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Singularity formation for time-like extremal hypersurfaces

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ABSTRACT

We derive self-similar string solutions in a graph representation, near the point of singularity formation, which can be shown to extend to point-like singularities on M(em)branes, as well as to the radially symmetric case.

1. Introduction

The space–time volume in D-dimensional Minkowski space of a time-like $(M+1)$-dimensional hypersurface $\mathcal{M}$, that at any time t can be described as a graph over $\mathbb{R}^M$, $z(t,\mathbf{x})$, is

$$S[z] = \int \sqrt{1 - \dot{z}^2 + (\nabla z)^2}\, d^M x\, dt. \tag{1}$$

Stationary points of (1) correspond to manifolds $\mathcal{M}$ whose mean curvature vanishes. In the simplest case, $M = 1$ and $D = 3$ ($z = z(t,x)$, $\dot{z} = \partial z/\partial t$, $z' = \partial z/\partial x$), the Lagrange equation reads

$$\ddot{z}\left(1 + z'^2\right) - z''\left(1 - \dot{z}^2\right) = 2\dot{z}z'\dot{z}'. \tag{2}$$

This so-called Born–Infeld equation [1] describes the motion of a string in the plane, and has been studied for more than 40 years [2–4]; recent work related to higher dimensional time-like zero mean curvature hypersurfaces includes [5–8].

Here we show how (2) can develop singularities in finite time, starting from smooth initial data. While generic singularity formation of strings in *four*-dimensional Minkowski space is well understood (cf. [9–14]), we are not aware of any literature obtaining the swallowtail singularity that we find in *three* dimensions. This is done first via the self-similar ansatz

$$z(t,x) = z_0 - \hat{t} + \hat{t}^{\alpha} h\left(\frac{x}{\hat{t}^{\beta}}\right) + \cdots, \tag{3}$$

where $\hat{t} := t_0 - t \to 0$ (the dots are indicating lower order terms), and then via Taylor expansion of the general parametric solution for closed strings near the point where the singularity first forms.

Inserting (3) into (2) one finds the similarity equation

$$h''\left(2\alpha h - \frac{(\alpha+1)^2}{4}\xi^2\right) = (\alpha - 1)\left[h'^2 + \alpha h - \frac{3}{4}(\alpha+1)\xi h'\right]. \tag{4}$$

The above ansatz is consistent provided $\beta = (1+\alpha)/2 > 1$, and (3) is an asymptotic solution of (2) if the similarity equation is satisfied. For consistency with a finite outer solution of (2), the profile h must satisfy

$$h(\xi) \propto A_{\pm}\xi^{\frac{2\alpha}{\alpha+1}} \quad \text{for } \xi \to \pm\infty \tag{5}$$

(for a general discussion of matching self-similar solutions to the exterior see [15]).

The ansatz (3) is formally consistent for a continuum of similarity exponents $\alpha \geqslant 1$ and for any solution of the similarity equation (4). However, by considering the regularity of solutions of (4) in the origin $\xi = 0$ the similarity exponent must be one of the sequence

$$\alpha = \alpha_n = \frac{n+1}{n}, \quad n \in \mathbb{N}, \tag{6}$$

certainly if $h(0) = 0 = h'(0)$, and presumably in general (i.e. *all* relevant solutions of (4)). Of this infinite sequence, we believe that only $\alpha = 2$ is realized for generic initial data; indeed, in this case

$$\xi = \zeta + c\zeta^3/3, \qquad h(\xi) = \zeta^2/2 + c\zeta^4/4, \tag{7}$$

which we will deduce from a parametric string solution corresponding to (2).

The importance of the similarity solution (3) lies in the fact that it can be generalized to arbitrary dimensions, in particular to membranes. We find that the same type of singular solution is observed in any dimension, even having the same spatial structure (7).

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Section 2 is devoted to solving the similarity equation (4). In this analysis we confine ourselves to a description of the solution for times $t < t_0$. In Section 3 we consider the relevance of our results to higher dimensional cases with codimension $D - (M + 1) = 1$. In Section 4 we compare our results to an exact solution for closed strings. While this analysis is confined to $D = 3$, it does permit to consider times before and after the swallowtail singularity.

2. The similarity equation

A way of satisfying (4) is to demand

$$L^2 := h'^2 + 2\alpha h - (\alpha+1)\xi h' = 0 \tag{8}$$

(differentiating e.g. $(1+\alpha)\xi = h' + 2\alpha h/h'$ one can eliminate h'', reducing (4) to an identity, as long as $h' \neq 1$).

The transformation

$$h(\xi) = \xi^2 g(\xi) = \xi^2\left(\frac{(1+\alpha)^2}{8\alpha} - \frac{v^2}{2\alpha}\right) \tag{9}$$

yields

$$\begin{aligned} -\frac{d\xi}{\xi} &= \frac{v\,dv}{v^2 \pm \alpha v + (\alpha^2-1)/4} \\ &= \frac{1}{2}\left(\frac{\alpha+1}{v \pm \frac{\alpha+1}{2}} - \frac{\alpha-1}{v \pm \frac{\alpha-1}{2}}\right)dv, \end{aligned} \tag{10}$$

i.e. (choosing the lower sign)

$$\frac{|v - (\alpha+1)/2|^{\alpha+1}}{|v - (\alpha-1)/2|^{\alpha-1}} = \frac{E}{\xi^2}. \tag{11}$$

This yields solutions $v \in [(\alpha-1)/2, (\alpha+1)/2)$,

$$\begin{aligned} v &\approx \frac{\alpha-1}{2} + \left(\frac{\xi^2}{E}\right)^{\frac{1}{\alpha-1}} + \cdots \quad \text{as } \xi \to 0, \\ v &\approx \frac{\alpha+1}{2} - \left(\frac{E}{\xi^2}\right)^{\frac{1}{\alpha+1}} + \cdots \quad \text{as } \xi \to \pm\infty, \end{aligned} \tag{12}$$

i.e.

$$\begin{aligned} &h(\xi) \geqslant 0, \quad h(0) = 0, \\ &h(\xi) \propto \xi^2/2 \quad \text{as } \xi \to 0, \\ &h(\xi) \propto \frac{1+\alpha}{2\alpha} E^{\frac{1}{1+\alpha}} \xi^{\frac{2\alpha}{1+\alpha}} \quad \text{as } \xi \to \pm\infty. \end{aligned} \tag{13}$$

Note that these solutions are consistent with the growth conditions (5).

To solve the *second order equation* (4) we note that $\tilde{h}(\xi) := ch(\xi/\sqrt{c})$ solves (4), if h does, and that

$$\frac{h'}{\xi} - \frac{2h}{\xi^2} = \frac{1}{\alpha} f\left(\sqrt{\frac{(\alpha+1)^2}{4} - 2\alpha\frac{h(\xi)}{\xi^2}}\right) \quad \equiv \left(\frac{1}{\alpha} f(v)\right) \tag{14}$$

reduces (4) to

$$\begin{aligned} &-\left(v^2 - \frac{(\alpha+1)^2}{4}\right)\left(v^2 - \frac{(\alpha-1)^2}{4}\right) \\ &\quad = f\left(\alpha v f' - (\alpha-1) f - (\alpha+2)v^2 + (\alpha^2-1)(\alpha-2)/4\right), \end{aligned} \tag{15}$$

and

$$\frac{d\xi}{\xi} = -\frac{v\,dv}{f(v)} = \frac{\alpha\,dg}{f}. \tag{16}$$

The growth condition (5) implies that h grows less than quadratically at infinity. Thus we deduce from (14) that f vanishes at $(\alpha+1)/2$. Furthermore, from a direct calculation using the growth exponent (5) we find the first derivative, yielding the initial conditions

$$f\left(\frac{\alpha+1}{2}\right) = 0, \qquad f'\left(\frac{\alpha+1}{2}\right) = 1. \tag{17}$$

Using (17), (15) yields a polynomial solution

$$\begin{aligned} f(v) &= \left(v - \frac{(\alpha+1)}{2}\right)\left(v - \frac{(\alpha-1)}{2}\right) \\ &= v^2 - \alpha v + \frac{\alpha^2-1}{4}, \end{aligned} \tag{18}$$

i.e. (11), which corresponds to the first order equation (8), but also an infinity of other solutions (a Taylor expansion around $v_\infty = (\alpha+1)/2$ shows that (15) leaves $f''((\alpha+1)/2)$ undetermined, when (17) holds). We note that (15) also has the solution $f_-(v) = f(-v)$, and for the special case $\alpha = 2$ another pair of polynomial solutions,

$$\tilde{f}(v) = (v+3/2)(v-1/2) = v^2 + v - 3/4, \tag{19}$$

and $\tilde{f}_-(v) = \tilde{f}(-v)$. While the asymptotic behavior following from (19) is in disagreement with (5), integration methods similar to those used by Abel [16] perhaps permit a complete reduction of (15) to quadratures.

In any case, (15) can be simplified in various ways. For $\alpha = 2$, e.g. it reduces to

$$yy' = y - \frac{1}{4v^{5/2}}\left(v^2 - 9/4\right)\left(v^2 - 1/4\right) \tag{20}$$

via

$$f(v) = \sqrt{v}\,y\left(4v^{3/2}/3\right). \tag{21}$$

The solution (18), which is consistent with the growth condition (5), is equivalent to the solution (11) of (8) given before. If one investigates the behavior of the solution in the origin (either using (11) directly or by series expansion of (8)), one finds that only for $\alpha = \alpha_n$ (cf. (6)) a smooth solution is possible. Thus the first consistent solution is found for $n = 1$ or $\alpha = 2$. Higher order solutions $n = 2, 3, \ldots$ are also possible in principle. They have the property that apart from $f''(0)$, the first non-vanishing derivative is $f^{(2n+2)}(0)$. However, we believe that they correspond to non-generic initial conditions, whose derivatives have corresponding properties of vanishing up to a certain order. To demonstrate this point, one would have to perform a stability analysis of the corresponding solution [15]. In the string picture discussed below this can be shown explicitly, as higher order solutions correspond to non-generic initial data.

3. Higher dimensions

The solutions of (8), found to govern singularities of (2), also apply to higher dimensions. The reason is that the left-hand side of (8) is the leading order term of

$$\mathcal{L}^2 = 1 - \dot{z}^2 + z'^2. \tag{22}$$

In other words, the asymptotic singular solutions discussed above have $\mathcal{L}^2 = 0 +$ lower order. In fact, differentiating (22) with respect to t and x one easily shows that $\mathcal{L}^2 = 0$ provides solutions of (2). In higher dimensions, differentiating $1 - z^\alpha z_\alpha = 0$ gives $z^\alpha z_{\alpha\beta} = 0$, and hence

$$\left(1 - z_\alpha z^\alpha\right)\Box z + z^\beta z^\alpha z_{\alpha\beta} = 0. \tag{23}$$

Thus solutions of $\mathcal{L}^2 = 0$ also solve the M-brane equation (23) in arbitrary dimensions.

For the special case of radially symmetric membranes:

$$\ddot{z}\left(1 + z'^2\right) - z''\left(1 - \dot{z}^2\right) - 2\dot{z}z'\dot{z}' = \frac{z'}{r}\left(1 - \dot{z}^2 + z'^2\right) \equiv \frac{z'}{r}\mathcal{L}^2. \tag{24}$$

Insert the radial version of the ansatz (3),

$$z(t, r) = -\hat{t} + \hat{t}^\alpha h\left(\frac{r - r_0}{\hat{t}^\beta}\right) + \cdots, \tag{25}$$

into (24). If $r_0 \neq 0$, the entire right-hand side of (24) is of lower order in $\hat{t}$, and the similarity equation (4) remains the same. Geometrically, this corresponds to the singularity forming along a circular ridge of radius r_0.

If on the other hand $r_0 = 0$, i.e. the singularity forms along the axis, the right-hand side is of the same order, and the similarity equation becomes

$$h''\left(2\alpha h - \frac{(\alpha + 1)^2}{4}\xi^2\right) + (1 - \alpha)\left[h'^2 + \alpha h - \frac{3}{4}(\alpha + 1)\xi h'\right]$$
$$= -\frac{h'}{\xi}\left[h'^2 + 2\alpha h - (\alpha + 1)\xi h'\right]. \tag{26}$$

This equation can in principle have solutions different from (4). For solutions of (8), however, the expression in angular brackets in (26) vanishes, hence solutions of (8) also solve (26). Thus (25), (7) describe a point-like singularity on a membrane. These observations straightforwardly generalize to higher M-branes, $M > 2$.

4. Parametric string solution

Let us now compare our findings with the solution of closed bosonic string motions given by Eq. (50) of [17] (note that the definitions of f and g are changed by $\pi/4$, and that the constant λ is chosen to be 1):

$$\dot{\mathbf{x}}(t, \varphi) = \sin(f - g)\begin{pmatrix} -\sin(f + g) \\ \cos(f + g) \end{pmatrix}, \tag{27}$$

$$\mathbf{x}'(t, \varphi) = \cos(f - g)\begin{pmatrix} \cos(f + g) \\ \sin(f + g) \end{pmatrix}, \tag{28}$$

where $f = f(\varphi + t)$ and $g = g(\varphi - t)$. From (28) one finds the curvature

$$k(t, \varphi) = \frac{f' + g'}{\cos(f - g)}. \tag{29}$$

The hodograph transformation

$$(t, \varphi) \to t = x^0, \qquad x = x^1(t, \varphi),$$
$$x^2(t, \varphi) = z\left(t, x^1(t, \varphi)\right), \tag{30}$$

implying $\dot{z} = \dot{y} - \dot{x}y'/x'$, $z' = y'/x'$ $(\partial\phi/\partial x^0 = -\dot{x}/x'$, $\partial\phi/\partial x^1 = 1/x')$ permits to go between the parametric string picture (27)–(29) and the graph description (2). In particular,

$$1 - \dot{z}^2 + z'^2 = \left(\frac{\cos(f - g)}{\cos(f + g)}\right)^2 \tag{31}$$

is manifestly non-negative in the parametric string-description, while for solutions of (2) one has to demand it explicitly — leading e.g. to the exclusion of solutions with $h'(0) = 0$, $h(0) < 0$.

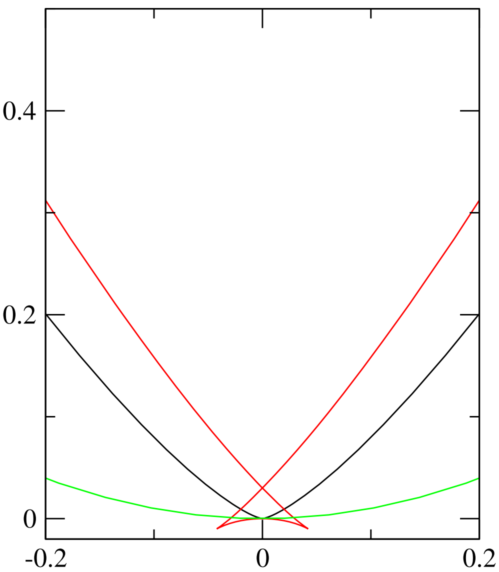

Fig. 1. The formation of a swallowtail, as described by (37). Shown is a smooth minimum ($\hat{t} > 0$), a minimum with a 4/3 singularity ($\hat{t} = 0$), and a swallowtail or double cusp ($\hat{t} < 0$).

Let us give an explicit example of $\mathbb{M}_2 \subset \mathbb{R}^{1,2}$ being at $t = 0$ a regular graph, while for $t = 1$ a curvature singularity has developed. Let $\mathbb{M}_2$ be described by $\mathbf{x}(\varphi, t)$, as defined by (27), (28), with $\varphi \in \mathbb{R}$, $t \geqslant 0$. Let

$$f(w) = \begin{cases} \arctan w & \text{for } w \geqslant \epsilon, \\ \chi_\epsilon(w)\arctan w & \text{for } 0 \leqslant w \leqslant \epsilon < 0, \\ 0 & \text{for } w \leqslant 0, \end{cases} \tag{32}$$

where $\chi_\epsilon(w \geqslant \epsilon) = 1$, $\chi_\epsilon(w \leqslant 0) = 0$, and $\chi_\epsilon(0 < w < 0)$ such that $f'(w) \geqslant 0$. We also assume that $g(w) = -f(-w)$. A simple calculation then shows that for $\varphi \in [-t + \epsilon, t - \epsilon]$ one obtains $(\mathbf{x}_0(t = 0, u = 0) = 0)$

$$x(\varphi, t) = -\varphi + \arctan(\varphi + t) + \arctan(\varphi - t),$$
$$y(\varphi, t) = \ln\sqrt{\left(1 + (\varphi + t)^2\right)\left(1 + (\varphi - t)^2\right)},$$
$$k(\varphi, t) = \frac{2}{\sqrt{(1 + (\varphi + t)^2)(1 + (\varphi - t)^2)}}\frac{\varphi^2 + 1 + t^2}{\varphi^2 + 1 - t^2}. \tag{33}$$

Note that for $t > 1$ this is no longer a graph.

The example (32) underlies a general structure that can be uncovered by a local expansion of the functions f and g around the singularity. Namely, as seen from (29), the singularity occurs when $\cos(f - g)$ vanishes. We are interested in describing the situation when this first occurs. Then a Taylor expansion yields

$$f(\zeta) = f_0 + f_1\zeta + f_2\zeta^2/2 + O\left(\zeta^3\right),$$
$$g(\zeta) = g_0 + g_1\zeta + g_2\zeta^2/2 + O\left(\zeta^3\right), \tag{34}$$

where $f_0 - g_0 = \pi/2$. Without loss of generality the parametrization can be chosen such that the singularity occurs for $\zeta = 0$. Then to leading order in ζ we find

$$f - g = \pi/2 + (f_1 - g_1)\varphi + (f_1 + g_1)t + \cdots. \tag{35}$$

Without loss of generality we can assume that the singularity occurs at $\varphi = 0$, since φ is simply a parameter. For $\varphi = 0$, $f - g$ then assumes the singular value for $t = 0$, i.e. we have $t_0 = 0$. But the expansion (34) must in fact obey the constraint $f_1 = g_1$, otherwise there will be a φ such that $f - g$ becomes critical at some *earlier* time $t < t_0$, contradicting our assumption of capturing the earliest time a singularity occurs.

Now using $f_1 = g_1$ to leading order the expansions are

$$f - g \approx \pi/2 - 2f_1\hat{t} + (f_2 - g_2)\varphi^2/2,$$
$$f + g \approx f_0 + g_0 + 2f_1\varphi. \tag{36}$$

It is clear from the first equation that φ is of the order $\hat{t}^{1/2}$; hence the entire expression (33) can be expanded in powers of $\hat{t}$, using $\varphi \propto \hat{t}^{1/2}$. Deriving the corresponding expressions for x' and y', and using the integrability condition (27), (28), one obtains

$$\begin{pmatrix} x \\ y \end{pmatrix} = R(\omega)\begin{pmatrix} 2a\hat{t}\varphi + 2b\varphi^3/3 \\ -\hat{t} + 2a^2\hat{t}\varphi^2 + ab\varphi^4 \end{pmatrix} \tag{37}$$

where $a = f_1$ and $b = (g_2 - f_2)/4$. The rotation matrix R is

$$R = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}, \quad \omega = f_0 + g_0. \tag{38}$$

For the case that $\omega = 0$, i.e. up to a rotation in space, (37) corresponds exactly to the similarity equation (3) with similarity solution (7), putting $\zeta = 2a\varphi/\hat{t}^{1/2}$. Furthermore, the free constant c in (7) can be identified as $c = b/(4a^3) > 0$. Since φ is of order $\hat{t}^{1/2}$, it follows that x is of order $\hat{t}^{3/2}$ and y of order $\hat{t}^2$, implying that the exponents are $\alpha = 2$ and $\beta = 3/2$. The case $\omega \neq 0$ is not included in the analysis of Section 2, since the corresponding curve is no longer a graph, as implied by (3).

The curve described by (37) is shown in Fig. 1, for the case of $w = 0$, and disregarding the spatial translation of z by the term $-\hat{t}$. In catastrophe theory, the curve that results for $\hat{t} < 0$ is known as the “swallowtail” [18]. The same swallowtail curve also appears as the shape of a wavefront in geometrical optics [19,20]. For $\hat{t} > 0$ the curve is smooth, while for $\hat{t} = 0$ a rather mild singularity develops; at the origin, $y \propto x^{4/3}$. After the singularity ($\hat{t} < 0$) *two* cusp singularities are formed, and the curve self-intersects. It is easy to confirm that both cusps behave locally like $x_r \propto y_r^{2/3}$, but where the axes are rotated by an angle of $\hat{t}^{1/2}/c^{1/2}$ relative to the orientation of the original swallowtail. This means the orientation of the cusps is asymptotically at a right angle to the swallowtail. The important point is that the cusps, which are born out of the swallowtail, exist for a finite amount of time $t > t_0$, rather than existing only for some particular singular time. Note that in our earlier analysis based on the similarity description (3) we focused on the time before the first singularity alone. To describe the regime $t > t_0$, and thus the *continuation* through the initial swallowtail singularity, would require a separate ansatz for $\hat{t} < 0$ in the similarity formulation.

In [9, pp. 157–160] (see also [10–14]) the dynamics of closed bosonic strings $\mathbf{x}(t,\varphi)$ is considered in 3 space dimensions (rather than in 2, discussed above), and the general solution is written in the form

$$\mathbf{x}(t,\varphi) = \frac{1}{2}\big[\mathbf{a}(\varphi - t) + \mathbf{b}(\varphi + t)\big]. \tag{39}$$

The vector-valued functions $\mathbf{a}$ and $\mathbf{b}$ are arbitrary up to the constraint $\mathbf{a}'^2 = \mathbf{b}'^2 = 1$. An important conclusion then is that generically a $y \propto x^2/3$ cusp singularity forms at a time t_s, but that the curve is regular for some time interval $t \neq t_s$. However, as we have seen above, this scenario is incorrect for a curve embedded in two-dimensional space, in which case a cusp exists for a finite interval in time. In higher dimensions, on the other hand, the swallowtail singularity becomes “unfolded” into directions out of its plane, and the generic situation of [9–14] applies.

In the case of non-generic initial conditions other solutions are possible. For example, instead of (34)

$$f(\zeta) = \pi/4 + a\zeta - b\zeta^{2n}, \tag{40}$$

where $n \in \mathbb{N}$ but $n > 1$. Only even powers $2n$ are allowed, otherwise the singularity occurs for all φ at the same time, i.e. it is no longer point-like. If the leading order term is not linear but itself of higher order, the resulting similarity profile becomes singular at the origin, cf. (A.6). Repeating the above calculation along the same lines, we find

$$x' = 2a\hat{t} + 2b\varphi^{2n}, \qquad y' = 4a^2\hat{t}\varphi + 4ab\varphi^{2n+1}, \tag{41}$$

which is equivalent to the symmetric shape function

$$\xi = \zeta + 2d(n+1)\zeta^{2n+1}/(2n+1),$$
$$h = \zeta^2/2 + d\zeta^{2n+2}. \tag{42}$$

The corresponding similarity exponent is $\alpha = \alpha_n$, as given by (6). We thus retrieve the exact same solutions identified by our previous analysis, based on a similarity description.

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Appendix A. Additional solutions of the similarity equation

It is possible to construct many more solutions to the similarity equation (8), which are all defined on the real line, but which we reject since they either contradict (5) or are not smooth. The simplest case is

$$h(\xi) = \frac{\xi^2}{2}, \tag{A.1}$$

which is a solution for *any* α, but evidently does not satisfy the matching condition (5).

Recall that for $\alpha = \alpha_n$, (11) furnishes smooth solutions on the real line. On the other hand, while for $\alpha = 3$ the second derivative of the resulting solution is well-defined, the third derivative is discontinuous. Namely, for $E = 4$ (e.g.) one finds that for $\xi > 0$,

$$h(\xi) = -\frac{2}{3}\big(\xi + 2(1+\xi) - 2(1+\xi)^{3/2}\big),$$
$$h'(\xi) = 2(\sqrt{1+\xi} - 1) > 0,$$
$$h''(\xi) = 1/\sqrt{1+\xi}, \tag{A.2}$$

so that

$$h'''(\xi) = \begin{cases} -(1+\xi)^{-3/2}/2 & \text{for } \xi > 0, \\ (1+\xi)^{-3/2}/2 & \text{for } \xi < 0. \end{cases} \tag{A.3}$$

Other solutions whose scaling exponent is not from the set (6), but which have well-defined second derivatives, can be found from the parametric string solution as described in Section 4. If the expansion of f does not start with a linear term as in (34), but at higher order, e.g.

$$f(\zeta) = \pi/4 + \zeta^3/2 - b\zeta^4, \tag{A.4}$$

one finds

$$x' = 3\hat{t}\varphi^2 + 2b\varphi^4, \qquad y' = 3\hat{t}\varphi^5 + 2b\varphi^7. \tag{A.5}$$

Integrating (A.5), the result once more conforms with (3), with a similarity exponent of $\alpha = 4$, and the similarity function has the parametric form

$$\xi = \zeta^3 + 2b\zeta^5/5,$$
$$h = \zeta^6/2 + b\zeta^8/4. \tag{A.6}$$

It is confirmed easily that (A.6) solves (8) with $\alpha = 4$, but the third derivative of $h(\xi)$ is singular at the origin.

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