Multifractal scaling from nonlinear turbulence dynamics: Analytical methods

Jens Eggers
Universität Gesamthochschule Essen, Fachbereich Physik, 45117 Essen, Germany
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We consider a general class of stochastic models of a turbulent cascade. They are based solely on the classical conceptions of local interactions and energy conservation. We show that such a model must necessarily exhibit strongly non-Gaussian fluctuations on small scales. This intermittent behavior is characterized by multifractal scaling. We develop analytical methods to calculate the anomalous scaling exponents without adjustable parameters and give numerical values for a specific model studied previously.

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I. INTRODUCTION

Perhaps the most famous example of self-similarity and scaling in a physical system is the velocity field of fully developed turbulent flow [1–3]. Its scaling properties are described by the structure functions

$$D^{(m)}(r) = \langle |\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t) - \boldsymbol{u}(\boldsymbol{x},t)|^m\rangle \sim r^{\zeta(m)}. \quad (1.1)$$

The angular brackets refer to an average over different realizations of the turbulent flow field. It is believed that at sufficiently high Reynolds numbers the ensemble of $\boldsymbol{u}$ fields is translationally invariant and locally isotropic [4]. This means the ensemble of velocity *differences* $\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t) - \boldsymbol{u}(\boldsymbol{x},t)$ is invariant under translations and rotations for scales $|\boldsymbol{r}| \ll L$, where L is the length scale of energy input.

The structure functions are found to scale with a spectrum of exponents $\zeta(m)$, which is independent of boundary conditions [5], i.e., of the mechanism of energy input. Thus one of the main tasks of turbulence theory is to understand this spectrum and its universality on the basis of the Navier-Stokes equation.

The first step towards a physical understanding of the scaling exponents was taken by Kolmogorov and Obukhov [4,6]. Their argument rests on the assumption that energy is fed into the flow at a finite rate on large scales L, which is transported locally (in scale) towards smaller scales. Eventually this cascade is terminated by viscous dissipation. Since the coupling is local, it is tempting to assume that the flow statistics eventually become independent of large scale flow features on scales $r \ll L$. Moreover, time and velocity scales are related by

$$\tau_r \approx r/v_r \quad (1.2)$$

due to the scale invariance of the Euler equation [7]. This is equivalent to the assumption that the motion of turbulent eddies is unaffected by viscosity for high Reynolds numbers. Hence, in the absence of any other scale, the moments of the velocity on scale r, say

$$\Delta u(\boldsymbol{r}) = |\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})|,$$

would be determined by the mean energy flux through scale r alone. By energy conservation the flux is independent of scale and equal to the mean energy dissipation rate ϵ. This means

$$D^{(m)}(r) = \langle [\Delta u(\boldsymbol{r})]^m\rangle = b^{(m)}(\epsilon r)^{m/3}, \quad (1.3)$$

where the $b^{(m)}$ are universal constants. The exponent spectrum in this classical theory therefore is $\zeta_{\rm cl}(m) = m/3$. With this understanding one can identify a length scale where inertial and viscous forces are balanced. This is the Kolmogorov scale

$$\eta = (\nu^3/\epsilon)^{1/4}. \quad (1.4)$$

Hence (1.3) should be observed for $\eta \ll r \ll L$.

However, it was pointed out by Landau and Lifshitz [8] that (1.3) cannot be correct as it stands. Their argument may be stated as follows. Assume that the average energy input $\epsilon(t)$ is changing very slowly over a total time interval T, e.g., there is a slow change of weather conditions in atmospheric turbulence. Then the structure functions

$$D^{(m)}(r,t) = b^{(m)}[\epsilon(t)r]^{m/3} \quad (1.5)$$

are also slowly time dependent. The average in $D^{(m)}(r,t)$ is computed over time intervals $\Delta t \ll T$, where $\epsilon(t)$ is almost constant, yet Δt is assumed large enough for the averages to converge.

On the other hand, one could compute $D^{(m)}(r)$ over the total time interval T, giving $D_T^{(m)}(r) = b^{(m)}\bar{\epsilon}^{m/3}r^{m/3}$. The overbar denotes a time average over the time interval T. But this must coincide with the time average of the time dependent structure function (1.5): $\overline{D^{(m)}(r)} = D_T^{(m)}(r)$. By virtue of (1.5) this would mean that $\bar{\epsilon}^{m/3} = \overline{\epsilon^{m/3}}$, which for general $\epsilon(t)$ cannot be true of course. Hence $b^{(m)}$ cannot be universal.

Indeed, experiments have given results in disagreement with (1.3) [5,9]. Especially for high moments, the exponents show deviations $\delta\zeta(m) = \zeta(m) - m/3$ from classical scaling. By dimensional arguments, this is only possible if (1.3) is generalized to

$$D^{(m)}(r) = b^{(m)}(\epsilon r)^{m/3}(r/L)^{\delta\zeta(m)}. \tag{1.6}$$

Note the singularity for $L \to \infty$, which is the trademark of multifractals [2,10,3]. This means that universality, or independence from large scale features, is a very subtle issue indeed: while $\delta\zeta(m)$ is universal, there is a *very strong* dependence of local statistics on the input scale.

Hence there are the following issues.

(1) What is the dynamical basis of Landau's argument and do corrections to classical scaling indeed take the form (1.6)?

(2) Supposing one has demonstrated the scaling form (1.6), how does one calculate the correction exponents $\delta\zeta(m)$ from the dynamical equations?

Answering these questions on the basis of the Navier-Stokes equation alone seems to be a problem far out of reach at present. Therefore, we construct a general class of stochastic models, which include the most important features of Navier-Stokes dynamics, yet are simple enough to be analytically tractable.

Large scale flow features of size L are represented by their energy E_1. They decay into eddies of size $L/2$, which have the energy E_2, and so on. Hence the turbulent flow field is described by a sequence of energies $E_i, i = 1, \ldots, N$. The features incorporated into the dynamics are energy conservation and local scale invariance. For simplicity, we restrict ourselves to the framework of Langevin equations with white noise.

Within the model, we are able to *prove* that intermittency corrections must have the multifractal form (1.6). Moreover, for the model we calculate the exponents $\delta\zeta(m)$ *without adjustable parameters*. It turns out that the $\delta\zeta(m)$ are necessarily different from zero, i.e., there necessarily are intermittency corrections. That means Kolmogorov's classical theory [4] is inconsistent with the assumptions it is originally based on.

Our method relies on two ideas. First, we formulate a perturbation theory which expands around a state of constant energy flux in the inertial range. Higher orders take fluctuations in the energy flux into account and induce corrections to classical scaling.

Second, we formulate the problem in terms of *ratios* of energies E_{l+1}/E_l in adjacent steps of the turbulent cascade. From the solution of the model it turns out that the distribution of E_{l+1}/E_l is independent of large scale flow features. This ensures the scaling form (1.6) as well as universality of the exponent corrections $\delta\zeta(m)$.

The paper is organized as follows. In Sec. II we introduce the present class of models, which are generalizations of a turbulence model proposed recently in Ref. [11], referred to as I hereafter. Section III prepares our analytical treatment. First, we derive a set of nonlinear equations for static (equal time) correlations between different levels of the cascade. Then we show how this nonlinear system can be treated by setting up a perturbation expansion in the noise strength.

Sections IV and V contain the main results. In the former, we introduce a description in terms of energy ratios. By establishing that the distribution of energy ratios is scale independent, we show that moments of the energy have the multifractal scaling form (1.6). We also verify the Kolmogorov structure equation [4]. In Sec. V we give an analytical expression for the intermittency exponents $\delta\zeta(m)$ by solving the perturbation equations to lowest nontrivial order. This is done by using the scaling form established in Sec. IV.

The Discussion relates our work to Kolmogorov's classical paper on intermittency [1]. In the framework of our model we are able to give detailed microscopic interpretations of the famous Kolmogorov similarity hypotheses. Next we test the quality of our results obtained from perturbation theory by comparing with numerical simulations. Even to lowest order in the noise strength we obtain excellent agreement. Finally, we discuss related work.

II. MODEL

The physical assumptions incorporated into our model are [11] (i) local scale invariance, (ii) energy conservation, and (iii) equipartition of energy in equilibrium. A few comments on these assumptions are in order. By local scale invariance we mean that at a given scale r, $L > r > 2^{-N}L$, there is just a single length and time scale. Hence local quantities are related by dimensional analysis, analogous to (1.2). This scale invariance is of course broken at the largest scale L and at the cutoff scale $2^{-N}L$. The exchange of energy between turbulence elements is only possible for eddies of comparable size; hence we allow only local energy transfer.

The physical reasoning behind the third assumption is explained in detail in [13]. If the system is not driven, it tends to distribute energy equally among its modes. If energy is fed into the largest scale, this tendency becomes the driving force behind the cascading of energy: In order to equilibrate, energy has to flow into the next smaller scale, where it again produces a nonequilibrium situation. Hence energy flows to even smaller scales and so on down to dissipation scales. At the same time this will tend to bound fluctuations, since energy gradients between neighboring levels cannot get too large. Hence the size of fluctuations at a given distance from the highest level will be finite.

We will now try to develop the most general Langevin description consistent with the above assumptions. To incorporate energy conservation and local coupling, we write

$$\partial_t E_l = T_{l-1\to l} - T_{l\to l+1}, \tag{2.1}$$

where E_l is the energy of the lth mode in a chain of turbulence elements. The rate of change of E_l is determined by the influx from above (which may be positive *and* negative) minus the outflux to the lower mode. The highest mode, labeled 1, represents an eddy of size L; at each step the size is cut in half.

In reality, the spatial structure of the flow field should be represented too, allowing a given eddy to decay into several smaller eddies instead of just one. In I we demonstrate how such a tree structure of eddies can be represented by a simple chain with *renormalized* chain elements. Hence, while spatial structure is extremely important to account for intermittency effects quantitatively [12], the linear chain is sufficiently general as an ansatz.

$T_{l\to l+1}$ is split into a deterministic and a stochastic part:

$$T_{l\to l+1}(t) = 2^l E_l^{3/2}\psi\left(\frac{E_{l+1}}{E_l}\right) + 2^{(l+1)/2}E_l^{5/4}\chi\left(\frac{E_{l+1}}{E_l}\right)\xi_{l+1}(t), \qquad (2.2)$$

$$\langle \xi_l(t)\xi_k(t')\rangle = 2\delta_{lk}\delta(t-t'). \qquad (2.3)$$

The Itô definition of (2.2) is assumed [14]. This can be done without loss of generality, since Stratonovich's case would only result in a redefinition of the arbitrary scaling function ψ. This is the most general form consistent with the local scaling (1.2): $T_{l\to l+1}$ has dimensions of (energy)/(time) and ξ_{l+1} has dimensions of $1/(\text{time})^{1/2}$. To ensure stability, some conditions on ψ and χ are needed. In accordance with (iii) we demand that

$$\psi(1) = 0\,,$$
$$\psi'(V) < 0\,, \quad 0 < V < \infty. \qquad (2.4)$$

Hence the system is always driven towards the equilibrium state $E_{l+1}/E_l = 1$. We further demand that $\chi(V)$ decays sufficiently fast for $V \to 0$ and $V \to \infty$. This ensures that the energy stays finite.

This completes the description of the model. It is identical to the Langevin equation derived in I, but with a specific form of ψ and χ. We remark that the derivation in I is only valid to lowest order in the expansion parameter φ_0, but even the higher order terms must conform with (2.2), since it is the most general case.

In I, extensive simulations of the turbulent state of this model are reported. To reach a turbulent state, the highest level is driven with a constant energy input ϵ:

$$\partial_t E_1 = \epsilon - T_{1\to 2}. \qquad (2.5)$$

At a level N energy was extracted by an eddy viscosity [15]. This was merely a device of simulational convenience; the exact dissipative mechanisms are inconsequential to the inertial range properties to be investigated here. In these simulations, a stable turbulent state was reached consistent with the phenomenology of Sec. I. The energy scales as in (1.6), where $D^{(m)}(r)$ has to be identified with $\langle E_l^{m/2}\rangle$, $r = 2^{-l}L$. The exponents are close to the classical values, but significant corrections $\delta\zeta(m) \neq 0$ are observed. In the following sections we will outline how $\delta\zeta(m)$ may be computed from the scaling functions ψ and χ.

III. STATIONARY MOMENTS

We will now derive a set of equations for the stationary, static correlation functions generated by the equations of motion (2.1)–(2.3). To solve this nonlinear set of equations, a perturbation theory is set up.

We write the equations of motion (2.1) and (2.2) as

$$\partial_t E_l = h_l(E_{l-1}, E_l, E_{l+1}) + g_{lk}(E_{l-1}, E_l, E_{l+1})\xi_k(t), \qquad (3.1)$$

$$h_l = 2^l\left\{\frac{1}{2}E_{l-1}^{3/2}\psi\left[\frac{E_l}{E_{l-1}}\right] - E_l^{3/2}\psi\left[\frac{E_{l+1}}{E_l}\right]\right\}, \qquad (3.2)$$

$$g_{lk} = 2^{l/2}\left\{\delta_{lk}E_{l-1}^{5/4}\chi\left[\frac{E_l}{E_{l-1}}\right] - \delta_{lk-1}\sqrt{2}E_l^{5/4}\chi\left[\frac{E_{l+1}}{E_l}\right]\right\}. \qquad (3.3)$$

For $l = 1$ we have

$$h_1 = \epsilon - 2E_1^{3/2}\psi\left[\frac{E_2}{E_1}\right], \qquad (3.4)$$

$$g_{1k} = -2\delta_{2k}E_1^{5/4}\chi\left[\frac{E_2}{E_1}\right]. \qquad (3.5)$$

Since we are looking for correlations in the stationary state, the time derivative of all expectation values should be zero. Hence, for any collection of n levels $l_i, i = 1,\ldots,n$, we obtain

$$\partial_t\left\langle\prod_{i=1}^n E_{l_i}\right\rangle = 0.$$

Using the equations of motion (3.1) this means

$$0 = \partial_t\left\langle\prod_{i=1}^n E_{l_i}\right\rangle = \sum_{i=1}^n\left\langle(\partial_t E_{l_i})\prod_{\substack{m=1\\m\neq i}}^n E_{l_m}(t)\right\rangle = \sum_{i=1}^n\left\langle h_{l_i}(t)\prod_{\substack{m=1\\m\neq i}}^n E_{l_m}(t)\right\rangle + \sum_{i=1}^n\left\langle g_{l_ik}(t_\epsilon)\xi_k(t)\prod_{\substack{m=1\\m\neq i}}^n E_{l_m}(t)\right\rangle. \qquad (3.6)$$

Here we have set $t_\epsilon = t-\epsilon$ and take the limit $\epsilon \to 0+$. Furthermore, we use the notation $h_l(t) = h_l(E_{l-1}(t), E_l(t), E_{l+1}(t))$; $g_{lk}(t)$ is defined analogously. It remains to calculate the second bracket involving the random force. To this end we observe that in the limit $\epsilon \to 0+$ [14]

$$E_{l_m}(t) = E_{l_m}(t_\epsilon) + g_{l_m l}(t_\epsilon)\int_{t-\epsilon}^t \xi_l(t')dt'.$$

Hence

$$\left\langle g_{l_ik}(t_\epsilon)\xi_k(t)\prod_{\substack{m=1\\m\neq i}}^n E_{l_m}(t)\right\rangle$$

$$= \left\langle g_{l_ik}(t_\epsilon)\prod_{\substack{m=1\\m\neq i}}^n E_{l_m}(t_\epsilon)\right\rangle \langle\xi_k(t)\rangle$$

$$+\sum_{\substack{j=1\\j\neq i}}^n \left\langle g_{l_ik}(t_\epsilon)g_{l_jl}(t_\epsilon)\prod_{\substack{m=1\\m\neq i,j}}^n E_{l_m}(t_\epsilon)\right\rangle$$

$$\times\int_{t-\epsilon}^t \langle\xi_l(t')\xi_k(t)\rangle dt' + O(\epsilon).$$

Here we used the fact that terms with double integrations are of higher order in ϵ and that $E_l(t_\epsilon)$ and $\xi(t')$ are statistically independent for $t' > t_\epsilon$. Noticing that $\langle\xi_k(t)\rangle = 0$ and $\int_{t-\epsilon}^t\langle\xi_l(t')\xi_k(t)\rangle dt' = \delta_{lk}$, (3.6) takes the form

$$\sum_{i=1}^n \left\langle h_{l_i}(t)\prod_{\substack{m=1\\m\neq i}}^n E_{l_m}(t)\right\rangle + \sum_{\substack{i,j=1\\i\neq j}}^n \left\langle D_{l_il_j}(t)\prod_{\substack{m=1\\m\neq i,j}}^n E_{l_m}(t)\right\rangle = 0, \quad (3.7)$$

$$D_{ij} = g_{ik}g_{jk}. \quad (3.8)$$

Thus for every n-tuple of levels $(l_1,\ldots,l_n)$ we have generated a nonlinear equation for static correlation functions involving the energies E_{l_i-1}, E_{l_i}, and E_{l_i+1}, where i runs from 1 to n.

This is the set of equations fundamental for the rest of this paper. It completely determines the static (multivariate) distributions of energies in our model. We will solve it perturbatively in a low noise expansion [16]. This means we expand around the deterministic part of the stationary solution $E_l^{(0)}$, obtained by switching off the noise term in the equation of motion (3.1); hence

$$h_l(E_{l-1}^{(0)}, E_l^{(0)}, E_{l+1}^{(0)}) = 0. \quad (3.9)$$

By energy conservation $\epsilon = \langle T_{l\to l+1}\rangle$, so the form of the energy transfer (2.2) leads to

$$E_l^{(0)} = C2^{-(2/3)l} \quad (3.10)$$

with

$$C = \left[\frac{\epsilon}{\psi(2^{-2/3})}\right]^{2/3}. \quad (3.11)$$

Note that the zeroth order solution (3.10) carries a finite but time independent energy transfer rate ϵ in the inertial range.

Since corrections to classical scaling are known to be small [5], one can hope to be able to expand in the noise strength. This is done most conveniently by multiplying the noise term in the equation of motion (3.1) with a parameter λ, expanding the solution in λ, and setting $\lambda = 1$ at the end of the calculation. Higher orders in λ thus contain fluctuations of the energy flux around ϵ.

It is convenient to scale the energies by their zeroth order values $E_l^{(0)}$:

$$e_l = \frac{E_l}{E_l^{(0)}}\quad,\quad l = 1,\ldots,N, \quad (3.12)$$

and to write h_l and $D_{l_1l_2}$ as functions of e_l:

$$\bar h_l(\{e_k\}) = (E_l^{(0)})^{-1}h_l(\{E_k\}),$$
$$\bar D_{l_1l_2}(\{e_k\}) = (E_{l_1}^{(0)}E_{l_2}^{(0)})^{-1}D_{l_1l_2}(\{E_k\}). \quad (3.13)$$

Deviations of e_l from 1 must be of higher order in λ; hence we write fluctuations x_l in e_l as

$$e_l = 1 + \lambda x_l. \quad (3.14)$$

Higher orders in λ are also contained in the variable x_l. By induction one shows that the fundamental equation (3.7) may now be written in the form

$$\sum_{i=1}^n \left\langle \bar h_{l_i}\prod_{\substack{m=1\\m\neq i}}^n \lambda x_{l_m}\right\rangle + \lambda^2\sum_{\substack{i,j=1\\i\neq j}}^n \left\langle \bar D_{l_il_j}\prod_{\substack{m=1\\m\neq i,j}}^n \lambda x_{l_m}\right\rangle = 0. \quad (3.15)$$

Now everything has to be written as an expansion in λ:

$$\bar h_i = \sum_{|\boldsymbol{m}|=1}^\infty A_i^{\boldsymbol{m}}\lambda^{|\boldsymbol{m}|}\boldsymbol{x}^{\boldsymbol{m}},$$
$$\bar D_{ij} = \sum_{|\boldsymbol{m}|=0}^\infty B_{ij}^{\boldsymbol{m}}\lambda^{|\boldsymbol{m}|}\boldsymbol{x}^{\boldsymbol{m}},$$

where $\boldsymbol{m}$ is a multiple index of order $|\boldsymbol{m}|$. Both $\boldsymbol{x}$ and $\boldsymbol{m}$ are vectors whose components run over all levels of the cascade: $\boldsymbol{x} = (x_1,\ldots,x_N)$ and $\boldsymbol{m} = (m_1,\ldots,m_N)$. By Taylor's theorem

$$A_i^{\boldsymbol{m}} = \frac{1}{|\boldsymbol{m}|!}\frac{\partial^{m_1}}{\partial {x_1}^{m_1}}\cdots\frac{\partial^{m_N}}{\partial {x_N}^{m_N}}\bar h_i|_{\{e_l=1\}} \quad (3.16)$$

and

$$B_{ij}^{\boldsymbol{m}} = \frac{1}{|\boldsymbol{m}|!}\frac{\partial^{m_1}}{\partial {x_1}^{m_1}}\cdots\frac{\partial^{m_N}}{\partial {x_N}^{m_N}}\bar D_{ij}|_{\{e_l=1\}}\quad. \quad (3.17)$$

Due to the definition of $E_l^{(0)}$, the expansion of $\bar h_i$ starts with $|\boldsymbol{m}| = 1$. Finally, the moments of $\boldsymbol{x}$ are expanded into a power series in λ:

$$\langle\boldsymbol{x}^{\boldsymbol{m}}\rangle \equiv \langle x_1^{m_1}\cdots x_N^{m_N}\rangle = \sum_{i=0}^\infty \lambda^i M_{\boldsymbol{m}}^{(i)}.$$

To write the perturbative equations, it is convenient to introduce some notation: Let $\boldsymbol{m}^{(i)} = (m_1^{(i)},\ldots,m_N^{(i)})$ be the multiple index with components $m_p^{(i)} = \sum_{\substack{k=1\\k\neq i}}^n \delta_{pl_k}$. Accordingly the components of $\boldsymbol{m}^{(i,j)}$ are defined as $m_p^{(i,j)} = \sum_{\substack{k=1\\k\neq i,j}}^n \delta_{pl_k}$.

Then, inserting the expressions for $\bar{h}_i$, $\bar{D}_{ij}$, and $\langle \boldsymbol{x}^{\boldsymbol{m}} \rangle$ into (3.15), we find the fundamental system of perturbative equations to be

$$\sum_{i=1}^{n} \sum_{|\boldsymbol{m}|=1}^{k+1} A_{l_i}^{\boldsymbol{m}} M_{\boldsymbol{m}+\boldsymbol{m}^{(i)}}^{(k-|\boldsymbol{m}|+1)} + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \sum_{|\boldsymbol{m}|=0}^{k} B_{l_i l_j}^{\boldsymbol{m}} M_{\boldsymbol{m}+\boldsymbol{m}^{(i,j)}}^{(k-|\boldsymbol{m}|)} = 0. \quad (3.18)$$

The system (3.18) is the perturbative version of the fundamental equation (3.7). For given k, it contains all moments $\langle \boldsymbol{x}^{\boldsymbol{m}} \rangle$ up to kth order.

Each of the *linear* equations in (3.18) can be inverted to solve for the $M_{\boldsymbol{m}}^{(i)}$. Hence we have shown that the fundamental system (3.7) can indeed be solved and expressions for $\langle \boldsymbol{x}^{\boldsymbol{m}} \rangle$ can be found.

On the other hand, the equations in (3.18) rapidly become very cumbersome at higher orders. Therefore, in the next section we will use quite different methods to show that the solutions thus found have a very simple scaling structure to all orders in λ. This greatly simplifies the calculation of solutions of (3.18), as will be demonstrated in Sec. V to lowest order in λ.

IV. SCALING SOLUTIONS

We now want to demonstrate that solutions of the fundamental equations (3.7) form a multifractal. In terms of a single cascade level l this means that the moments of E_l have the scaling form

$$\langle E_l^{m/2} \rangle = b^{(m)} (\epsilon 2^{-l})^{m/3} (2^{-l})^{\delta\zeta(m)} \quad (4.1)$$

for l in the inertial range $1 \ll l \ll N$. This scaling is the complete analog of the scaling of structure functions $D^{(m)}(r)$, $r = 2^{-l}L$, as in (1.6). In the normalized energies e_l the classical scaling behavior has been divided out, so that we are left with

$$\langle e_l^{m/2} \rangle = b^{(m)} [\psi(2^{-2/3})]^{m/3} (2^{-l})^{\delta\zeta(m)}. \quad (4.2)$$

The key to the scale invariance of solutions (4.1) and (4.2) lies in the local scale invariance of the model equations in the inertial range. This is expressed by the homogeneity properties of $\bar{h}_l$ and $\bar{D}_{l_1 l_2}$:

$$\bar{h}_l(\{ce_l\}) = c^{3/2}\bar{h}_l(\{e_l\}) \quad , \quad l \geq 2$$

$$\bar{D}_{l_1 l_2}(\{ce_l\}) = c^{5/2}\bar{D}_{l_1 l_2}(\{e_l\}). \quad (4.3)$$

This means the fundamental equation (3.7) is invariant under the transformation $E_k \to cE_k$ if $l_i \geq 2$ for all $i = 1, \ldots, n$. But scale invariance is of course *broken* by the energy input on level 1. If ϵ were zero, and thus our equations were describing a system in thermal equilibrium, $\bar{h}_1$ would be a homogeneous function also. The model would then be globally scale invariant and the power law structure of solutions a trivial matter. To still demonstrate scaling in a *nonequilibrium* situation with $\epsilon \neq 0$ is the fundamental problem.

To make use of local scale invariance, we write the normalized energy (3.12) as a multiplicative process [17]

$$e_{l+1} = a_l e_l \quad , \quad (4.4)$$

where a_l is a stochastic variable. This ansatz is standard in the theory of multifractals [18,19], but the point here is to show that it actually solves the fundamental equation (3.7). Note that (4.4) should not be read as $e_{l+1}(t) = a_l(t)e_l(t)$, but is a statement about the *ensemble* of e_l's.

Using (4.4) we can write e_l as

$$e_l = e_1 \prod_{j=1}^{l-1} a_j. \quad (4.5)$$

Now, for any collection of levels $(l_1, \ldots, l_n)$ with $l_i \geq 2$ we can rewrite the fundamental equation (3.7) in the homogeneous form

$$\left\langle e_1^{n+1/2} \left(\prod_{i=1}^{l_1-3} a_i^{n+1/2} \right) H^{(\Delta l_1,\ldots,\Delta l_n)}(a_{l_1-2}, a_{l_1-1}, \ldots, a_{l_n-1}) \right\rangle = 0, \quad (4.6)$$

with

$$H^{(\{\Delta l_i\})}(a_{l_1-2}, a_{l_1-1}, \ldots, a_{l_n-1}) = \left\{ \sum_{i=1}^{n} 2^{-2l_1/3}\bar{h}_{l_i} \prod_{\substack{m=1\\ m\neq i}}^{n} y_{l_m} + \sum_{\substack{i,j=1\\ i\neq j}}^{n} 2^{-2l_1/3}\bar{D}_{l_i l_j} \prod_{\substack{m=1\\ m\neq i,j}}^{n} y_{l_m} \right\} \Bigg|_{\{y_k=\prod_{j=l_1-2}^{k-1} a_j\}} ,$$

$\Delta l_i = l_i - l_1$, $i = 1, \ldots, n$. Without loss of generality we will henceforth assume that $l_1 \leq l_2 \leq \cdots \leq l_n$. Note that the only explicit dependence of H on level number is through *differences* Δl_i between two levels, as the combinations $2^{-2l_1/3}\bar{h}_{l_i}$ and $2^{-2l_1/3}\bar{D}_{l_i l_j}$ only depend on Δl_i.

If it were not for correlations between different levels, it would be straightforward to demonstrate multifractal scaling (4.1) and (4.2). For if the a_j were uncorrelated, it follows from (4.5) that

$$\langle e_l^{m/2} \rangle = \langle e_1^{m/2} \rangle \prod_{j=1}^{l-1} \langle a_j^{m/2} \rangle.$$

If moreover the averages are independent of j, $\langle a_j^{m/2} \rangle \equiv \langle a^{m/2} \rangle$, we have

$$\langle e_l^{m/2}\rangle = \langle e_1^{m/2}\rangle\langle a^{m/2}\rangle^{l-1},$$

which is equivalent to (4.2) if one identifies $\delta\zeta(m) = -\ln(\langle a^{m/2}\rangle)/\ln 2$.

To show that the average $\langle a_j^{m/2}\rangle$ is independent of j we use the fundamental equation in its homogeneous form (4.6): For uncorrelated a_j we have

$$\langle e_1^{n+1/2}\rangle \left(\prod_{i=1}^{l_1-3}\langle a_i^{n+1/2}\rangle\right)\langle H^{(\{\Delta l_i\})}(a_{l_1-2}, a_{l_1-1}, \dots, a_{l_n-1})\rangle = 0,$$

and hence

$$\langle H^{(\{\Delta l_i\})}(a_{l_1-2}, a_{l_1-1}, \dots, a_{l_n-1})\rangle = 0, \quad \Delta l_i = l_i - l_1. \tag{4.7}$$

This is a set of equations for the multipliers a_j alone, each of which is *local* in level indices. This means that the equation for a set of levels $(l_1, \dots, l_n)$ in the variables $a_{l_1-2}, \dots, a_{l_n-1}$ is identical to the equation for levels $(l_1+l, \dots, l_n+l)$ in the variables $a_{l_1+l-2}, \dots, a_{l_n+l-1}$. From this "translational invariance" it immediately follows that the distribution of each a_j must be the same since the set of equations (4.7) completely determines this distribution.

At this point it should be noted that even if the a_j are uncorrelated, the correlation in the energies e_l is long ranged and in fact does not decay at all. Namely,

$$\frac{\langle e_1 e_l\rangle - \langle e_1\rangle\langle e_l\rangle}{\langle e_1\rangle\langle e_l\rangle} = \frac{\langle e_1^2\rangle - \langle e_1\rangle^2}{\langle e_1\rangle^2} \neq 1$$

even for $l \to \infty$. Hence only in a variables the local or scaling structure of solutions becomes apparent.

But we now hasten to add that the a variables *are* in fact correlated, as to be expected from the structure of the fundamental equation, which couples neighboring levels. Still the above very simple analysis essentially goes through, as correlations between a variables *decay* exponentially with the level separation between them. Accordingly, we will generalize from the case of uncorrelated a variables to the case of multivariate distributions which factor for widely separated levels.

More formally, we will show that

$$P(a_{i_1+i}, \dots, a_{i_p+i}) = P(a_{i_1}, \dots, a_{i_p}) \tag{4.8}$$

(translational invariance) and

$$P(a_{i_1}, \dots, a_{i_q}, a_{i_{q+1}}, \dots, a_{i_p})$$

$$= P(a_{i_1}, \dots, a_{i_q})P(a_{i_{q+1}}, \dots, a_{i_p}),$$

$$i_1 \le \dots \le i_p,\ i_{q+1} - i_q \ge n_d \tag{4.9}$$

(correlation decay) for some decorrelation distance n_d.

We will proceed in two steps. First we show that a distributions with (4.8) and (4.9) imply multifractal scaling (4.1) and (4.2). Then we show that distributions with (4.8) and (4.9) actually solve the fundamental equation. This means we have shown that solutions of our model are multifractals. Moreover, the validity of (4.8) and (4.9) will be checked explicitly in perturbation theory in Sec. V.

As usual, distributions that factor are best treated in a cumulant expansion. To this end we define stochastic variables b_j via

$$a_j = e^{\lambda b_j} \quad , \quad j = 1, \dots, N. \tag{4.10}$$

We now expand moments in λ and remember that to zeroth order in λ we have $a_j = 1$, consistent with (3.14).

To simplify notation we disregard fluctuations in e_1 and set $e_1 = 1$, giving, with (4.5),

$$\langle e_l^{m/2}\rangle = \left\langle \exp\left[\lambda\frac{m}{2}\sum_{i=1}^{l-1} b_i\right]\right\rangle.$$

Writing this as

$$\langle e_l^{m/2}\rangle = \exp\left[\sum_{p=1}^{\infty}\frac{\lambda^p}{p!}C_p(m)\right], \tag{4.11}$$

we obtain

$$C_p(m) = \frac{\partial^p}{\partial\lambda^p}\ln\left\{\left\langle \exp\left[\lambda\frac{m}{2}\sum_{i=1}^{l-1} b_i\right]\right\rangle\right\}\Bigg|_{\lambda=0}. \tag{4.12}$$

Repeated application of the chain rule shows that $C_p(m)$ can be written as

$$C_p(m) = \left(\frac{m}{2}\right)^p \sum_{i_1,\dots,i_p=1}^{l-1} C(b_{i_1}, \dots, b_{i_p}), \tag{4.13}$$

where $C(b_{i_1}, \dots, b_{i_p})$ is the pth order cumulant of $b_{i_1}, \dots, b_{i_p}$. By definition,

$$C(b_{i_1}, \dots, b_{i_p}) = (-i)^p\frac{\partial^p \ln\phi(\lambda_{i_1}, \dots, \lambda_{i_p})}{\partial\lambda_{i_1}\cdots\partial\lambda_{i_p}}, \tag{4.14}$$

where $\phi(\lambda_{i_1}, \dots, \lambda_{i_p})$ is the generating functional

$$\phi(\lambda_{i_1}, \dots, \lambda_{i_p}) = \left\langle \exp\left[i\sum_{j=1}^{p}\lambda_{i_j} b_{i_j}\right]\right\rangle. \tag{4.15}$$

To analyze (4.13) we use the fact that cumulants $C(b_{i_1}, \dots, b_{i_p})$ vanish if the variables $b_{i_1}, \dots, b_{i_p}$ fall into two classes $b_{i_1}, \dots, b_{i_q}$ and $b_{i_{q+1}}, \dots, b_{i_p}$ for which the generating functional factorizes. This follows directly from the definition (4.14). But (4.9) implies that this must always be the case for $i_p - i_1 \geq pn_d$ and $i_1 \leq i_2 \leq \cdots \leq i_p$. Hence all terms in the sum (4.13) vanish except in a band of width pn_d around the diagonal $i_1 = i_2 = \cdots = i_p$. Also, (4.8) implies that $C(b_{i_1+i},\ldots,b_{i_p+i}) = C(b_{i_1},\ldots,b_{i_p})$; hence in the limit $l \to \infty$ each of the $C_p(m)$ is proportional to l. This means we can write $C_p(m) = \bar{C}_p(m)l$, where $\bar{C}_p(m)$ is a constant. Inserting this into (4.11) gives

$$\langle e_l^{m/2}\rangle = \left\{\exp\left[\sum_{p=1}^{\infty}\frac{\lambda^p}{p!}\bar{C}_p(m)\right]\right\}^l .$$

But this is precisely the multifractal scaling law (4.2). The value of the correction exponent can be read off to be

$$\delta\zeta(m) = -\frac{1}{\ln 2}\sum_{p=1}^{\infty}\frac{\lambda^p}{p!}\bar{C}_p(m). \tag{4.16}$$

This completes the first step of our program to demonstrate multifractality. Now we show that distributions with translational invariance and finite correlation distance solve the fundamental equation.

To this end we decompose $H^{(\{\Delta l_i\})}(a_{l_1-2},a_{l_1-1},\ldots,a_{l_n-1})$ into its positive and negative parts:

$$H^{(\{\Delta l_i\})}(a_{l_1-2},a_{l_1-1},\ldots,a_{l_n-1}) = H^+ - H^-$$

$$\text{with } H^+,H^- > 0.$$

The indices on $H^\pm$ have been dropped for notational convenience. Then positive and negative parts can be represented in the form

$$H^\pm = H_0^\pm e^{\lambda h^\pm}, \tag{4.17}$$

where $h^\pm$ are again stochastic variables and $H_0^\pm$ are constants. Repeating the above procedure, the fundamental equation (4.6) can be expressed through a variables:

$$\left\langle\left(\prod_{i=1}^{l_1-3} a_i^{n+1/2}\right)H^+\right\rangle = \left\langle\left(\prod_{i=1}^{l_1-3} a_i^{n+1/2}\right)H^-\right\rangle. \tag{4.18}$$

Both sides of this equation are now expanded in cumulants:

$$\left\langle\left(\prod_{i=1}^{l_1-3} a_i^{n+1/2}\right)H^\pm\right\rangle = H_0^\pm\left\langle\exp\left[\lambda(n+1/2)\sum_{i=1}^{l_1-3} b_i + \lambda h^\pm\right]\right\rangle$$

$$= H_0^\pm \exp\left\{\sum_{p=1}^{\infty}\frac{\lambda^p}{p!}\left[(n+1/2)^p\sum_{i_1,\ldots,i_p=1}^{l_1-3} C(b_{i_1},\ldots,b_{i_p}) + \sum_{q=0}^{p-1}(n+1/2)^q\sum_{i_1,\ldots,i_q=1}^{l_1-3}\bar{C}_p^\pm(b_{i_1},\ldots,b_{i_q},h^\pm)\right]\right\}. \tag{4.19}$$

Here we have broken up the expression into terms with cumulants including only b variables and those with the new variables $h^\pm$:

$$\bar{C}_p^\pm(b_{i_1},\ldots,b_{i_q},h^\pm)$$

$$= (-i)^p\frac{\partial^q}{\partial\lambda_{i_1}\ldots\partial\lambda_{i_q}}\frac{\partial^{p-q}}{\partial\lambda^{p-q}}\ln\bar{\phi}^\pm(\lambda_{i_1},\cdots,\lambda_{i_q},\lambda) \tag{4.20}$$

with the generating functional

$$\bar{\phi}^\pm(\lambda_{i_1},\ldots,\lambda_{i_q},\lambda) = \left\langle\exp\left[i\left(\sum_{j=1}^{q}\lambda_{i_j}b_{i_j} + \lambda h^\pm\right)\right]\right\rangle. \tag{4.21}$$

The cumulants $C(b_{i_1},\ldots,b_{i_p})$ appear on both sides of (4.18) and therefore drop out. Since $h^\pm$ only depends on a_{l_1-2} through a_{l_n-1}, $\bar{C}_p^\pm(b_{i_1},\ldots,b_{i_q},h^\pm)$ will vanish if all indices $i_1,\ldots,i_q$ are smaller than l_1-n_d-1 because of finite correlation length (4.9). But since none of the indices i_1 through i_q should be separated by more than the decorrelation distance n_d, $\bar{C}_p^\pm(b_{i_1},\ldots,b_{i_q},h^\pm)$ in fact vanishes if *any* of the $i_1,\ldots,i_q$ are smaller than l_1-qn_d-1. This means for large l_i the fundamental equation can be written as

$$H_0^+\exp\left\{\sum_{p=1}^{\infty}\frac{\lambda^p}{p!}\sum_{q=0}^{p-1}(n+1/2)^q\sum_{i_1,\ldots,i_q=l_1-qn_d-1}^{l_1-3}\bar{C}_p^+(b_{i_1},\ldots,b_{i_q},h^+)\right\}$$

$$= H_0^-\exp\left\{\sum_{p=1}^{\infty}\frac{\lambda^p}{p!}\sum_{q=0}^{p-1}(n+1/2)^q\sum_{i_1,\ldots,i_q=l_1-qn_d-1}^{l_1-3}\bar{C}_p^-(b_{i_1},\ldots,b_{i_q},h^-)\right\}. \tag{4.22}$$

In this form the fundamental equation is again both *local* in the a variables $a_j = e^{\lambda b_j}$ and does not depend explicitly on level numbers l_i, but only on *differences* $l_i - l_1$ between level numbers. Hence solutions automatically will be translationally invariant and correlations between distant levels will decay. This completes the proof of multifractality for the present class of stochastic models.

We close this section by also showing that

$$\zeta(3) = 1, \tag{4.23}$$

consistent with the Kolmogorov structure equation [4]. Indeed, for the special case of only one level l we have

$$H^{(0)}(a_{l-2}, a_{l-1})$$
$$= C^{1/2}\left\{\psi(2^{-2/3}a_{l-2}) - a_{l-2}^{3/2}\psi(2^{-2/3}a_{l-1})\right\},$$

so that the fundamental equation for a single level becomes

$$\left\langle \left(\prod_{i=1}^{l-3} a_i^{3/2}\right)\psi(2^{-2/3}a_{l-2})\right\rangle$$
$$= \left\langle \left(\prod_{i=1}^{l-2} a_i^{3/2}\right)\psi(2^{-2/3}a_{l-1})\right\rangle. \tag{4.24}$$

Bringing both sides of this equation into a form analogous to (4.19), one immediately finds that the right hand side contains one additional factor of $2^{-\delta\zeta(3)}$, implying $2^{-\delta\zeta(3)} = 1$ or $\delta\zeta(3) = 0$. Since $\delta\zeta(3) = \zeta(3) - 1$ this implies (4.23).

V. FIRST ORDER SOLUTION

In this section we compute the exponent spectrum of turbulent energies without adjustable parameters to lowest nontrivial order in λ. We proceed in two steps.

First, we show that to lowest order the spectrum of exponent corrections $\delta\zeta(m) = \zeta(m) - m/3$ is a quadratic polynomial in m. Using $\delta\zeta(3) = 0$ the whole spectrum can be written in terms of a single parameter μ. Here μ is the famous dissipation exponent introduced by Kolmogorov [1].

Second, we compute μ for our model by solving the perturbative equations (3.18) for the correlation matrices of energies. This is fairly straightforward because the structure of correlations is known from the results of the preceding section.

We derive the form of $\delta\zeta(m)$ by expanding in powers of λ:

$$\delta\zeta(m) = \sum_{j=1}^{\infty} \lambda^{2j}\zeta_j^{(m)}. \tag{5.1}$$

Since the fundamental equation for turbulent fluctuations x_j, (3.15), is invariant under a change of sign in λ, (5.1) contains only even powers.

Using the scaling structure of e_l derived in Sec. IV, we find

$$\langle e_l^{m/2}\rangle = 1 - l\lambda^2(\ln 2)\zeta_1^{(m)} + O(\lambda^4). \tag{5.2}$$

Here and for the rest of this section we disregard fluctuations in e_1 and set $e_1 = 1$. On the other hand, e_l can be written as $e_l = 1 + \lambda x_l$; hence

$$\langle e_l^{m/2}\rangle = \sum_{i=0}^{m/2}\binom{m/2}{i}\lambda^i\langle x_l^i\rangle.$$

But the lowest order expansion of the moments $\langle x_l\rangle$ and $\langle x_l^2\rangle$ is $\langle x_l\rangle = \lambda\langle x_l\rangle^{(1)} + O(\lambda^3)$ and $\langle x_l^2\rangle = \langle x_l x_l\rangle^{(0)} + O(\lambda^2)$. Hence, comparing coefficients of order λ^2 we have

$$-l(\ln 2)\zeta_1^{(m)} = (m/2)\langle x_l\rangle^{(1)} + [m(m-2)/8]\langle x_l x_l\rangle^{(0)}.$$

This equation says that $\langle x_l\rangle^{(1)}$ and $\langle x_l x_l\rangle^{(0)}$ must be linear functions of l for large l :

$$\langle x_l\rangle^{(1)} = \sigma_1 l,$$
$$\langle x_l x_l\rangle^{(0)} = \sigma_2 l, \tag{5.3}$$

and hence

$$\zeta_1^{(m)} = -\frac{m}{2\ln 2}\left[\sigma_1 + \left(\frac{m}{4} - \frac{1}{2}\right)\sigma_2\right]. \tag{5.4}$$

Moreover, using $\delta\zeta(3) = 0$, and hence $\zeta_1^{(3)} = 0$, one of the linear coefficients σ_1, σ_2 can be eliminated:

$$\sigma_1 = -\frac{1}{4}\sigma_2. \tag{5.5}$$

This means we can write $\delta\zeta(m)$ to lowest order as

$$\delta\zeta(m) = -\mu\frac{m}{18}(m-3), \tag{5.6}$$

where the dissipation exponent μ can be expressed through model parameters:

$$\mu = \frac{9}{4}\frac{\sigma_2}{\ln 2}. \tag{5.7}$$

Here we have set $\lambda = 1$. Equation (5.6) is the famous lognormal approximation to the exponent spectrum [1]. It comes out of the lowest order of our perturbation theory. Of course, this is only a satisfactory approximation for small m. To higher orders in λ, higher powers in m will appear.

It now remains to calculate the exponent μ. To this end one only needs to analyze the matrix $\langle x_{i_1}x_{i_2}\rangle$. It is determined by a special case ($k = 0, n = 2$) of the perturbative system (3.18):

$$A_{j_1}^i\langle x_i x_{j_2}\rangle^{(0)} + A_{j_2}^i\langle x_i x_{j_1}\rangle^{(0)} = -2B_{j_1 j_2} \tag{5.8}$$

with

$$A_j^i = \frac{\partial}{\partial x_i}\bar{h}_j|_{\{e_l=1\}},$$
$$B_{j_1 j_2} = \bar{D}_{j_1 j_2}|_{\{e_l=1\}}. \tag{5.9}$$

All indices run from 1 to N and summation over like indices is assumed. In Appendix A we show that the following sum rules can be derived from the homogeneity properties of $\bar{h}_j$ and $\bar{D}_{j_1 j_2}$ and from energy conservation:

$$\sum_{i=1}^{N} A_j^i = 0 \tag{5.10}$$

and

$$\sum_{i=1}^{N} 2^{-2i/3} B_{ij} = 0. \tag{5.11}$$

Since all couplings are local, A_j^i and B_{ij} must be tridiagonal, and in view of the above constraints

$$A_j^i = A\, 2^{(i+j)/3} \{\delta_{ij} - \gamma_1 \delta_{ij-1} - \gamma_2 \delta_{i-1j}\}, \tag{5.12}$$

where γ_1 and γ_2 are related by

$$1 - 2^{-1/3}\gamma_1 - 2^{1/3}\gamma_2 = 0. \tag{5.13}$$

The matrix B_{ij} is symmetric; hence

$$B_{ij} = B\, 2^{(i+j)/3} \left\{\delta_{ij} - \frac{1}{2^{1/3} + 2^{-1/3}} (\delta_{ij-1} + \delta_{i-1j})\right\}. \tag{5.14}$$

The dependence of $\langle x_{i_1} x_{i_2} \rangle^{(0)}$ on i_1 and i_2 is completely determined by translational invariance of the moments $\langle b_{k_1} b_{k_2} \rangle$, which was demonstrated in Sec. IV. One just has to remember that $e_l = 1 + \lambda x_l$ and $a_j = 1 + \lambda b_j + O(\lambda^2)$; cf. (4.10). Hence $x_l = \sum_{j=1}^{l-1} b_j + O(\lambda)$ and

$$\langle x_{i_1} x_{i_2} \rangle^{(0)} = \sum_{k_1, k_2 = 1}^{i_1 - 1, i_2 - 1} \langle b_{k_1} b_{k_2} \rangle^{(0)}, \tag{5.15}$$

where $\langle b_{k_1} b_{k_2} \rangle^{(0)}$ is the zeroth order term in an expansion in λ. Since $\langle b_{k_1} b_{k_2} \rangle^{(0)}$ depends only on $k = |k_1 - k_2|$ we introduce

$$\bar{b}_k = \left(1 - \frac{\delta_{k0}}{2}\right) \langle b_{k_1} b_{k_2} \rangle^{(0)}. \tag{5.16}$$

Then for asymptotic i_1 and i_2 a straightforward calculation shows that

$$\langle x_{i_1} x_{i_2} \rangle^{(0)} = \sum_{k=0}^{\infty} \bar{b}_k [\min(i_1 - 1, i_2 - i - 1) + \min(i_1 - i - 1, i_2 - 1)]. \tag{5.17}$$

Here we assumed that $\bar{b}_k \to 0$ for $k \to \infty$, so that the sum converges.

First, we notice that this expression explicitly solves the lowest order fundamental equation (5.8). Namely, the left hand side can be written as

$$A\, 2^{(j_1 + j_2)/3} \sum_{l=0}^{\infty} C_{|j_1 - j_2| l} \bar{b}_l. \tag{5.18}$$

Here and henceforth we let the total number of levels go to infinity. The exact form of the matrix $\boldsymbol{C}$ will be given in Appendix B. Comparing with the right hand side of (5.8) and using the special form of the matrix $\boldsymbol{B}$, we find

$$\bar{b}_k = -2\frac{B}{A} \left\{ (\boldsymbol{C}^{-1})_{k0} - \frac{1}{2^{1/3} + 2^{-1/3}} (\boldsymbol{C}^{-1})_{k1} \right\}. \tag{5.19}$$

Second, it is apparent from (5.17) that the total amplitude σ_2 of $\langle x_l x_l \rangle^{(0)}$ is given by $\sigma_2 = 2\sum_{k=0}^{\infty} \bar{b}_k$. In Appendix B it will be shown that $\bar{b}_k$ decays exponentially with the level distance k, so this sum converges. Using (5.7) we finally find for the dissipation exponent μ:

$$\mu = -\frac{9}{2 \ln 2} \Gamma c^{(0)}(\Delta) \left\{ 1 - \frac{2 g_1(\Delta)}{2^{1/3} + 2^{-1/3}} \right\}. \tag{5.20}$$

The only parameters appearing in these expressions are the ratios $\Delta = \sqrt{\gamma_1/\gamma_2}$ and $\Gamma = B/A$. Both can be computed from model parameters $\psi(2^{-2/3})$, $\psi'(2^{-2/3})$, and $\chi^2(2^{-2/3})$:

$$\Delta = 2^{1/3} \sqrt{1 - 3 \times 2^{-1/3} \psi/\psi'},$$

$$\Gamma = \frac{2(1 + 2^{2/3})\chi^2}{2^{1/3}\psi' - 3\psi/2}. \tag{5.21}$$

The functions $c^{(0)}(\Delta)$ and $g_1(\Delta)$ are calculated in Appendix B. Equations (5.6) and (5.20) constitute the central result of this section: They allow the analytical calculation of the exponent spectrum, starting from the dynamical equations.

The stability constraints $\psi > 0$ and $\psi' < 0$ [cf. (2.4)] imply that $\Gamma < 0$ and $\Delta > 2^{1/3}$. Analysis of $c^{(0)}(\Delta)$ and $g_1(\Delta)$ shows that this implies $\mu > 0$. Hence the dissipation exponent μ is always nonzero: In the present system with local scale invariance and energy conservation multifractal exponent corrections $\delta\zeta(m) \neq 0$ are inevitable. Here we have to exclude the trivial case $\chi^2 = 0$, which would mean that there are no fluctuations, which is absurd for a turbulent flow.

Moreover, since $\bar{b}_k$ behaves like $(2^{1/3}/\Delta)^{2k}$ (see Appendix B), $\bar{b}_k$ will decay for large level separations. Applying this to correlations between a variables, we find

$$\frac{\langle a_1 a_k \rangle - \langle a_1 \rangle \langle a_k \rangle}{\langle a_1 \rangle \langle a_k \rangle} = \lambda^2 \bar{b}_{k-1} + O(\lambda^4) \to 0$$

for $k \to \infty$. Hence a variables decorrelate for large level separations as opposed to the original energy variables; cf. our discussion in Sec. IV.

The physical reason for the decorrelation of a variables lies in the separation of time scales between distant levels. The large scale motion of low levels acts on the higher levels like a slowly varying energy input $\epsilon(t)$. As seen from the scaling law (4.1) and (4.2), this $\epsilon(t)$ enters the statistics of two neighboring levels e_l and e_{l+1} in exactly the same way, and therefore drops out of *ratios* $a_l = e_{l+1}/e_l$. Hence the statistics of a_l is independent of large scale motion, whereas e_l depends very strongly on it. This is the essence of Landau's argument about the nonuniversality of small scale statistics.

VI. DISCUSSION

We have seen that the presence of fluctuations automatically leads to intermittency and exponent corrections $\delta\zeta(m) \neq 0$. Our assumptions about the dynamics of the model were as innocuous as possible: Interactions are local, and the noise term has no memory and a Gaussian distribution. Hence the buildup of large, non-Gaussian fluctuations is a necessary consequence of local cascading and scale invariance. In a sense, classical theory is disqualified by precisely the assumptions it is built upon.

The conceptual difference of our work to the usual multiplicative models of turbulence [3,18,19] should be noted: Starting from the microscopic dynamics (2.1) we *show* that solutions have multiplicative form. This is what is *assumed* by tenet in the usual models. The distribution of multipliers, which are the a variables in our work, just appear as fit parameters, whereas we compute them from dynamical quantities. Besides offering insight into the dynamical origins of intermittency growth, this allows the conclusion, independent of model parameters, that intermittency must always be present as soon as there are fluctuations. In multiplicative models $\delta\zeta(m) = 0$ for all m always appears as a special case and is completely consistent with the presence of fluctuations.

As mentioned earlier, this inconsistency was already noticed by Landau and prompted Kolmogorov to put forward a refined theory [1]. In his paper he formulates his assumptions in terms of three "similarity hypotheses" for turbulent flow. It is extremely revealing to discuss our results in terms of three hypotheses for turbulent flow which are just slight modifications of the ones originally proposed by Kolmogorov. They correspond precisely to the main statements of the present paper, which we are now able to *demonstrate* in the framework of our model.

Consider a set of points $\boldsymbol{x}^{(k)}$ in a turbulent flow, $\boldsymbol{u}(\boldsymbol{x}, t)$, whose distances to a reference point $\boldsymbol{x}$ scale as

$$|\boldsymbol{x}^{(k)} - \boldsymbol{x}| \approx 2^{-k} L, \tag{6.1}$$

and $k \gg 1$ is assumed. Now consider the distribution of the values of

$$\frac{|\boldsymbol{u}(\boldsymbol{x}^{(k+1)}) - \boldsymbol{u}(\boldsymbol{x})|}{|\boldsymbol{u}(\boldsymbol{x}^{(k)}) - \boldsymbol{u}(\boldsymbol{x})|}. \tag{6.2}$$

This is the analog of our variable a_k. Two points should be noted: Kolmogorov's original variables are somewhat different in that he looks at ratios of velocity variables at a level k and a fixed reference level 0. With this definition, there are correlations remaining in the lower lying levels, so we do not expect those variables to decorrelate. Second, in our model we use *energies* to define a_k instead of velocity differences. We think energies are preferable since as conserved quantities they are "slow" variables of the system and have less local fluctuations. In particular, they do not have the problem of zeros in the denominator of (6.2).

We now formulate our hypotheses.

First similarity hypothesis. The distribution of values of (6.2) only depends on the Reynolds number. In the context of our model, the Reynolds number is $\mathrm{Re} = 2^{4N/3}$, where N is the total number of levels and hence $\eta = 2^{-N} L$ a cutoff scale. This is precisely what we demonstrated in Sec. IV: The distribution of large scale energies drops out of the fundamental equation; cf. (4.22).

Second similarity hypothesis. For large $\mathrm{Re} \gg 1$ the distribution given in the first hypothesis does not depend on Re. (This is also supposed to mean that it does not depend on level number k.) The independence of Re, or of N for $N \to \infty$ in the case of our model, is again a consequence of the local structure of the fundamental equation (4.22). As a corollary, the distribution of a_k is independent of k, which is what we called "translational invariance"; cf. (4.8).

Third similarity hypothesis. If k_1 and k_2 are widely separated, $k_1 \ll k_2$, the corresponding distributions of (6.2) are statistically independent. This is what we called "finite correlation length"; cf. (4.9). In Sec. V we computed the correlation between a_{k_1} and a_{k_2} and showed that $(\langle a_{k_1} a_{k_2}\rangle - \langle a_{k_1}\rangle\langle a_{k_2}\rangle)/(\langle a_{k_1}\rangle\langle a_{k_2}\rangle)$ indeed decays exponentially with $|k_1 - k_2|$.

In the same paper [1], Kolmogorov also suggests that the correction exponents $\delta\zeta(m)$ of the mth order structure functions have the parabolic form $\delta\zeta(m) = -\mu m(m-3)/18$. This is precisely our equation (5.6), obtained from lowest order perturbation theory. Although it is clear from our model that higher order corrections in m must also appear in $\delta\zeta(m)$, it is nevertheless a very good approximation for most practical purposes.

But still, since numerical values of exponents have only been computed to lowest order in perturbation theory, it is natural to ask how well they compare with simulations of the full nonlinear equations. In I, simulations of our model have already been performed for a specific choice of scaling functions $\psi(V)$ and $\chi(V)$. We show that for finite energy input $\epsilon > 0$ a turbulent state is reached. It is well described by multifractal scaling of the form (4.1) and numerical values of the exponent corrections $\delta\zeta(m)$ are determined. The model parameters reported in I give $\psi(2^{-2/3}) = 5 \times 10^{-4}$, $\psi'(2^{-2/3}) = -1.08 \times 10^{-3}$, and $\chi^2(2^{-2/3}) = 2.02 \times 10^{-5}$. With our expression (5.20) the dissipation exponent μ can easily be computed to be $\mu = 0.0163$. Using the parabolic form of $\delta\zeta(m)$, we find $\delta\zeta(18) = -0.245$, while the value obtained from simulation in I was $\delta\zeta(18) = -0.25 \pm 0.002$. Given a moment of such high order the agreement is extremely good.

This strongly suggests that our perturbation theory correctly approximates intermittent fluctuations of the energy transfer. We reiterate that this is because the zeroth order problem we expand about is already a "turbulent" state with finite energy flux. This sets it aside from the usual perturbation theories (see, for example, [21,22]), which expand about a diffusive problem without inertial range. It seems highly unlikely that perturbation theory, even if taken to arbitrarily high order, can "cross over" between two such extremely different states of motion.

However, even for the present perturbation theory the quality of agreement with numerical simulation will depend on the noise level chosen in the model. It seems desirable to choose the noise level so as to resemble the size of fluctuations in the large scale motion of real turbulence and then compare the values of the exponent corrections. Unfortunately, this is difficult to do in the case of the linear chain discussed in this paper. More likely, real turbulence corresponds to a situation where interactions are allowed to "branch out" and one eddy can decay into several smaller eddies. This different type of interaction leads to much larger exponent corrections at the same noise level, as shown in I.

To account for this effect, we recently introduced the concept of "eddy competition" [12]. In this physical picture, an eddy decays into eight smaller eddies. The subeddies compete for energy, so one can grow very strongly at the expense of its siblings. In a simple chain, only moderate growth is possible, since it is at the expense of only a single eddy "mother." This is seen explicitly in the expression (5.20). The two terms in the curly brackets almost cancel, where the second comes from the depletion of energy in the mother eddy. It would be interesting to perform the same calculation in the branched system, to see eddy competition at work.

Another interesting point would be to study *temporal* correlations within our model. Besides the predictions of classical scaling theories, nothing is known about those quantities. In particular, considering the scale dependence of correlations times, it seems natural to ask if they conform with classical predictions or if they carry intermittency corrections themselves.

Finally, we would eventually like to introduce a "real" viscosity ν into our model. It would correspond to an additional dissipative term $-\nu 2^{2l} E_l$ in the energy balance (2.1) and therefore results in a natural realization of a cutoff scale η. All possible questions of dissipation range statistics could then be treated consistently.

Many of the conclusions drawn in the present paper are somewhat parallel to Kraichnan's [23]. He concludes that classical theory can only be consistent if there is significant *spatial* mixing, which counteracts the buildup of fluctuations. His cascade model is not formulated in terms of microscopic variables, but is a phenomenological equation of motion for the probability distributions. Another way to get around the conclusions drawn here is to allow interactions which are *not* local in k space, but which contain significant direct coupling between small and large scales.

Benzi *et al.* [17] consider the chaotic dynamics of a cascade where every step is represented by a complex velocity amplitude u_l. The nonlinear coupling is quadratic like the Navier-Stokes case, so naive perturbation theory would lead to classical scaling. Decomposing u_l into the modulus ρ_l (which corresponds to $\sqrt{e_l}$ in the present model) and phases Θ_l, they end up with equations remarkably similar to ours. Solutions are obtained numerically.

In conclusion, multifractals are the natural framework in which to describe a scale invariant, local cascade. There are methods at hand to calculate their scaling properties from microscopic dynamics. To put them to use in turbulence, details of the flow field have to be considered.

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APPENDIX A: SUM RULES

Here we derive the sum rules (5.10) and (5.11) for the matrices A_j^i and B_{ij}. We will do that in a slightly more general framework, obtaining sum rules which relate different orders of perturbation theory. The essential ingredients are the homogeneity properties (4.3) of $\bar{h}_l$ and $\bar{D}_{l_1 l_2}$, combined with the constraints of energy conservation. These sum rules ensure that perturbation theory conforms with the scaling (4.1) at every order.

Let us introduce expansion coefficients through

$$A_i^{\mu_1,\ldots,\mu_m} = \frac{1}{m!}\frac{\partial}{\partial x_{\mu_1}}\cdots\frac{\partial}{\partial x_{\mu_m}}\bar{h}_i|_{\{e_l=1\}} \tag{A1}$$

and

$$B_{ij}^{\mu_1,\ldots,\mu_m} = \frac{1}{m!}\frac{\partial}{\partial x_{\mu_1}}\cdots\frac{\partial}{\partial x_{\mu_m}}\bar{D}_{ij}|_{\{e_l=1\}} \quad . \tag{A2}$$

Note that this is just an alternative notation of (3.16) and (3.17), which is more convenient for the present purposes. Differentiating Eqs. (4.3) with respect to c and using the chain rule we find

$$\sum_{\mu=1}^{N} A_i^{\mu_1,\ldots,\mu_m,\mu} = \frac{3/2-m}{m+1} A_i^{\mu_1,\ldots,\mu_m},$$
$$\sum_{\mu=1}^{N} B_{ij}^{\mu_1,\ldots,\mu_m,\mu} = \frac{5/2-m}{m+1} B_{ij}^{\mu_1,\ldots,\mu_m}. \tag{A3}$$

This connects the coefficients appearing at a given order of perturbation theory with the previous order. Since $\bar{h}_i|_{\{e_l=1\}} = 0$, which expresses conservation of energy at mean field level, and using (A3) recursively down to zeroth order it follows that

$$\sum_{\mu_1,\ldots,\mu_m=1}^{N} A_i^{\mu_1,\ldots,\mu_m} = 0. \tag{A4}$$

It will immediately be noticed that (5.10) is just a special case of (A4) for $m = 1$.

To obtain (5.11), we use energy conservation in the fluctuating part of the Langevin equation (3.1). Namely, by inspection of g_{lk}, (3.3), it follows that

$$\sum_{l=1}^{N} g_{lk}(E_{l-1}, E_l, E_{l+1}) = 0$$

for all values of the energies. In terms of the normalized function $\bar{D}_{ij}$ this means that

$$\sum_{i=1}^{N} 2^{-2i/3}\bar{D}_{ij} = 0, \tag{A5}$$

and a corresponding expression exists for the sum over j. The relation (A5) can be expressed through a class of sum rules for the expansion coefficients $B_{ij}^{\mu_1,\ldots,\mu_m}$:

$$\sum_{i=1}^{N} 2^{-2i/3} B_{ij}^{\mu_1,\ldots,\mu_m} = 0 \tag{A6}$$

for all $\mu_1, \ldots, \mu_m$. Equation (5.11) is just the zeroth order case $m = 0$ of (A6).

APPENDIX B: CALCULATION OF EXPONENTS

In this appendix we supply the explicit calculation of the dissipation exponent μ. To this end we have to evaluate the amplitude

$$\sigma_2 = 2\sum_{l=0}^{\infty} \bar{b}_k,$$

where $\bar{b}_k$ is defined by (5.16). Using (5.17) as an ansatz for $\langle x_{i_1} x_{i_2}\rangle^{(0)}$, the left hand side of (5.8) can be written as

$$A\, 2^{(j_1+j_2)/3} \sum_{l=0}^{N} C_{|j_1-j_2|l}\bar{b}_l \quad ,$$

where the matrix C_{nl} is

$$C_{nl} = \begin{cases} (2^{n/3}+2^{-n/3})[2^{-1/3}\gamma_1 - 2^{1/3}\gamma_2], & l \geq n+1 \\ 2^{n/3}[2^{-1/3}\gamma_1(1+\delta_{n0})] \\ +2^{-n/3}[2^{2/3}\gamma_1 - 2^{1/3}\gamma_2(1-\delta_{n0})], & l = n \\ 2 \times 2^{-n/3}[2^{-1/3}\gamma_1 - 2^{1/3}\gamma_2], & l \leq n-1 \end{cases} .$$

For the moment we keep the total number of levels N finite.

In view of (5.19) we need to calculate the quantities

$$c^{(m)} = (1+\delta_{0m})\sum_{l=0}^{N} (\boldsymbol{C}^{-1})_{lm} \tag{B1}$$

in the limit of large N. Then σ_2 is given by

$$\sigma_2 = -2\frac{B}{A}\left\{c^{(0)} - \frac{2c^{(1)}}{2^{1/3}+2^{-1/3}}\right\}. \tag{B2}$$

To find the inverse of $\boldsymbol{C}$, we write $\boldsymbol{C}$ as

$$\boldsymbol{C} = \boldsymbol{E} + \boldsymbol{u}\otimes\boldsymbol{v}$$

with $\boldsymbol{E}$ being a triangular matrix

$$E_{nl} = \begin{cases} 0, & l \geq n+1 \\ d_n, & l = n \\ e_n, & l \leq n-1, \end{cases} \tag{B3}$$

$$\begin{aligned} d_n &= (2^{n/3}2^{1/3}\gamma_2 + 2^{-n/3}2^{-1/3}\gamma_1)(1+\delta_{n0}), \\ e_n &= (2^{n/3} - 2^{-n/3})(2^{1/3}\gamma_2 - 2^{-1/3}\gamma_1), \\ u_n &= 2^{n/3} + 2^{-n/3}, \\ v_n &= 2^{-1/3}\gamma_1 - 2^{1/3}\gamma_2. \end{aligned}$$

According to the Sherman-Morrison formula [20] we have

$$\boldsymbol{C}^{-1} = \boldsymbol{E}^{-1} - \frac{(\boldsymbol{E}^{-1}\boldsymbol{u})\otimes(\boldsymbol{v}\boldsymbol{E}^{-1})}{1+\alpha} \quad , \quad \alpha = \boldsymbol{v}\boldsymbol{E}^{-1}\boldsymbol{u}.$$

This formula greatly simplifies for the sum

$$c^{(m)} = \frac{1+\delta_{0m}}{1+\alpha}\sum_{l=0}^{N} (\boldsymbol{E}^{-1})_{lm} \quad . \tag{B4}$$

Since $\boldsymbol{E}$ is triangular, it is easy to find $\boldsymbol{E}^{-1}$, giving

$$\begin{aligned} c^{(m)} &= \left(\prod_{j=1}^{m} g_j\right) c^{(0)}, \\ c^{(0)} &= \left\{ d_N \prod_{j=1}^{N} g_j + \frac{2^{-1/3}\Delta - (2^{-1/3}\Delta)^{-1}}{2^{-1/3}\Delta + (2^{-1/3}\Delta)^{-1}} \right. \\ &\quad \left. \times \left[1 + \sum_{j=1}^{N} (2^{j/3} + 2^{-j/3}) \prod_{k=1}^{j} g_k\right] \right\}^{-1}, \\ g_k &= \frac{2^{k/3}/\Delta + (2^{k/3}/\Delta)^{-1}}{2^{(k-1)/3}\Delta + (2^{(k-1)/3}\Delta)^{-1}}, \quad \Delta = \sqrt{\frac{\gamma_1}{\gamma_2}}. \end{aligned} \tag{B5}$$

In special cases (B5) can be evaluated explicitly, giving, for example, $c^{(0)} = 1/2$ for $\Delta = 2^{2/3}$, $N \to \infty$. Setting $\Gamma \equiv B/A$ and using $c^{(1)} = g_1 c^{(0)}$, one finds (5.20). It will be noticed that for $N \to \infty$, $c(0)$ depends on the ratio Δ alone. For numerical values of $c^{(0)}$ at general Δ the sum in (B5) must be evaluated numerically.

Finally, we want to find the behavior of $\bar{b}_k$ for large k. Since $\bar{b}_k$ essentially behaves like $(\boldsymbol{C}^{-1})_{km}$ at constant m, in view of (B1) we just have to take differences between two $c^{(m)}$'s at different N. Analysis of the N dependence of $c^{(m)}$ [cf. (B5)] then reveals that $\bar{b}_k$ is proportional to $(2^{1/3}/\Delta)^{2k}$.

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