Air Entrainment through Free-Surface Cusps

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In many industrial processes, such as pouring a liquid or coating a rotating cylinder, air bubbles are entrapped inside the liquid. We propose that this is due to air being drawn into the narrow channel of a cusp singularity that generically forms on free surfaces. Since the width of the cusp is exponentially small in the driving strength, even the minute viscosity of air is enough to destroy the stationary solution, and a sheet emanates from the cusp's tip, through which air is entrained. Our analytical theory is confirmed by quantitative comparison with numerical simulations of the flow equations, and is found to be in qualitative agreement with experimental observation.

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Air bubbles are a ubiquitous presence in fluid flow, appearing when pouring a liquid into a beaker, when beating an egg, or in river streams. This aeration is often desirable, for example, to promote chemical reactions [1], yet in many industrial processes entrainment of air bubbles is detrimental to the quality of the product, rendering the flow unsteady. For example, it is the single most important factor limiting the speed at which paints or coatings can be applied to a solid surface [2,3]. But in spite of its importance, no general understanding of air entrainment exists, except for the rather special circumstance that the free surface conspires to enclose an air bubble from all sides, as was recently found for a disturbed water jet [4].

In recent years great efforts have been devoted to an understanding of various singularities that form on free surfaces, for example, in drop formation [5,6], drop coalescence [7], erupting jets [8], and crumpling of paper [9]. In addition to these three-dimensional singularities, free surfaces turn out to be extremely susceptible to the formation of cusps [3,10–12], which is the generic structure that forms in two-dimensional viscous flow [13]. Examples are drop impact on a surface [14], jets impinging on a pool of liquid [1], and the coating of a prewetted solid cylinder [3].

However, cusp solutions should be expected to depend on the presence of an outer fluid or gas like air in a very singular fashion, since the air is drawn into a very narrow passage by the external flow. Thus even an air viscosity that is exponentially small in the driving strength is enough to destroy the stationary solution, similar to the singular nature of pattern selection in growth phenomena [15,16]. Indeed, the air sheet that forms out of the cusp seems to represent a broader class of phenomena, characterized by a pattern being born out of a singularity. Other examples are "electric jets," which shoot out of fluid cones (so-called Taylor cones) formed by strong electric fields [17,18], jets emanating from tipped ends of bubbles in shear flows [19], "tip streaming," or spouts formed by planar interfaces "selective withdrawal" [20].

In this Letter, we show that the presence of two-dimensional cusp singularities on the free surface results in a generic mechanism for air entrainment. In addition, the present theory hopefully is a first step towards understanding other, three-dimensional problems, and will help explain why the appearance of secondary structures out of singularities occurs in so many systems.

For purposes of illustration, consider now the particular example of a two-dimensional cusp that forms when a thin stream of a viscous silicone oil is poured into a container of the same fluid. Since the falling liquid drags other fluid away from the surface, a dip is produced around the fluid stream. Increasing the flow rate above a critical value, this dip is no longer smooth, but a singular point on the surface is approached with two vertical tangents. A cross section of this cusped profile is shown as a black silhouette in Fig. 1a, the outer wall of the free surface ending in a vertical tangent at the cusp point. For clarity, the lighting is chosen such that the free surface appears opaque, so the falling jet is indicated only symbolically to guide the eye, but not visible directly.

Increasing the flow rate still further, there is a second critical value where the stationary profile of Fig. 1a ceases to exist and a sheet of air shoots out from the tip of the cusp. The bottom picture shows this dynamical structure $1/60$th of a second after the stationary solution has vanished. A thin air sheet now forms the wall of a transparent fluid cylinder. The details of this dynamical structure, such as the bell-shaped opening at its lower rim, are not the subject of this paper, but only the loss of the static shape that leads to it. The cylinder eventually grows to about 10 times the length shown, and is unstable to the formation of bubbles at its lower end, so the liquid pool quickly becomes contaminated by bubbles of a broad variety of sizes.

Since the air sheet near the cusp is of micron thickness, the curvature with which it is wrapped around the impinging jet is of no consequence and the cusp can be viewed as a two-dimensional object. In this spirit, Joseph *et al.* [11] performed experiments with a two-dimensional flow produced by two counterrotating cylinders submerged below the surface of a very viscous liquid. If the cylinders are placed sufficiently close to each other, a cusp forms in a symmetrical position between the cylinders [12]. Letting

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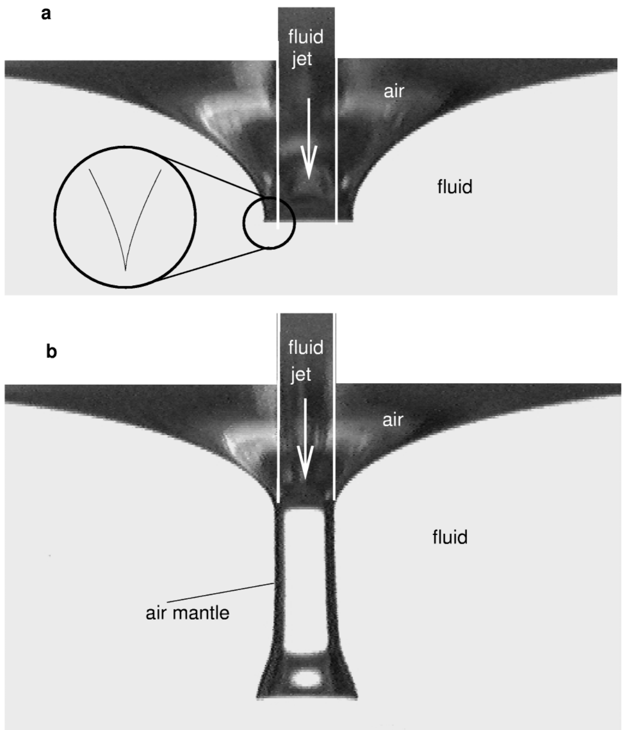

FIG. 1. (a) Cross section of the stationary air-fluid interface produced by a thin (1 mm) stream of viscous oil poured into a deep pool of the same fluid. The position of the cusp is marked by a circle. (b) A hollow cylinder of air forms after a sheet of air shot out from the cusp at a slightly higher flow rate (photograph by Itai Cohen).

the distance between the rollers go to zero, Jeong and Moffatt found a family of exact solutions to this problem, obeying the local scaling form

$$h(y) = \kappa^{-3/4} H(y\kappa^{1/2}), \tag{1}$$

where κ is the curvature at the tip (see Fig. 2), and

$$H(\xi) = \sqrt{a\xi}\,(\xi + \sqrt{2/a}) \tag{2}$$

is universal up to the constant a.

Here and in all of the following, lengths are nondimensionalized using some external length scale of the problem, such as the distance between the rotating cylinders in the Joseph experiment, or the radius of the impinging jet in the pouring experiment of Fig. 1. A change in the definition of this external length scale will be reflected in a different value for the constant a in (2). The self-similar structure of the solution (1) is typical for flows near singularities [6,8,21] which involve very small scales, far removed from any external scale. Physically (1) means that the shape of the interface is *independent* of scale, up to a rescaling of the axes with a typical local scale of the solution, which is the radius of curvature $R = \kappa^{-1}$ at the tip. The spatial dependence enters only in the form of the similarity variable $\xi = y\kappa^{1/2}$. Away from the tip the functional form of the interface is $h(y) = \sqrt{a}\, y^{3/2}$, so the walls are parallel to the y axis asymptotically.

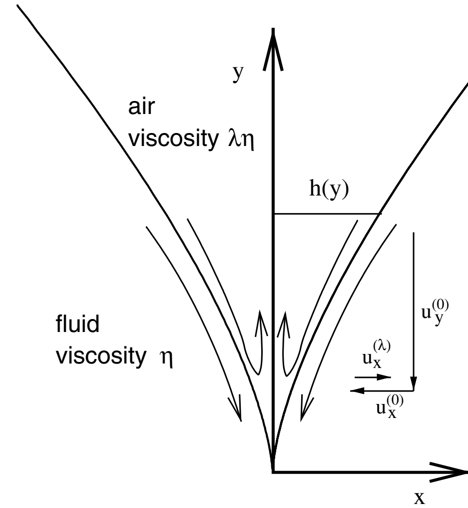

FIG. 2. The local shape of the cusp, cut perpendicular to the sheet of air. The variable y is the distance from the tip.

The other crucial property of singularities is that their shape is universal, i.e., independent of the particular type of flow that generates the cusp. Thus our theory, based on the stability of such a singular structure, will be equally general. Indeed, Antonovskii [22] discovered yet another class of exact solutions, but where the cusp is formed on the surface of a circular bubble. A local analysis reveals that the scaling function H is identical to the one in [12] except for a different value of the numerical constant a, confirming the expectation that the flow on small scales is universal, independent of the particular features of the driving flow.

A crucial and fascinating property of all cusp solutions is that the tip curvature κ grows *exponentially* with the capillary number $\mathrm{Ca} = \eta U/\gamma$, where η is the fluid viscosity, γ the surface tension, and U is a typical velocity scale of the external flow. The physical origin of this exponential sensitivity to driving lies in the interplay between the external flow and surface tension. Without any medium inside the cusp, the y component of the velocity field $u_y^{(0)}(y)$, which is parallel to the cusp surface, meets no resistance and thus corresponds to a *downward* motion with velocity U. However, the extremely high curvature at the tip produces a local *upward* motion, which cancels U and allows the tip of the cusp to be a stagnation point. The strength of the point forcing at the tip is the pull 2γ exerted by the almost vertical walls of the cusp. Since the velocity field generated by a point source in two-dimensional Stokes flow is logarithmic, which has to be smoothed out over the scale $R = \kappa^{-1}$ of the tip, the upward velocity is $u_{\mathrm{up}} \approx (2\gamma/\eta)\ln R$. Thus the stationarity condition at the tip leads to $\kappa = c_1 \exp[c_2 Ca]$, where c_1 and c_2 are constants that depend on the flow characteristics. Note that the contribution from the surface tension is important only in a small region around the tip, while the mean flow along the walls of the cusp is constant up to logarithmic corrections. It is this flow that draws the air into the cusp.

Thus as the strength of the driving flow increases relative to the surface tension, the size of the tip may easily reach microscopic dimensions [12] if the effect of the air is not taken into account. Without it, stable solutions are predicted to exist for all capillary numbers [23], in disagreement with experiment. Moffatt suggested [24] that this is because all previous analyses neglect the viscosity $\lambda\eta$ of the air being drawn into the cusp by the flow $u_y^{(0)}(y)$ parallel to the cusp surface. The viscosity contrast λ thus measures the viscosity of the air relative to that of the fluid. The air entering a narrow space and having to escape again generates a so-called lubrication pressure $p_{\rm lub}(y)$ inside the cusp, whose derivative with respect to the distance y from the cusp is

$$p'_{\rm lub} = 3\lambda\eta u_y^{(0)}(y)/h^2(y) \tag{3}$$

by Reynolds' theory [25]. Since the cusp narrows as $h(y) \sim y^{3/2}$, the lubrication pressure pushes the walls apart according to $p_{\rm lub} \sim y^{-2}$, just as it would keep separated to narrowly spaced mechanical parts.

Figure 3 proves by direct numerical simulation that this is enough to destroy the stationary solution found for $\lambda = 0$. We use a boundary integral code [26,27], optimized to resolve the cusp between two merging cylinders [28], neglecting the fluid inertia. Starting from Antonovskii's solution with $\kappa_0 = 10^4$, λ is increased in steps of 2.5×10^{-5}, pushing the interface forward, but only every fourth profile is shown. At $\lambda = 5.5 \times 10^{-4}$, no more stationary solution is found, but instead air enters the fluid forming a narrow

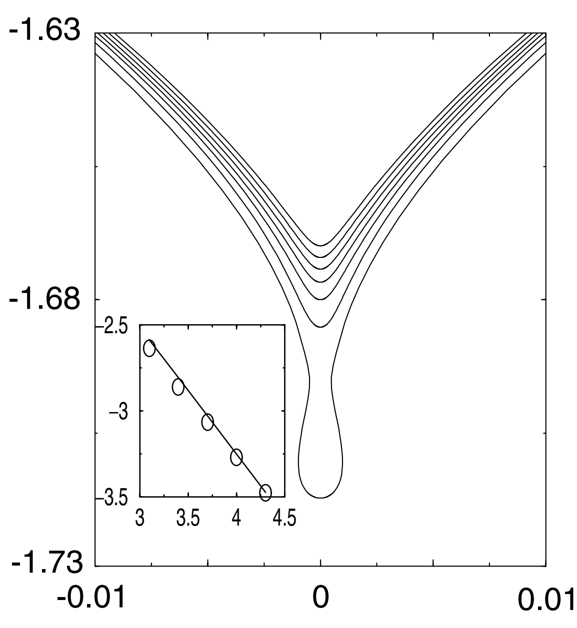

FIG. 3. A boundary integral simulation of a bubble in the flow proposed by Antonovskii for $\epsilon = 5$ and Ca $= 0.0992$ [22]. The undisturbed bubble radius is used to nondimensionalize all lengths in the problem. As λ is increased, the tip is pushed forward, but becomes narrower at a given y. The lowest profile is nonstationary. The inset shows the critical value of λ beyond which there is no more stationary solution for a given curvature.

sheet, as seen in Fig. 1 and observed in earlier experiments [29]. An important consequence is that in a physically correct description which incorporates the effect of the air (or some other gas atmosphere), molecular dimensions are never reached, so continuum theory remains valid throughout [30].

To describe the influence of the air analytically, note that the extra transverse velocity field $u_x^{(\lambda)}(y)$ generated by the air pressure can simply be added to $u_x^{(0)}(y)$ as given in [12], since Stokes' equation is linear. Geometrically, the cusp looks like a two-dimensional *crack* entering the fluid, a problem well studied in linear elasticity [31]. Borrowing Muskhelishvili's result, we can now write $u_x^{(\lambda)}(y)$ as

$$\begin{aligned} u_x^{(\lambda)}(y) &= \int_0^\infty p(y')m(y'/y)\,dy', \\ m(x) &= (1/2\pi)\ln[(1+\sqrt{x})/(1-\sqrt{x})]. \end{aligned} \tag{4}$$

But our free-surface problem is of course nonlinear, since the free surface has to follow the streamlines of the flow, which are modified by $u_x^{(\lambda)}$. Namely, the inverse slope of the interface is

$$h' = (u_x^{(0)} + u_x^{(\lambda)})/u_y^{(0)}, \tag{5}$$

where $u_x^{(0)}$ and $u_y^{(0)}$ are known [12] and $u_x^{(\lambda)}$ is calculated from h as outlined above. Our approach of combining lubrication theory with results for thin cracks is similar to that employed in [32,33], for the propagation of magma-filled fissures in the earth's mantle. Note that while $u_x^{(0)}$ has to point inward towards the cusp, $u_x^{(\lambda)}$ results from the lubrication pressure and points away from the cusp (cf. Fig. 2). Thus, at a given distance y from the tip, h' becomes smaller and the channel *narrows*. Owing to (3) the lubrication pressure is increased, further increasing $u_x^{(\lambda)}$, so this nonlinear feedback eventually destroys the cusp solution, as seen in Fig. 3.

It is extremely useful to recast Eqs. (3)–(5) in the scaling variable $\xi = y\kappa^{1/2}$, cf. (1). First, from (5) and since $u_y^{(0)}$ is a constant up to logarithmic corrections, $u_x^{(0)}$ must scale as $\kappa^{-3/4}\kappa^{1/2} = \kappa^{-1/4}$. From (3) $p_{\rm lub}$ is estimated as $p_{\rm lub} \sim \lambda\kappa$, and thus $u_x^{(\lambda)} \sim \lambda\kappa^{1/2}$ from integrating once. The two opposing velocities become comparable at some critical value of the parameter $r = \lambda\kappa^{3/4}$. Thus (3)–(5) can be recast in similarity variables, leading to an integral equation for the correction $H_c(\xi)$ to the unperturbed surface profile $H(\xi)$:

$$H_c(\xi) = -\frac{3r\xi}{u_y^{(0)}(\xi/\kappa^{1/2})}\int_0^\infty \frac{u_y^{(0)}(\zeta/\kappa^{1/2})M(\zeta/\xi)}{[H(\zeta)+H_c(\zeta)]^2}\,d\zeta\,, \tag{6}$$

where $M'(\zeta) = m(\zeta)$. It is a simple matter to solve (6) numerically, giving increasingly large corrections $H_c(\xi)$ to the profile as r is raised. Since H_c is negative, the denominator in the integrand of (6) *decreases,* leading to a further increase in the absolute magnitude of the correction, in accordance with the qualitative argument given above. Owing to this nonlinear feedback, a solution ceases to exist above a critical value of r, which has a weak dependence on κ due to the logarithmic dependence of $u_y^{(0)}$ on its argument. Hence for the flow parameters of Fig. 3, we predict that the stationary cusp is lost when the curvature reaches a critical value of $\kappa_{\mathrm{cr}} \approx 0.45\lambda^{-4/3}$. This approximation is hardly distinguishable from the result of the full solution of (6), which in the inset of Fig. 3 is seen to be in good agreement with numerical simulations for various values of λ. Because of the relationship between curvature and capillary number, this translates into the anticipated critical value $\mathrm{Ca}_{\mathrm{cr}}$ above which stationary solutions are no longer found. At low viscosities, the capillary number never even reaches the critical value for the *formation* of a cusp, so an unperturbed water jet does not entrain air [4].

In conclusion, we have incorporated the effect of an outer fluid like air into the theoretical description of a cusp. This allows for the first quantitative description of air entrainment through surface singularities. A description of the resulting sheet of air and its stability remains to be done. Other instabilities may occur at a three-phase boundary, for example, when the solid to be coated is dry, a problem studied in [2,34]. In the presence of surfactant, still another mechanism for the loss of stationary solutions has been suggested by Siegel [23,35].

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