0.668$. In §4 we will prove the bound that we need, namely

LEMMA 3.3.

$$c(-1, 0, -0.5 + 0.668i) > 1/3.1.$$

This completes the proof of Lemma 3.1. □

Now we follow the same argument that we used to obtain Proposition 1.2 from Lemma 2.3, except that we use the parameters $r = 1.4$ and $s = -0.28$. This yields the following quantitative bound.

Proposition 3.4. *For any degree $d \geq 2$, and $1 \leq N \leq d-2$, we have*

$$S(p) \leq \max\left(3.1,\ \ 4\left(\frac{1+1.4^{-N}}{2}\right)^{2/N},\ \ 4 - \frac{3.472N - 3.6}{3.1d - 2.232N - 4}\right).$$

We must now estimate the minimum value of the above bound as we vary N. Since the second quantity in the maximum in Proposition 3.4 is an increasing function of N and the third argument is a decreasing one, we should take N close to the point where they are equal. For large d this occurs where N is around $2.25\sqrt{d}$.

To obtain the simple bound stated in Theorem 1.1, we first check separately the cases $8 \leq d \leq 12$. The following table gives the bound of Proposition 3.4, rounded up to 4 decimal places, together with the appropriate value of N, and also gives the bound of Theorem 1.1, rounded down to 4 decimal places.

d	N	Bound in Proposition 3.4	Bound in Theorem 1.1
8	4	3.1753	3.1999
9	5	3.2455	3.2457
10	5	3.2455	3.2844
11	5	3.2735	3.3176
12	6	3.3096	3.3467

For $d \geq 13$, we take $N = \lfloor 2.05\sqrt{d} \rfloor > 2.05\sqrt{d} - 1$, and then we can check algebraically that for $d \geq 13$ we have

$$4 - \frac{3.472N - 3.6}{3.1d - 2.232N - 4} \leq 4 - \frac{3.472(2.05\sqrt{d} - 1) - 3.6}{3.1d - 2.232(2.05\sqrt{d} - 1) - 4} \leq 4 - \frac{2.263}{\sqrt{d}}.$$

We also check that $N \geq 7$ and hence

$$\left(\frac{1+1.4^{-N}}{2}\right) \leq \left(1 - \frac{5}{4N}\right)^{N/2},$$

so

$$4\left(\frac{1+1.4^{-N}}{2}\right)^{2/N} \leq 4 - \frac{5}{N} \leq 4 - \frac{5}{2.05\sqrt{d}} < 4 - \frac{2.263}{\sqrt{d}}.$$

This completes the proof of Theorem 1.1.

4. *Proof of Lemma 3.3*

We use a numerical method to estimate the minimal capacity c given by

$$c = c(-1, 0, -0.5 + 0.668i) = c(i, -i, 1.336)/2\,.$$

Jenkins' proof of Lemma 3.2 uses his solution of the problem of minimising the modulus of any point omitted by a Bieberbach-Eilenberg function, and in principle this gives a numerical method for evaluating $c(-1, 0, -1/2 + it)$. It is also possible to use Kuz'mina's results to evaluate $c(-1, 0, -1/2 + it)$ numerically. However, we will use different method, using a classical formula for the extremal mapping, in order to estimate $c(-1, 0, -1/2 + it)$ with error bounds. In each of these methods there is an implicitly defined parameter that must be solved for.

Our chosen method is based on the following well-known results about the extremal configuration for $c(x_1, x_2, x_3)$ (see for example [**7**, **9**]).

Lemma 4.1. *Let x, y, z be three distinct points of $\mathbb{C}$. There exists a unique compact connected set $E \subset \mathbb{C}$ such that $x_1, x_2, x_3 \in E$ and $\operatorname{cap}(E) = c(x_1, x_2, x_3)$. This set consists of three real-analytic arcs meeting at some point a; these arcs are trajectories of the quadratic differential*

$$\frac{(z-a)\,dz^2}{(z-x_1)(w-x_2)(w-x_3)}.$$

If x_1, x_2, x_3 are not collinear then $E \setminus \{x_1, x_2, x_3\}$ is contained in the interior of the convex hull of $\{x_1, x_2, x_3\}$. Moreover, a conformal mapping φ from $\mathbb{C} \setminus E$ onto $|u| > 1$ is given by

$$u = \varphi(w) = \exp \int_{x_1}^{w} \left(\frac{z-a}{(z-x_1)(z-x_2)(z-x_3)} \right)^{1/2} dz\,,$$

where the path of integration only meets E at x_1, and we choose the appropriate single-valued branch of the square root on the complement of E.

We apply these results to the case of where $x_1 = t$ is a positive real, with $t > 1$, and $x_2 = i$, $x_3 = -i$. Then we find $a \in \mathbb{R}$, because the extremal configuration is unique, and $0 < a < t$. Moreover, the conformal mapping φ^{-1} from $|u| > 1$ onto $\mathbb{C} \setminus E$ has boundary values $\varphi^{-1}(1) = t$ and $\varphi^{-1}(-1) = a$. We therefore have (with the positive value of the square root taken in each integral)

$$\log \varphi(R) = \int_t^R \left(\frac{z-a}{(z-t)(z^2+1)} \right)^{1/2} dz$$

$$\log -\varphi(-R) = \int_{-R}^{a} \left(\frac{z-a}{(z-t)(z^2+1)} \right)^{1/2} dz$$

We will use the following lemma to compute the value of a with error bounds.

LEMMA 4.2. *For $R \in (t, \infty]$ and $y \in [0, t)$, define*

$$I(y,R) = \int_t^R \left(\frac{z-y}{(z-t)(z^2+1)}\right)^{1/2} dz - \int_{-R}^y \left(\frac{z-y}{(z-t)(z^2+1)}\right)^{1/2} dz\,.$$

Then we have

(i) $I(y) = \lim_{R\to\infty} I(y,R)$ *exists for each* $y \in [0,t)$,

(ii) $|I(y,R) - \lim_{R\to\infty} I(y,R)| \le \frac{t}{R-t}\sqrt{\frac{R^2}{R^2-1}}$, *for all* $y \in [0,t)$,

(iii) $\lim_{R\to\infty} I(y,R)$ *is a monotone decreasing function of* $y \in [0,t)$,

(iv) $\lim_{R\to\infty} I(a,R) = 0$.

To prove parts (i) and (ii) we need to estimate $|I(y,R') - I(y,R)|$ uniformly for all $R' > R$. It is simple to check that

$$I(y,R') - I(y,R) = \int_R^{R'} \left(\frac{1}{z^2-1}\right)^{1/2} \left(\left(\frac{z-a}{z-t}\right)^{1/2} - \left(\frac{z+a}{z+t}\right)^{1/2}\right) dz\,,$$

and the bound (ii) easily follows, with (i) as a corollary. For part (iii) we consider the integral

$$I'(y,R) = \int_t^R \left(\frac{z-y}{(z-t)(z^2+1)}\right)^{1/2} dz - \int_{y-R}^y \left(\frac{z-y}{(z-t)(z^2+1)}\right)^{1/2} dz\,,$$

which clearly has the same limit as $I(y,R)$ as $R \to \infty$. By a change of variable to $z - y$ in the second integral here it is easy to see that $I'(y,R)$ is monotone decreasing in y, for each fixed value of R. Finally, part (iv) follows from considering the Laurent expansion of φ about ∞, which shows that

$$\begin{aligned}\log(\varphi(R)) - \log(-\varphi(R))) &= \log(c(t,i,-i)R + O(1)) - \log(c(t,i,-i)R + O(1)) \\ &= O(1/R) \quad \text{as } R \to \infty.\end{aligned}$$

For efficient numerical approximation of $I(y,R)$ using a package such as Maple, it seems helpful to make the substitution $z = \sinh\theta$, in order to express $\lim_{R\to\infty} I(y,R)$ as

$$\int_{\sinh^{-1}(t)}^{\infty} \left(-1 + \sqrt{1 + \frac{t-a}{\sinh\theta - t}}\right) d\theta - \sinh^{-1}(t) - \int_{-\infty}^{\sinh^{-1}(a)} \left(-1 + \sqrt{1 + \frac{t-a}{\sinh\theta - t}}\right) d\theta - \sinh^{-1}(a)$$

Applying Lemma 4.2, we used Maple to compute the value of a in the case $t = 1.336$, obtaining $a = 0.5166430198$, correct to at least 6 decimal places. To do this, we used the θ formulation of the integrals, with limits $\theta = \pm 40$ corresponding to $R = \sinh(40)$, and asked Maple to compute the integrals correct to 12 decimal places, to guarantee that $I(0.5166430) > 10^{-8}$ and $I(0.5166431) < -(10^{-8})$.

Once we know a, we can easily compute $c(t,-i,i)$, using the integral formula for φ. We find $c(1.336, i, -i) = 0.647754557$, and this yields the required capacity estimate:

$$c(-1, 0, -0.5 + 0.668i) > 0.3238 > 1/3.1\,.$$

As a validation of the numerical method, we carried out the same computations for $t = \sqrt{3}$, in which case the capacity is known explicitly; we found that our method gave an error of less than 10^{-10} for the capacity. Professor Toshiyuki Sugawa has implemented an alternative numerical method, using Kuz'mina's description of $c(0, e^{i\theta}, e^{-i\theta})$ in terms of Jacobi elliptic functions. Professor Sugawa's Mathematica code gives an answer agreeing with ours to at least 10 decimal places for the case $t = 1.336$, and correct to 16 decimal places for the known case $L = 2$. It therefore seems likely that his code produces more accurate approximations than ours.

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