Origin of the Obukhov scaling relation in turbulence

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We introduce a cascade model for a turbulent velocity field. The cascade's length is determined by a local Reynolds number and fluctuates in space. We derive $\mu=2-\zeta(6)$ relating the dissipation correlation exponent μ with the scaling exponents $\zeta(m)$ of the velocity difference in accordance with experiment. Several applications are pointed out.

Ever since Obukhov and Kolmogorov published their classical papers on intermittency [1,2] in 1962, there have been major efforts [3–7] to understand the statistical properties of the dissipation field $\mathscr{E}(\boldsymbol{x},t)=\nu\partial_i u_j(\boldsymbol{x},t)\partial_i u_j(\boldsymbol{x},t)$ in high Reynolds number turbulence. The strongly fluctuating or "intermittent" character of $\mathscr{E}(\boldsymbol{x},t)$ is supposed to induce scaling corrections to the classical relation for the velocity field

$$D(r)=\langle\!\langle |\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)-\boldsymbol{u}(\boldsymbol{x},t)|^2\rangle\!\rangle =b\bar{\mathscr{E}}^{2/3}r^{2/3}\,. \tag{1}$$

Here $\langle\!\langle\ \rangle\!\rangle$ denotes the average over the statistical ensemble, and $\bar{\mathscr{E}}$ is the mean value of the dissipation rate per unit mass. Eq. (1) is to be used in the inertial range, $r\gtrsim 9\eta$, with $\eta=(\nu^3/\bar{\mathscr{E}})^{1/4}$ being the Kolmogorov microscale. The lognormal, β-, or statistical β-model [1–6] of intermittency imply another r-exponent, $\zeta(2)=\frac{2}{3}+B$, in eq. (1). The crucial question of how velocity and dissipation fields are interrelated, however, has not come beyond the original argument by Obukhov [1]. To reproduce this argument, let us consider the longitudinal mth order structure functions

$$D_{\parallel}^{(m)}(r)=\langle\!\langle [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)-\boldsymbol{u}(\boldsymbol{x},t)]_{\parallel}^{m}\rangle\!\rangle\propto r^{\zeta(m)}\,. \tag{2}$$

Assuming that the statistics of the velocity difference $\boldsymbol{v}(\boldsymbol{r};\boldsymbol{x})=\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r})-\boldsymbol{u}(\boldsymbol{x})$ only depends on the dissipation rate on scale r, defined by

$$\mathscr{E}_r(\boldsymbol{x},t)=\frac{3}{4\pi r^3}\int_{V_r}\mathscr{E}(\boldsymbol{x}+\boldsymbol{y},t)\,\mathrm{d}V(y)\,,$$

a typical velocity difference may be approximated by $v(r)\approx\mathscr{E}_r^{1/3}r^{1/3}$, giving

$$D_{\parallel}^{(m)}(r)=b_{\parallel}^{(m)}\langle\!\langle [\mathscr{E}_r(\boldsymbol{x})]^{m/3}\rangle\!\rangle r^{m/3}\,. \tag{3}$$

If $\langle\!\langle [\mathscr{E}_r(\boldsymbol{x})]^{m/3}\rangle\!\rangle$ has a power law dependence on r, (3) will obviously lead to corrections to the classical exponents $\zeta_{\mathrm{cl}}(m)=\frac{1}{3}m$. The special case $m=6$ directly gives the well known formula

$$\mu=2-\zeta(6)\,, \tag{4}$$

where μ is the exponent of the dissipation correlations,

$$\langle\!\langle \mathscr{E}(\boldsymbol{x}+\boldsymbol{r},t)\mathscr{E}(\boldsymbol{x},t)\rangle\!\rangle\propto\bar{\mathscr{E}}^2(r/L)^{-\mu}\,.$$

Eq. (4) is well supported by recent experiments [8–10], with $\mu\approx 0.2$. On the other hand, there are still considerable analyzing errors in both the measurements of $\zeta(6)$ and μ [9,10], leaving room for different interpretations [11]. Future experiments have to settle this.

The usual cascade models of refs. [3–7] are not able to produce relations like (4) by their very nature, since they are only stated in terms of the dissipation field. Instead, the present authors very recently introduced a cascade model for the velocity field itself [12]. Depending on model assumptions, we either have $\mu=0$, independent of the velocity correlations, if spatial fluctuations are not allowed, or the $\zeta(m)$ are linked to μ by $\mu=2\zeta(2)-\zeta(4)$, instead of (4), if the velocity cascades fluctuate spatially down to a smallest cascade level which is the same at all positions x.

In this Letter we derive (4) in the framework of a cascade model, which is a refinement of our previous one [12] without independently introducing a dissipation statistics. One still is not able to confirm that (4) or some other relation is valid for the Navier–Stokes dynamics, but we hope to have boiled down one of the most widely used relations of turbulence theory to its physical roots. We briefly repeat in the following the essential ideas of our model to make this Letter self-contained; for more details, see ref. [12].

The basic idea of our model is a Fourier–Weierstrass superposition of the velocity field. We represent a one-dimensional cut through a turbulent velocity field by

$$u(x)=\sum_{l=-N_x}^{N_L} u^{(l)}(x)\exp[\mathrm{i}\lambda^{-l}(x/L_0)]+\mathrm{c.c.}$$
$$=\sum_{k\in \mathrm{K}_x}\mu_k(x)\exp(\mathrm{i}kx),$$
$$\mathrm{K}_x=\{\pm L_0^{-1}\lambda^{-N_L},\ldots,\pm L_0^{-1}\lambda^{N_x}\}. \quad (5)$$

L_0 is the reference length scale. Eq. (5) corresponds to a decomposition of the flow field into contributions from different cascade levels l, running from the largest eddies, $l=N_L$, down to the smallest eddies, $l=-N_x$. The diameter of the eddies of level l is $2\pi L_0\lambda^l$, i.e., λ is the ratio of the sizes of two successive eddy generations. The important difference to our previous model [12] is that the cutoff scale $\eta_x=L_0\lambda^{-N_x}$ is allowed to fluctuate in space x via properly choosing N_x. The value of N_x is determined by the requirement

$$N_x=\text{largest } N \text{ with } u^{(-N)}(x)L_0\lambda^{-N}/\nu\geqslant \mathrm{Re}_{cr}. \quad (6)$$

ν is a parameter representing the viscosity and has the dimension of squared length over time, which together with L_0 sets the time scale. This choice of N_x means that the eddy decay process stops at some locally determined scale η_x, at which the dissipation term of the Navier–Stokes equation becomes comparable with the interaction term. In the model of ref. [12] a mean x-independent cutoff scale $\tilde{\eta}$ is chosen as an input parameter. Here, instead, the input parameter is Re_{cr}, which determines the mean cutoff length. Otherwise our model is unchanged: $u^{(l)}(x)$ represents the amplitude of an eddy of level l at position x, $u^{(l)}(x)$ being constant over each eddy diameter $2\pi L_0\lambda^l$. Assuming $\lambda=2$ for simplicity, there are 2^{N_L-l} eddies of this diameter on level l, except for those which have been eliminated by the requirement (6). The amplitudes of the eddies of level l will be denoted by $u_i^{(l)}$, $i=1, \ldots, 2^{N_L-l}$. $u_1^{(N_L)}\equiv u_L$ is the amplitude of the largest eddy, the amplitudes $u_1^{(N_L-1)}$ and $u_2^{(N_L-1)}$ of the next smaller generation are obtained by multiplication with contraction factors s_1 and s_2. Repeating this procedure, $u_i^{(l)}$ is given by $s_i u_{j(i)}^{(l+1)}$, $j(i)$ the integer of $\frac{1}{2}i$ or $\frac{1}{2}(i+1)$. The contraction factors s_i are assumed to be random, and are chosen from a common probability distribution $p(s)$. To assure statistical self-similarity, the various transitions are taken to be independent, so the two offsprings $u_i^{(l)}$ of an $(l+1)$-eddy $u_{j(i)}^{(l+1)}$ are obtained with independently chosen s. In ref. [12] we called this a multifractal model. The average over the ensemble of the $u_i^{(l)}$ is denoted by a single bracket $\langle\ \rangle$.

We can now proceed to calculate the local dissipation rate $\mathscr{E}(x)=15\nu(\partial u/\partial x)^2$ with $u(x)$ from (5), where we neglect the x-derivative of $u_k(x)$ as explained in ref. [12]. The close resemblance of a typical $\mathscr{E}(x)$ to a corresponding experimental signal is illustrated in fig. 1:

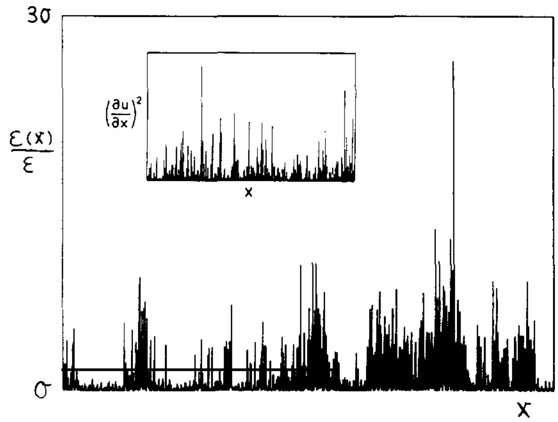

Fig. 1. The normalized dissipation field $\mathscr{E}(x)/\langle\langle\mathscr{E}(x)\rangle\rangle$ for our model with $\mathrm{Re}/\mathrm{Re}_{cr}=4000$. $p(s)$ was fitted to experiment, cf. ref. [12]. The mean number of levels is 9, $\mathrm{Re}_\lambda\approx 1000$. In the insert we show a measured dissipation field of the wake of a cylinder [13]. Re_λ is only about 90 according to ref. [8].

$$C_{\mathscr{E}}(r) \equiv \langle\!\langle \mathscr{E}(x)\mathscr{E}(x+r)\rangle\!\rangle$$
$$= (15\nu)^2 \sum_{k_1+k_2+k_3+k_4=0} \langle u_{k_1}(x)u_{k_2}(x)u_{k_3}(x+r)$$
$$\times u_{k_4}(x+r)\rangle k_1k_2k_3k_4 \exp[\mathrm{i}(k_3+k_4)r] .$$

Here $\langle\!\langle\ \rangle\!\rangle$ stands for $p(s)$-averaging and in addition a translational average over x. Because of the factors k_i, the sum is dominated by the terms with largest k_i. Thus $C_{\mathscr{E}}(r)$ is given (up to oscillatory terms) by

$$C_{\mathscr{E}}(r) \approx (15\nu)^2 L_0^{-4}$$
$$\times 4\langle \lambda^{2N_x}|u^{(-N_x)}(x)|^2 \lambda^{2N_{x+r}}|u^{(-N_{x+r})}|^2\rangle .$$

In the limit of high Reynolds numbers one may write according to (6): $u^{(-N_x)}(x) = \mathrm{Re}_{cr}\nu\lambda^{N_x}/L_0$,

$$C_{\mathscr{E}}(r) \propto \langle \lambda^{4N_x}\lambda^{4N_{x+r}}\rangle . \tag{7}$$

Condition (6) says that N_x is the largest number with

$$\sum_{i=1}^{N_L+N_x} \log(\lambda s_i^{-1})$$

still smaller than $R \equiv \log(\mathrm{Re}/\mathrm{Re}_{cr})$, with $\mathrm{Re} = Lu_L/\nu$. Hence N_x follows a first passage time distribution. The correlation between N_x and N_{x+r} comes in because the sequence $y_i = \log(\lambda s_i^{-1})$ for some position x is identical to the corresponding sequence for $x+r$ down to the level with eddy size about r, thus $l_r = \log_\lambda(r/2\pi L_0)$. We only consider the case $l_r \gg -\langle N_x\rangle$. For lower levels, $l<l_r$, the sequences are independent and the product in (7) factorizes. Let $w(n,z)$ denote the probability distribution of $z = \sum_{i=1}^{n} y_i$. Then

$$W(n,Y) = \int_Y^\infty [w(n,z) - w(n-1,z)]\,\mathrm{d}z \tag{8}$$

is the distribution to be beyond Y after precisely n steps. With $W(n,Y)$ expression (7) is easily evaluated:

$$C_{\mathscr{E}}(r) \propto \langle \lambda^{4N_x}\lambda^{4N_{x+r}}\rangle$$
$$= \lambda^{8(N_L-l_r)} \int_{-\infty}^{\infty} w(N_L - l_r, Y)\langle\lambda^{4N}\rangle^2_{R-Y}\,\mathrm{d}Y , \tag{9}$$

where

$$\langle\lambda^{4N}\rangle_Y = \sum_{N=1}^{\infty} W(N,Y)\lambda^{4N} .$$

Clearly, $w(n,y)$ and so $W(n,Y)$ and so $C_{\mathscr{E}}(r)$ depend on the probability distribution $p(s)$ for the reduction factors s. So far we were only able to calculate (9) analytically if $p(s)$ is a lognormal distribution. In ref. [12] we derived the expression

$$\zeta(m) = -\frac{\log\langle s^m\rangle}{\log\lambda} \tag{10}$$

for the structure function exponents. According to (10) a lognormal $p(s)$ gives a polynomial of second order for $\zeta(m)$, which is known [9] to give a very good representation of the experimental data up to $m=12$. We stress that the lognormal choice for $p(s)$ has only been made for computational convenience, since then $w(n,z)$ is a normal distribution. Note that $w(n,z)$ cannot be taken to be a normal distribution if $p(s)$ is not lognormal. The central limit theorem fails here for reasons similar to those encountered in the Yaglom model for the energy dissipation [3,4,12], see also ref. [14].

Writing $\langle y\rangle$ and σ^2 for the mean value and variance of the y_i we obtain

$$w(n,z) = \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left(-\frac{1}{2}\frac{(z-n\langle y\rangle)^2}{\sigma^2 n}\right).$$

For large Y, one may pass to a continuous variable $\xi = n\langle y\rangle/Y$. For this variable, $W(n,Y)$ becomes

$$\hat{W}(\xi,Y) = \sqrt{\frac{\langle y\rangle Y}{2\pi\xi}}\frac{1}{2\sigma}(1+1/\xi)$$
$$\times \exp\left(-\frac{\langle y\rangle Y}{2\sigma^2}\frac{(1-\xi)^2}{\xi}\right).$$

Hence the relative width $\sigma_{N_x}/(N_L+N_x)$ is $\sigma/\sqrt{\langle y\rangle R}$ which becomes increasingly smaller for large Reynolds numbers, and $l_r \gg -\langle N_x\rangle$ ensures $l_r > -N_x$ for all x. For practical purposes, σ_{N_x} is about one. The integral appearing in the expression for $\langle\lambda^{aN}\rangle_Y$,

$$\int_0^\infty (1+1/\xi)\exp\left(-\frac{\langle y\rangle Y}{2\sigma^2}\frac{(1-\xi)^2}{\xi}\right.$$
$$\left. + \frac{aY\xi}{\langle y\rangle}\log\lambda\right)\mathrm{d}\xi ,$$

can be evaluated exactly giving

$$\langle \lambda^{aN} \rangle_Y \propto \exp[Y(\langle y \rangle - \sqrt{\langle y \rangle^2 - 2a\sigma^2 \log \lambda})/\sigma^2] .$$

It is now easy to perform the integral in (9), which finally leads to

$$\mu = \frac{2\langle y \rangle}{\sigma^2 \log \lambda} (\langle y \rangle - \sqrt{\langle y \rangle^2 - 8\sigma^2 \log \lambda}) - 8 . \quad (11)$$

According to the Kolmogorov structure equation $\zeta(3)=1$. Then (10) gives

$$\zeta(m) = \tfrac{1}{3}m + \frac{\sigma^2}{\log \lambda} (\tfrac{3}{2}m - \tfrac{1}{2}m^2) . \quad (12)$$

Inserting (12) into (11) one arrives at the desired relation (4). For $p(s)$ a bimodal distribution, $p(s) = q\delta(s-s_a) + (1-q)\delta(s-s_b)$, we evaluated (9) numerically. Even for various values of q, s_a, s_b for which $p(s)$ is far from lognormality and in particular the saddle point approximation badly fails, we always reproduced (4). Therefore, even without general proof we believe the scaling relation (4) to be correct independent of $p(s)$.

Finally it is an easy exercise to repeat the analysis of ref. [12] and calculate other turbulent quantities of interest, such as the Taylor microscale λ_T or the Taylor Reynolds number Re_λ. We only report here our result for $g(q)$ defined by $\langle\langle \mathscr{E}^q \rangle\rangle / \langle\langle \mathscr{E} \rangle\rangle^q \propto Re_\lambda^{g(q)}$. It reads

$$g(q) = 3(1-2q) + \frac{2}{\mu}\{12 - 3[16 + 4(1-2q)\mu + \tfrac{1}{4}\mu^2]^{1/2}\} . \quad (13)$$

To lowest order in μ, (13) gives $\mu_S = \frac{9}{16}\mu$ and $\mu_k = \frac{3}{2}\mu$ for the exponents of skewness (with $q=\frac{3}{2}$) and kurtosis (with $q=2$). These (lowest order) $g(q)$ are identical with the results of the phenomenological theory of Wyngaard and Tennekes [15]. We finally remark that $\langle\langle \mathscr{E}^q \rangle\rangle$ does not exist for $q > (4+\frac{1}{2}\mu)^2/8\mu$. This is an artefact of the property of the lognormal distribution used in the calculation of (13), to allow arbitrarily large values of s.

Concluding, we remark that the scaling relation (4) certainly does not confirm the very existence of corrections to classical scaling. In ref. [11] from a numerical simulation of the Navier–Stokes equation we found chaotic fluctuations but, nevertheless, $p(s)$ to be statistically sharp. In that case (4) is trivially fulfilled with $\zeta(6)=2$ and $\mu=0$.

In a variety of other fields ultrametric structures similar to our model are presently discussed. Examples include spin glasses [16], neural networks [17], and precision dependent clustering in coupled map lattices [18]. In these systems, fluctuating cutoff scales arise in a natural way and corresponding relations between "inertial range" and "dissipation range" statistics is to be expected.

We would like to thank Charles Meneveau for pointing out the possible importance of a fluctuating cutoff scale. This concept has been used in different context in ref. [19], where additional applications are discussed.

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