

Fourier Analysis in Phase Space

Lecturer: Jens Marklof, University of Bristol

There is a beautiful mathematical world hidden underneath the humble Fourier transform, connecting the representation theory of the symplectic group with important applications in the analysis of PDE, number theory, quantum mechanics and signal processing. This introductory course is aimed at postgraduate students from a broad mathematical background, requiring few prerequisites beyond the standard foundational undergraduate lectures. Topics to be covered include:

1. Fourier integrals and pseudodifferential operators
2. Oscillatory integrals and stationary phase
3. Symplectic geometry and Hamiltonian dynamics
4. Heisenberg group and Schrödinger representation; Stone-von Neumann theorem
5. The Shale-Weil representation of the metaplectic group
6. Bargmann representation and Fock space
7. Poisson summation, theta functions and Gauss sums
8. Semiclassical measures and Egorov's theorem; quantum ergodicity

Assessment will be in the form of tailored individual or group projects that will explore variations and extensions of the topics covered in the lectures.

Recommended literature:

- Maciej Zworski, Semiclassical Analysis, Graduate Studies in Mathematics 138, American Mathematical Society, 2012
- Gerald B. Folland, Harmonic Analysis in Phase Space, Annals of Mathematics Studies 122, Princeton University Press, 1989
- Gérard Lion, Michèle Vergne, The Weil representation, Maslov index and theta series, Progress in Mathematics, Birkhäuser 1980

Projects:

- Pick something that interests you, fill in details and expand
- We will work on $\mathbb{R}^n$; you can adapt to other spaces, e.g.

$$(\mathbb{Z}/N\mathbb{Z})^n, \quad \mathbb{F}_q^n$$

challenge:
make uniform
in N resp. q

- No more than 10-15 pp latex

Lecture 1: Fourier integrals and pseudo-differential operators

§ 1 Fourier transform

For $f \in L^2(\mathbb{R}^n)$ define the FT $\hat{f}$ of f

$$\hat{f}(\gamma) = [\mathcal{F} f](\gamma) = \int_{\mathbb{R}^n} f(x) e(-x \cdot \gamma) dx$$

$$\text{with } e(x) := e^{2\pi i x}$$

Define inner product and norm

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^n} \overline{f_1}(x) f_2(x) dx, \quad \|f\|_2 = \sqrt{\langle f, f \rangle}$$

Thm 1.1

(i) $\mathcal{F}: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is unitary
i.e.

$$\langle f_1, f_2 \rangle = \langle \hat{f}_1, \hat{f}_2 \rangle$$

(ii) $\mathcal{F}^{-1} = \mathcal{P} \mathcal{F} = \mathcal{F} \mathcal{P}$, $\mathcal{P} f(x) = f(-x)$

$$\text{explicitly } \mathcal{F}^{-1} f(x) = \int_{\mathbb{R}^n} f(\gamma) e(x \cdot \gamma) d\gamma$$

Proof: D17. 

In particular (i) $\Rightarrow \boxed{\| \mathcal{F} f \|_2 = \| f \|_2}$

Also note

$$\boxed{\| \mathcal{F} f \|_\infty \leq \| f \|_1}$$

$$\sup_{y \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x) e(-x \cdot y) dx \right|$$

Now restrict to Schwartz class $\mathcal{S}(\mathbb{R}^n)$

$$= \{ f \in C^\infty(\mathbb{R}^n) : |f|_{\alpha, \beta} < \infty \forall \alpha, \beta \in \mathbb{Z}_{\geq 0}^n \}$$

with seminorm $(\Delta$ -wieq & absolute homog. but $|f|=0 \not\Rightarrow f=0$)

$$|f|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|$$

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad D^\beta = D_1^{\beta_1} \dots D_n^{\beta_n}$$

α, β "multi-indices"
($|\alpha| = \alpha_1 + \dots + \alpha_n$)

$$D_i = \frac{1}{2\pi i} \frac{\partial}{\partial x_i}$$

Also define multiplication operator M_j by

$$M_j f(x) = x_j f(x)$$

$$M^\alpha f(x) = x^\alpha f(x)$$

With this

$$|f|_{\alpha, \beta} = \| M^\alpha D^\beta f \|_\infty$$

Note:

$$\boxed{\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \quad 1 \leq p \leq \infty}$$

(1.1)

Thm 1.2

- (a) (b) (c)
- (i) $f \in \mathcal{S}(\mathbb{R}^n) \Rightarrow M_j f, D_j f, \mathcal{F}f \in \mathcal{S}(\mathbb{R}^n)$
- (ii) $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n) \Rightarrow f_1 \cdot f_2 \in \mathcal{S}(\mathbb{R}^n)$

Proof:

(i-a) $D_j f \in \mathcal{S}(\mathbb{R}^n)$ follows from def.

(i-b) $M_j f \in \mathcal{S}(\mathbb{R}^n)$ follows from the commutation relation (Heisenberg)

$$[D_j, M_k] := D_j M_k - M_k D_j = \frac{\delta_{jk}}{2\pi i} \quad (1.2)$$

Repeated application allows us to write

$$\begin{aligned} D_j^{\beta_j} M_j &= D_j^{\beta_j-1} \left(M_j D_j + \frac{1}{2\pi i} \right) \\ &= M_j D_j^{\beta_j} + \text{poly}(D_j) \end{aligned}$$

Claim now follows from def.

(use Δ -inequality)

(i-c) To prove $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^n)$ note that

$$M_j^{\alpha_j} D_j^{\beta_j} \mathcal{F} f(\gamma) =$$

"

$$= \gamma_j^{\alpha_j} \int_{\mathbb{R}^n} (-x_j)^{\beta_j} f(x) e(-x \cdot \gamma) dx$$

$$= (-1)^{\beta_j} \gamma_j^{\alpha_j} \int_{\mathbb{R}^n} [M_j^{\beta_j} f](x) e(-x \cdot \gamma) dx$$

repeated
integration
by parts

$$\frac{1}{\gamma_j^{\alpha_j}} \int_{\mathbb{R}^n} [D_j^{\alpha_j} M_j^{\beta_j} f](x) e(-x \cdot \gamma) dx$$

$$= (-1)^{\beta_j} [\mathcal{F} D_j^{\alpha_j} M_j^{\beta_j}] f(\gamma)$$

Thus

$$M^{\alpha} D^{\beta} \mathcal{F} = (-1)^{|\beta|} \mathcal{F} D^{\alpha} M^{\beta} \quad (1.3)$$

This yields

$$\| M^{\alpha} D^{\beta} \mathcal{F} f \|_{\infty} = \| \mathcal{F} D^{\alpha} M^{\beta} f \|_{\infty}$$

$$\leq \| D^{\alpha} M^{\beta} f \|_1 \stackrel{(1.1)}{<} \infty$$

$\in \mathcal{S}(\mathbb{R}^n)$ by (a), (b)

(ii) Use Leibniz rule.

Convolution

$$f_1 * f_2(x) = \int_{\mathbb{R}^n} f(x-x_1) f(x_1) dx_1$$

$$\mathcal{F}(f_1 \cdot f_2) = \mathcal{F}f_1 * \mathcal{F}f_2 \quad (1.4)$$

Proof:

$$\begin{aligned} f_1 \cdot f_2(x) &= \int \hat{f}_1(\gamma_1) \hat{f}_2(\gamma_2) e(x(\gamma_1 + \gamma_2)) d\gamma_1 d\gamma_2 \\ &= \int \hat{f}_1(\gamma_1 - \gamma_2) \hat{f}_2(\gamma_2) e(x\gamma_1) d\gamma_1 d\gamma_2 \\ &= \mathcal{F}^{-1}(\mathcal{F}f_1 * \mathcal{F}f_2)(x) \quad \square \end{aligned}$$

Example (Gaussian)

$Q \in M_{n \times n}(\mathbb{R})$ symmetric, pos. def.

$$f(x) = e^{-\pi x \cdot Q x}$$

$$\mathcal{F}f(\gamma) = \frac{1}{\sqrt{\det Q}} e^{-\pi \gamma \cdot Q^{-1} \gamma}$$

or (more symmetrically)

$$f(x) = (\det Q)^{1/4} e^{-\pi x \cdot Q x}$$

$$\mathcal{F}f(\gamma) = (\det Q^{-1})^{1/4} e^{-\pi \gamma \cdot Q^{-1} \gamma}$$

Proof:

$$\hat{F}(\gamma) = \int_{\mathbb{R}^n} e^{-\pi x \cdot Q x - 2\pi i x \cdot \gamma} dx$$

"Complete the square"

$$= \int_{\mathbb{R}^n} e^{-\pi (x + iQ^{-1}\gamma) \cdot Q (x + iQ^{-1}\gamma) - \pi \gamma \cdot Q^{-1}\gamma} dx$$

$$= e^{-\pi \gamma \cdot Q^{-1}\gamma} \int_{\mathbb{R}^n} e^{-\pi w \cdot Q w} dw$$

diagonalise Q

$$= \int_{\mathbb{R}^n} e^{-\pi (q_1 w_1^2 + \dots + q_n w_n^2)} dw$$

$q_i > 0$ ev's of Q

$$= \prod_i \frac{1}{\sqrt{q_i}} \left(\underbrace{\int_{\mathbb{R}} e^{-\pi t^2} dt}_{=1} \right)^n$$

$$\int_{\mathbb{R}^2} e^{-\pi(x_1^2 + x_2^2)} dx_1 dx_2$$

$$= 2\pi \int_0^\infty e^{-\pi r^2} r dr = \pi \int_0^\infty e^{-\pi r} dr = 1$$



Tempered distributions

form a space $\mathcal{S}'(\mathbb{R}^n)$ dual to $\mathcal{S}(\mathbb{R}^n)$

- $\mathcal{S}'(\mathbb{R}^n) := \{ \text{linear maps } \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{C} :$

$$\begin{array}{ccc} f_j \rightarrow f & & u(f_j) \rightarrow u(f) \\ u \in \mathcal{S}'(\mathbb{R}^n) & \Rightarrow & u \in \mathbb{C} \end{array}$$

- Define topology on $\mathcal{S}'(\mathbb{R}^n)$ by

$$u_j \rightarrow u \quad \text{if} \quad u_j(f) \rightarrow u(f) \quad \forall f \in \mathcal{S}(\mathbb{R}^n)$$

- Define $M^\gamma u, D^\gamma u, \mathcal{F}u \in \mathcal{S}'(\mathbb{R}^n)$

by

$$M^\gamma u(f) = u(M^\gamma f) \quad \forall f \in \mathcal{S}(\mathbb{R}^n)$$

$$D^\gamma u(f) = (-1)^{|\gamma|} u(D^\gamma f)$$

$$\mathcal{F}u(f) = u(\mathcal{F}f)$$

- It is helpful to identify u with a "function" via

$$u(f) = \int_{\mathbb{R}^n} f(x) g(x) dx$$

$$\begin{array}{ccc} M^\gamma u & \longleftrightarrow & M^\gamma g \\ D^\gamma u & \longleftrightarrow & D^\gamma g \\ \mathcal{F}u & \longleftrightarrow & \mathcal{F}g \end{array}$$

Example 1 Dirac measure $\mu = \delta_x$, $\delta_x(f) = f(x)$

$$\mathcal{F} \delta_x(f) = \delta_x \hat{f} = \hat{f}(x) = \int f(y) e(-xy) dy$$

Example 2 (imaginary Gaussian - exercise)

$Q \in M_{n \times n}(\mathbb{R})$ symmetric, $\det Q \neq 0$

$$\mu(f) = \int f(x) g(x) dx, g(x) = e\left(\frac{1}{2} x \cdot Q x\right)$$

$\operatorname{sgn} Q$

$= \#(\text{evs})$

$- \#(-\text{evs})$

$$\mathcal{F} g(y) = \frac{e^{i\frac{\pi}{2} \operatorname{sgn} Q}}{|\det Q|^{\frac{1}{2}}} e\left(-\frac{1}{2} y \cdot Q^{-1} y\right)$$

§ 2 Pseudodifferential operators

So far dealt with differential operators

$$D^q = \frac{1}{(2\pi i)^{|q|}} \frac{\partial}{\partial x^q}$$

Recall (1.3) which says in particular

$$M^q \mathcal{F} = \mathcal{F} D^q \quad D^\beta \mathcal{F} = (-1)^{|\beta|} \mathcal{F} M^\beta \quad (2.1)$$

This means that the Fourier transform intertwines multiplication by x and differentiation.

What about multiplication by $e(p \cdot x)$?

Define

$$\begin{aligned} S^q f(x) &= f(x+q) \\ T^p f(x) &= e(p \cdot x) f(x) \end{aligned} \quad (2.2)$$

Then

$$T^p \mathcal{F} = \mathcal{F} S^p \quad S^p \mathcal{F} = \mathcal{F} T^{-p} \quad (2.3)$$

Proof:

$$\begin{aligned} T^p \mathcal{F}^{-1} f(y) &= e(p \cdot y) \int_{\mathbb{R}^n} f(x) e(-x \cdot y) dx \\ &= \int f(x) e(-(x-p) \cdot y) dx \\ &= \int f(x+p) e(-x \cdot y) dx \\ &= \int S^{-p} f(x) e(-x \cdot y) dx \\ &= \mathcal{F}^{-1} S^{-p} f(y) \end{aligned}$$



Therefore (formally)

$$\begin{aligned} T^p &= e(p \cdot M) & M &= \begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix} \\ S^q &= \mathcal{F}^{-1} T^q \mathcal{F} \\ &= e(q \cdot D) & D &= \begin{pmatrix} D_1 \\ \vdots \\ D_n \end{pmatrix} \end{aligned}$$

What about the analogue of Heisenberg's commutation relations?

$$T^P S^Q f(x) = e(p \cdot x) f(x+q)$$

$$\begin{aligned} S^Q T^P f(x) &= T^P f(x+q) \\ &= e(p \cdot x + p \cdot q) f(x+q) \end{aligned}$$

and so

$$S^Q T^P = e(p \cdot q) T^P S^Q$$

or, more symmetrically

$$e\left(\frac{p \cdot q}{2}\right) T^P S^Q = e\left(-\frac{q \cdot p}{2}\right) S^Q T^P \quad (2.4)$$

So far we considered multiplication by x and by $e(p \cdot x)$, and the corresponding differentiation and translation operators.

Aim: To each $a \in \mathcal{S}(\mathbb{R}^{2n})$ ("symbol") associate operator $Op(a)$ ("quantisation") such that (formally - but see discussion of tempered distributions)

$$a(x, y) = e(p \cdot x) \mapsto Op(a) = T^P$$

$$a(x, y) = e(q \cdot y) \mapsto Op(a) = S^Q$$

Fourier expand:

$$a(x, y) = \int_{\mathbb{R}^{2n}} \hat{a}(q, p) e(p x + q y) dp dq$$

↓

$$Op(a) = \int_{\mathbb{R}^{2n}} \hat{a}(q, p) e\left(\frac{qp}{2}\right) T^p S^q dp dq$$

Weyl quantisation

— could use other conventions —

Consider action on $f \in \mathcal{S}(\mathbb{R}^n)$

$$Op(a) f(x)$$

$$= \int_{\mathbb{R}^{2n}} \hat{a}(q, p) e\left(\frac{qp}{2} + p x\right) f(x + q) dp dq$$

$$\begin{aligned} y &= x + q \\ q &= y - x \end{aligned}$$

$$= \int_{\mathbb{R}^{2n}} \hat{a}(y - x, p) e\left(\frac{x + y}{2} \cdot p\right) f(y) dp dy$$

$$\approx \iint_{\mathbb{R}^n \mathbb{R}^n} a(x', \frac{x + y}{2}) e((x - y) \cdot x') f(y) dy dx'$$

This is the usual formula for the Weyl quantisation of the symbol a .

Thm 2.1 Let $a \in \mathcal{S}(\mathbb{R}^{2n})$. Then $Op(a)$ can be extended to a continuous map

$$Op(a) : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$$

with

$$Op(a)^+ = Op(\bar{a}) \quad (2.5)$$

So real symbols correspond to formally self-adjoint ops

Proof:

$$Op(a) f(x) = \int_{\mathbb{R}^n} K_a(x, y) f(y) dy$$

with kernel

$$\begin{aligned} K_a(x, y) &= \int_{\mathbb{R}^n} a\left(\frac{x+y}{2}, y'\right) e((x-y) y') dy' \\ &= \left[\mathcal{F}^{-1} a\left(\frac{x+y}{2}, \cdot\right) \right] (x-y) \end{aligned}$$

$$\Rightarrow K_a(\cdot, \cdot) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \quad (2.6)$$

Define $Op(a)u$ with $u \in \mathcal{S}'(\mathbb{R}^n)$ by

$$Op(a)u(f) = \int_{\mathbb{R}^n} u(K_a(x, \cdot)) f(x) dx$$

$$\text{Set } h(x) = u(K_a(x, \cdot)), \quad (2.7)$$

Then $h \in \mathcal{S}(\mathbb{R}^n)$, by (2.6) and the linearity of u . Hence $\mathcal{O}_p(a)$ maps $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

To establish the continuity of this mapping, assume $u_j \rightarrow u$, i.e.

$$u_j(f) \rightarrow u(f) \quad \forall f \in \mathcal{S}(\mathbb{R}^n)$$

But then $h_j(x) \rightarrow h(x) \quad \forall x \in \mathbb{R}^n$

where $h_j(x) = h_j(\kappa_a(x, \cdot))$ and

g as above.

For the proof of (2.5) note that the kernel of $\mathcal{O}_p^+(a)$ is

$$\overline{\kappa_a(y, x)} = \kappa_{\bar{a}}(x, y) = \text{kernel of } \mathcal{O}_p(\bar{a})$$



We now turn to $\mathcal{O}_p(a)$ corresponding to tempered distributions.

For $u \in \mathcal{S}'(\mathbb{R}^{2n})$, identify with "fct" a via

Define $u(f) = \int_{\mathbb{R}^{2n}} f(x, y) a(x, y) dx dy$

$$Op(a) f(x) = \int_{\mathbb{R}^n} K_a(x, y) f(y) dy$$

where $K_a(\cdot, \cdot)$ corresponds to

$u_a \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ defined by

$$u_a(F) = \int F(x, y) K_a(x, y) dx dy$$

$\forall F \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$

Hence, for $f \in \mathcal{S}(\mathbb{R}^n)$, $Op(a) f$ defines an element in $\mathcal{S}'(\mathbb{R}^n)$ and we obtain:

Thm 2.2 Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$. Then $Op(a)$ can be extended to a continuous map
 $Op(a): \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$

We conclude this lecture by generalising the commutation relations (2.3).

$$\text{Set } \mathcal{J} = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}, \text{ i.e. } \mathcal{J} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix}$$

Thm 2.3 Let $a \in \mathcal{S}(\mathbb{R}^{2n})$ or $\in \mathcal{S}'(\mathbb{R}^{2n})$.

Then

$$\text{Op}(a) \mathcal{F} = \mathcal{F} \text{Op}(a \circ \mathcal{J}) \quad (2.8)$$

Proof: Recall

$$\text{Op}(a) = \int_{\mathbb{R}^{2n}} \hat{a}(p, q) e\left(\frac{ip}{2}\right) T^p S^q \, dp \, dq$$

$$\text{So } \text{Op}(a) \mathcal{F} =$$

$$= \int_{\mathbb{R}^{2n}} \hat{a}(p, q) e\left(\frac{ip}{2}\right) \underbrace{T^p S^q}_{\substack{(2.3) \\ = e\left(\frac{ip}{2}\right) \mathcal{F} S^p T^{-q}}} \mathcal{F} \, dp \, dq \quad \substack{(2.4) \\ = e\left(-\frac{ip}{2}\right) \mathcal{F} T^{-q} S^p}$$

$$= \mathcal{F} \int_{\mathbb{R}^{2n}} \hat{a}(q, -p) e\left(\frac{ip}{2}\right) T^p S^q \, dp \, dq$$

$$= \mathcal{F}_{\mathcal{O}_p}(a \circ \mathcal{J})$$

since

$$a \circ \mathcal{J}(x, y) = a(y, -x)$$

$$= \int \hat{a}(p, q) e^{(py - qx)} dq dp$$

$$= \int \hat{a}(q, -p) e^{(px + qy)} dq dp$$



Lecture 2: Oscillatory integrals and stationary phase

§ 3 Semiclassical Fourier transform

$$[\mathcal{F}_h f](y) = h^{-n/2} \int_{\mathbb{R}^n} f(x) e_h(-x \cdot y) dx$$

with $e_h(x) = e\left(\frac{x}{h}\right) = e^{\frac{2\pi i}{h} x}$,

Define dilation $A_h f(x) = h^{n/2} f(hx)$

Note $\|A_h f\|_2 = \|f\|_2$, i.e.,

A_h is unitary.

Furthermore

$$\mathcal{F}_h = A_h^{-1} \mathcal{F} = \mathcal{F} A_h \quad (3.1)$$

Proof:

$$\begin{aligned} A_h^{-1} \mathcal{F} f(y) &= h^{-n/2} \mathcal{F} f(h^{-1} y) \\ &= h^{-n/2} \int_{\mathbb{R}^n} f(x) e_h(-x \cdot y) dx \\ &= h^{n/2} \int_{\mathbb{R}^n} \underbrace{f(hx)}_{h^{-n/2} A_h f(x)} e(-xy) dx \end{aligned}$$

3-1



(3.1) $\Rightarrow \mathcal{F}_h$ is unitary, since $\mathcal{F}$ and A_h are.

Note also

$$\boxed{D_j A_h = A_h h D_j} \quad \boxed{h M_j A_h = A_h M_j} \quad (3.2)$$

Proof:

$$D_j A_h f(x) = h^{n/2} D_j f(hx) = h A_h D_j f(x)$$

$$M_j A_h f(x) = h^{-1} h x h^{n/2} f(hx) = h^{-1} A_h M_j f(x)$$

□

which in turn implies (cf. (2.1))

$$\boxed{M_j \mathcal{F}_h = \mathcal{F}_h h D_j} \quad \boxed{h D_j \mathcal{F}_h = -\mathcal{F}_h M_j} \quad (3.3)$$

Proof:

$$M_j \mathcal{F}_h = M_j \mathcal{F} A_h = \mathcal{F} D_j A_h = \mathcal{F} A_h h D_j$$

$$h D_j \mathcal{F}_h = h D_j \mathcal{F} A_h = -h \mathcal{F} M_j A_h = -\mathcal{F}_h M_j$$

□

Heisenberg

Thm 3.1 (Uncertainty principle)

For $f \in \mathcal{S}(\mathbb{R}^n)$ with $\|f\|_2 = 1$

$$\|M_j f\|_2 \|M_j \mathcal{F}_h f\|_2 \geq \frac{h}{4\pi}$$

Proof:

$$\begin{aligned} \|M_j f\|_2 \|M_j \mathcal{F}_h f\|_2 &= \|M_j f\|_2 \|\mathcal{F}_h^{-1} h D_j f\|_2 \\ &\stackrel{\mathcal{F}_h \text{ unitary}}{=} h \|M_j f\|_2 \|D_j f\|_2 \end{aligned}$$

Cauchy-Schwarz

$$\begin{aligned} \|M_j f\|_2 \|D_j f\|_2 &\geq |\langle M_j f, D_j f \rangle| \\ &\geq |\operatorname{Im} \langle M_j f, D_j f \rangle| \end{aligned}$$

Now

$$\langle M_j f, D_j f \rangle = \langle f, M_j D_j f \rangle$$

and also (use int. by parts)

$$\begin{aligned} &= \langle D_j M_j f, f \rangle \quad \int \overline{M_j D_j f} f \\ &\stackrel{(1.2)}{=} \langle (\frac{1}{2\pi i} + M_j D_j) f, f \rangle \\ &= -\frac{1}{2\pi i} + \overline{\langle f, M_j D_j f \rangle} \end{aligned}$$

Take average of these two expressions.

$$\operatorname{Im} \langle M_j f, D_j f \rangle = \frac{1}{4\pi} \quad \square$$

The uncertainty principle says that a fct f and its semiclassical FT $\mathcal{F}_h f$ can't be simultaneously highly localised.

Example:

$$f(x) = h^{-\frac{ns}{2}} \varphi(h^{-s}x), \varphi \in \mathcal{S}(\mathbb{R}^n), \|\varphi\|_2 = 1$$

$$\text{Then } \|f\|_2 = 1$$

and $\forall \varepsilon > 0 \exists \mathcal{H}_\varepsilon \subset \mathbb{R}^n$ compact s.t.

$$\int_{h^s \mathcal{H}_\varepsilon} |f(x)|^2 dx \geq 1 - \varepsilon.$$

"localised in $h^s \mathcal{H}_\varepsilon$ "

On the other hand

$$\begin{aligned} \mathcal{F}_h f(y) &= h^{-\frac{n}{2} - \frac{ns}{2}} \int_{\mathbb{R}^n} \varphi(h^{-s}x) e_n(-xy) dx \\ &= h^{-\frac{n(1-s)}{2}} \hat{\varphi}(h^{1-s}y) \end{aligned}$$

is localised in $h^{1-s} \mathcal{H}_\varepsilon \Rightarrow$ best simultaneous localisation for $s = \frac{1}{2}$

Asymptotics as $h \rightarrow 0$:

Thm 3.2 Let $f \in \mathcal{S}(\mathbb{R}^n)$, $\gamma \in \mathbb{R}^n$, $\gamma^q \neq 0$.

$$|\mathcal{F}_h f(\gamma)| \leq |\gamma^q|^{-1} \|D^q f\|_1 h^{|\alpha| - n/2}$$

That is, $|\mathcal{F}_h f(\gamma)| \leq C_N h^N \quad \forall N \geq 1$.
 $\uparrow$ depends on γ, f .

Proof: $\mathcal{F}_h f(\gamma) = A_h^{-1} \mathcal{F} f(\gamma) = h^{-n/2} \mathcal{F} f(h^{-1}\gamma)$

$$\begin{aligned} |\mathcal{F} f(h^{-1}\gamma)| &= h^{|\alpha|} |\gamma^q|^{-1} \underbrace{|M^q \mathcal{F} f(h^{-1}\gamma)|}_{= |\mathcal{F} D^q f(h^{-1}\gamma)|} \\ &= |\mathcal{F} D^q f(h^{-1}\gamma)| \\ &\leq \|D^q f\|_1 \end{aligned}$$



Special case of oscillatory integral,
here with linear phase.

We consider more general phase functions
in §4.

§ 4. Stationary phase

For $f \in C_c^\infty(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n)$, $h > 0$
define the oscillatory integral

$$I_h = I_h(f, \varphi) = \int_{\mathbb{R}^n} f(x) e_h(\varphi(x)) dx$$

$$\|x\|^2 = \langle x, x \rangle = x_1^2 + \dots + x_n^2$$

Thm 4.1 (Non-stationary phase)

Assume $\inf_{x \in \text{supp } f} \|D\varphi(x)\| > 0$.

Then $\forall N \exists C_N$ s.t.

$$|I_h(f, \varphi)| \leq C_N h^N$$

Proof: Define linear operator L by

$$Lg(x) = \frac{1}{\|D\varphi(x)\|^2} D\varphi(x) \cdot Dg(x)$$

Then for $g_h(x) = e_h(\varphi(x))$

we have $Lg_h = h^{-1}g_h$

Work out the adjoint of L :

$$\begin{aligned}
& \int_{\mathbb{R}^n} f(x) [L g(x)] dx \\
&= \sum_j \int_{\mathbb{R}^n} \underbrace{\frac{f(x)}{\|D\varphi(x)\|^2} D_j \varphi(x)}_{F_j(x)} \cdot D_j g(x) dx \\
&= - \sum_j \int_{\mathbb{R}^n} D_j F_j(x) g(x) dx \\
&= \int_{\mathbb{R}^n} [L^* f](x) g(x) dx
\end{aligned}$$

$$\Rightarrow L^* f(x) = - D \cdot F(x)$$

$$\text{and } L^* f \in \mathcal{S}(\mathbb{R}^n)$$

$$\text{since } \|D\varphi(x)\|^{-2} D_j \varphi(x) \text{ is in } C^\infty(\mathbb{R}^n).$$

Now

$$\begin{aligned}
I_n &= \int_{\mathbb{R}^n} f(x) g_n(x) dx \\
&= h^N \int_{\mathbb{R}^n} f(x) [L^N g_n](x) dx \\
&= h^N \int_{\mathbb{R}^n} [(L^*)^N f](x) g_n(x) dx \\
&\leq h^N \| (L^*)^N f \|_1
\end{aligned}$$

What if $D\varphi(x) = 0$ for some x ?
 "stationary phase"

We start with the special case of the quadratic phase $\varphi(x) = \frac{1}{2} x \cdot Q x$

Note $D\varphi(x) = Qx$

$\wedge \det Q \neq 0$
 $\pm Q = Q$
 real

so $D\varphi(x) = 0 \iff x = 0$

$D = \frac{1}{2\pi i} \partial$

since $\det Q \neq 0$

Theorem 4.2 (Quadratic stationary phase)

$$I_h(f, \varphi) = h^{n/2} \frac{e^{i\frac{\pi}{4} \operatorname{sgn} Q}}{|\det Q|^{1/2}} \quad *$$

$$* \left(\sum_{k=0}^{N-1} \frac{(-h\pi i)^k}{k!} (D \cdot Q^{-1} D)^k f(0) + O_f(h^N) \right)$$

For explicit dependence on f see proof

Proof: Example 2 in § 1 yields

we also
 $\int \bar{f} g \, dx$
 $= \int \widehat{\bar{f}} g \, dy$

$$I_h(\bar{f}, \varphi) = \int_{\mathbb{R}^n} \overline{f(x)} e_h\left(\frac{1}{2} x \cdot Q x\right) dx$$

$$= h^{n/2} \frac{e^{i\frac{\pi}{4} \operatorname{sgn} Q}}{|\det Q|^{1/2}} \int_{\mathbb{R}^n} \widehat{\bar{f}}(y) e\left(-\frac{h}{2} y \cdot Q^{-1} y\right) dy$$

$$=: J_h(f)$$

$$J_h(f) = \sum_{k=0}^{N-1} \frac{(-h\pi i)^k}{k!} \int_{\mathbb{R}^n} \widehat{\bar{f}}(y) (y \cdot Q^{-1} y)^k dy$$

+ $R_{h,N}(f)$
 error term

$$= \int \widehat{\bar{f}}(y) (y \cdot Q^{-1} y)^k dy = \int (M \cdot Q^{-1} M)^k \mathfrak{F} f(y) dy$$

$$\stackrel{M\mathfrak{F}=\mathfrak{F}D}{=} (D \cdot Q^{-1} D)^k f(0)$$

$$= (D \cdot Q^{-1} D)^k \bar{f}(0)$$

Now replace $f \rightarrow \bar{f}$ to get formula
 for $I_h(f, \varphi)$.

Error term:

$$R_{h,N}(f) = \int_0^h \frac{(-\pi i)^N}{(N-1)!} (h-t)^{N-1}$$

$$\int_{\mathbb{R}^n} \overline{\widehat{f}(\gamma)} (\gamma \cdot Q \gamma)^N e(-\frac{t}{2} \gamma \cdot Q' \gamma) d\gamma dt$$

So

$$|R_{h,N}(f)| \leq \frac{(-\pi i)^N}{N-1} h^N \int_0^1 (1-t)^{N-1} dt \cdot$$

$$\int_{\mathbb{R}^n} |\widehat{f}(\gamma)| (\gamma \cdot Q' \gamma)^N d\gamma$$

$$\leq C_N h^N$$

NB:

$$\leq C \max_{|\alpha| \leq 2N+n+1} \|D^\alpha f\|_1$$

since $\|\hat{u}\|_1 \leq C \max_{|\alpha| \leq n+1} \|D^\alpha u\|_1$

Exercise.

Def: $\varphi \in C^\infty(\mathbb{R}^n \rightarrow \mathbb{R})$ has non-degenerate critical point at $x_0 \in \mathbb{R}^n$ if $\partial \varphi(x_0) = 0$ and $\det \partial^2 \varphi(x_0) \neq 0$.

$$\partial_i = \frac{\partial}{\partial x_i} \quad \partial^2_{i,j} = \frac{\partial^2}{\partial x_i \partial x_j}$$

$$\text{sgn } \partial^2 \varphi(x_0) := \# \text{ +ve evs} - \# \text{ -ve evs.}$$

Thm 4.2 (Morse Lemma)

$\varphi \in C^\infty(\mathbb{R}^n \rightarrow \mathbb{R})$ with non-deg. crit. pt. at x_0 . Then $\exists$ neighbourhoods $0 \in U \subset \mathbb{R}^n$, $x_0 \in V \subset \mathbb{R}^n$ and a diffeo

$$\gamma: V \rightarrow U$$

s.t.

$$(\varphi \circ \gamma^{-1})(x) = \varphi(x_0) + \frac{1}{2} x \cdot Q_0 x$$

where

$$Q_0 = \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & -\mathbb{1}_{n-r} \end{pmatrix}$$

$$r = \# \text{ +ve vs of } \partial^2 \varphi(x_0)$$

Proof: See Zworski Theorem 3.15.
4-6

Thm 4.3 (General stationary phase)

Let $f \in C_c^\infty(\mathbb{R}^n)$, x_0 non-deg. crit. pt of φ , $\partial\varphi(x) \neq 0 \quad \forall x \in \text{supp } f \setminus \{x_0\}$.

Then there exist diff operators A_{2k} of order $\leq 2k$ s.t.

$$I_h(f, \varphi) = \sum_{k=0}^{N-1} [A_{2k} f](x_0) h^{k + \frac{n}{2}} e_n(\varphi(x_0)) + O_{f, \varphi}(h^{N + \frac{n}{2}})$$

where

$$A_0 = \frac{e^{\frac{i\pi}{4} \text{sgn } \partial^2 \varphi(x_0)}}{|\det \partial^2 \varphi(x_0)|^{1/2}}$$

Proof: W.l.o.g. $x_0 = 0$, $\varphi(0) = 0$, $\partial\varphi(0) = 0$.

Let $\chi \in C_c^\infty(\mathbb{R}^n)$ s.t. $\text{supp } \chi \subset \mathcal{U}$ and

$\chi(x) = 1$ in a neighbourhood of 0.

$$I_h(f, \varphi) = I_h(\chi \cdot f, \varphi) + I_h((1-\chi) \cdot f, \varphi)$$

apply Thm 4.1
 $= O_h(h^M)$

$$I_h(\underbrace{x \cdot f}_{\tilde{f}}, \varphi) = \int_{\mathbb{R}^n} \tilde{f}(x) e_h(\varphi(x)) dx$$

$x \mapsto \gamma^{-1}x$

$$= \int_{\mathbb{R}^n} \tilde{f}(\gamma^{-1}x) e_h(\varphi(\gamma^{-1}x)) |\det \partial(\gamma^{-1}x)| dx$$

Morse Lemma

$$= \int_{\mathbb{R}} \tilde{f}(x) e_h\left(\frac{i}{2} x \cdot Q_0 x\right) dx$$

$$\tilde{f}(x) = \tilde{f}(\gamma^{-1}x) |\det \partial(\gamma^{-1}x)|$$

Now apply quadratic stationary phase,
Theorem 4.2.



Lecture 3: Symplectic geometry and Hamiltonian dynamics

§ 5. Linear symplectic geometry

The symplectic vector space (V, σ) is a vector space $V \cong \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ with the symplectic form

$$\sigma(z_1, z_2) = z_1 \cdot J z_2 \quad (5.1)$$

$$J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$$

Explicitly $\sigma(z_1, z_2) = x_1 y_2 - y_1 x_2$, $z_j = \begin{pmatrix} x_j \\ y_j \end{pmatrix}$
e.g. $\sigma(x, y) = x \cdot y = -\sigma(y, x)$

Note:

(i) $\sigma(z_1, z_2) = -\sigma(z_2, z_1)$
"antisymmetric"

(ii) $\sigma(z_1, z_2) = 0 \quad \forall z_2 \Rightarrow z_1 = 0$
"non-degenerate"

(iii) ${}^t J = -J = J^{-1}$

For linear subspace $L \subset V$ set

$$L^\perp = \{ z \in V : \mathcal{G}(z, w) = 0 \ \forall w \in L \}$$

"orthogonal complement w.r.t. $\mathcal{G}$ "

Note: (i) $\dim L + \dim L^\perp = 2n$

$$(ii) (L^\perp)^\perp = L$$

$$(iii) (L_1 + L_2)^\perp = L_1^\perp \cap L_2^\perp$$

$$(iv) (L_1 \cap L_2)^\perp = L_1^\perp + L_2^\perp$$

Proof: Exercise.

Def: L is called Lagrangian if $L^\perp = L$.

If L is Lagrangian then

$$(i) \mathcal{G}(z, w) = 0 \ \forall z, w \in L.$$

(L is totally isotropic subspace)

$$(ii) \text{ If } z \in V \text{ s.t. } \mathcal{G}(z, L) = 0,$$

then $z \in L$

(L is maximal totally isotropic)

Example: $L = \mathbb{R}^n \times \{0\} \leftrightarrow \{0\} \times \mathbb{R}^n$.

Thm 5.1 If L Lagrangian, $\exists$ Lagrangian $\tilde{L}$ s.t. $L \oplus \tilde{L} = V$.

Proof: Let $\tilde{L}$ be maximal amongst tot. isotr. subspaces $\tilde{L}_0$ s.t. $\tilde{L}_0 \cap L = \{0\}$.

Then $\tilde{L}^\perp + \underbrace{L^\perp}_{=L} = (\underbrace{\tilde{L} \cap L}_{=\{0\}})^\perp = V$. Now

suppose $\exists z \in \tilde{L}^\perp$ s.t. $z \notin \tilde{L} + L$.

Then $\forall w \in \tilde{L}$ we have $\omega(\mathbb{R}z, w) = 0$

So $\tilde{L} + \mathbb{R}z$ is tot. isotrop. and

$\tilde{L} \subsetneq \tilde{L} + \mathbb{R}z$ is not maximal - contradiction!

So $\tilde{L}^\perp \subset \tilde{L} + L$ and therefore

$V = \tilde{L}^\perp + L \subset \tilde{L} + L \Rightarrow \tilde{L} + L = V$. 

Example: $L = \mathbb{R}^n \times \{0\}$ $\tilde{L} = \{0\} \times \mathbb{R}^n$

But there can be others...

$L, \tilde{L}$ are called transverse Lagrangian planes.

If $\langle b_1, \dots, b_n \rangle$ forms a basis of L , then there is a basis $\langle \tilde{b}_1, \dots, \tilde{b}_n \rangle$ of $\tilde{L}$ s.t.

$$\sigma(b_i, \tilde{b}_j) = \delta_{ij}.$$

To see this note that

$$\begin{aligned} L &\xrightarrow{\sim} \tilde{L}^* \\ z &\mapsto \sigma(z, \cdot) \end{aligned}$$

parametrises the space of lin. functional $\tilde{L}^*$ forms on $\tilde{L}$, with basis $\langle \sigma(b_j, \cdot) \rangle$. $\tilde{L}$ is dual to $\tilde{L}^*$, and $\langle \tilde{b}_j \rangle$ the dual basis.

$\langle b_1, \dots, b_n, \tilde{b}_1, \dots, \tilde{b}_n \rangle$ "symplectic basis" of V .

More generally, a symplectic manifold (M, σ) is $2n$ -dim smooth manifold with a symplectic (differential) form

$$\sigma(dx, dy) = dx \wedge dy$$

i.e. looks locally like $(\mathbb{R}^{2n}, \sigma)$

Symplectic group of (V, σ) :

$$Sp(V, \sigma) = \{ g \in GL(V) : \sigma(gz, gw) = \sigma(z, w) \}$$

In the standard basis $V \simeq \mathbb{R}^{2n}$ this becomes the matrix group

$$\begin{aligned} Sp(n, \mathbb{R}) &= \{ g \in GL(2n, \mathbb{R}) : gz \cdot Jgw = z \cdot Jw \} \\ &= \{ g \in GL(2n, \mathbb{R}) : {}^t g J g = J \} \end{aligned}$$

Thm 5.2

$$(i) \ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n, \mathbb{R}) \text{ iff } \left\{ \begin{array}{l} {}^t c a = {}^t a c \\ {}^t a d - {}^t c b = \mathbb{1}_n \\ {}^t d b = {}^t b d \end{array} \right\}$$

$$(ii) \ g^{-1} = \begin{pmatrix} {}^t d & -{}^t b \\ -{}^t c & {}^t a \end{pmatrix}$$

Proof: (i) $\underbrace{\begin{pmatrix} {}^t a & {}^t c \\ {}^t b & {}^t d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\begin{pmatrix} -{}^t c & {}^t a \\ -{}^t d & {}^t b \end{pmatrix}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$(ii) \ \begin{pmatrix} {}^t d & -{}^t b \\ -{}^t c & {}^t a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} {}^t d a - {}^t b c & {}^t d b - {}^t b d \\ -{}^t c a + {}^t a c & -{}^t c b + {}^t a d \end{pmatrix}$$



Thm 5.3

$Sp(V, \omega)$ acts transitively on pairs $(L, \tilde{L})$ of transverse Lagrangian subspaces.

Proof: Let $e_1, \dots, e_n, \tilde{e}_1 = e_{n+1}, \dots, \tilde{e}_n = e_{2n}$ be the standard (symplectic) basis (with $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ etc. the standard unit vec's) and $b_1, \dots, b_n, \tilde{b}_1, \dots, \tilde{b}_n$ be the symplectic basis for $L \oplus \tilde{L} = V \cong \mathbb{R}^{2n}$.

Then

$$g = \begin{pmatrix} b_1 & \dots & b_n & \tilde{b}_1 & \dots & \tilde{b}_n \end{pmatrix} \in Sp(n, \mathbb{R})$$

since

$$0 = \omega(e_i, e_j) = \omega(g e_i, g e_j) = \omega(b_i, b_j) = 0$$

$$\delta_{ij} = \omega(e_i, \tilde{e}_j) = \omega(g e_i, g \tilde{e}_j) = \omega(b_i, \tilde{b}_j) = \delta_{ij}$$

$$0 = \omega(\tilde{e}_i, \tilde{e}_j) = \omega(g \tilde{e}_i, g \tilde{e}_j) = \omega(\tilde{b}_i, \tilde{b}_j) = 0.$$

$$\Rightarrow g^{-1} L = \mathbb{R}^n \times \{0\} \quad g^{-1} \tilde{L} = \{0\} \times \mathbb{R}^n$$



Examples of symplectic matrices:

$$(i) \quad \begin{pmatrix} \mathbb{1}_n & x \\ 0 & \mathbb{1}_n \end{pmatrix} \quad x \in M_{n \times n}(\mathbb{R}), \quad {}^t x = x$$

$$(ii) \quad \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \quad a \in M_{n \times n}(\mathbb{R}), \quad \det a \neq 0$$

$$(iii) \quad J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$$

Thm 5.4

maximal compact
in $Sp(n, \mathbb{R})$

$$\begin{aligned} \kappa: U(n) &\longrightarrow Sp(n, \mathbb{R}) \cap O(2n) \\ u &\longmapsto \begin{pmatrix} \operatorname{Re} u & -\operatorname{Im} u \\ \operatorname{Im} u & \operatorname{Re} u \end{pmatrix} \end{aligned}$$

is a group isomorphism.

$$\text{Recall: } U(n) = \{ u \in GL(n, \mathbb{C}) : {}^t \bar{u} u = \mathbb{1}_n \}$$

$$O(n) = \{ r \in GL(n, \mathbb{R}) : {}^t r r = \mathbb{1}_n \}$$

Proof: Identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto x + iy$$



Thm 5.5 (Iwasawa decomposition)

$\forall g \in Sp(n, \mathbb{R}) \quad \exists! \quad x, a \in M_{n \times n}(\mathbb{R})$ symm.
with $a > 0$ (= pos. def.) and $u \in U(n)$
such that

$$g = \begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} k(u)$$

Proof: See A. Terras, Harmonic Analysis
on Symmetric Spaces and Applications II,
p. 277 (Cartan decomposition) & p. 287.

or "polar"

$$g = \underset{\substack{\uparrow \\ \text{pos. def.}}}{P} k(u)$$



See also:

G.B. Folland, Harmonic Analysis in Phase
Space, p. 172 for proof of Cartan
decomposition (Prop 4.3) and maximal
compactness of $k(U(n))$ (Prop 4.5).

§6 Symplectomorphisms & Hamiltonian dynamics

$U, V \in \mathbb{R}^{2n}$ open, $\alpha: U \rightarrow V$ smooth

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \mapsto w = \begin{pmatrix} u \\ v \end{pmatrix}$$

Def: α is symplectic (a "symplectomorphism") if $\alpha^* \omega = \omega$ (6.1)

with the push-forward

$$\alpha^* \omega(z_1, z_2) = \omega(\alpha z_1, \alpha z_2)$$

Write (6.1) as

$$dx \wedge dy = du \wedge dv$$

Thm 6.1

$\alpha: U \rightarrow V$ is symplectic iff

$$\partial \alpha = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \in \text{Sp}(n, \mathbb{R})$$

for every $\begin{pmatrix} x \\ y \end{pmatrix} \in U$.

Proof: $du \wedge dv = (A dx + B dy) \wedge (C dx + D dy)$

with $A = \partial_x u$, $B = \partial_y u$, $C = \partial_x v$, $D = \partial_y v$.

Then $du \wedge dv = dx \wedge dy$ iff $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(n, \mathbb{R})$



Generating functions

$$\varphi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \text{ smooth}$$

$$\text{s.t. } \det(\partial_{xu}^2 \varphi(x_0, u_0)) \neq 0.$$

$$\text{Set } y = \partial_x \varphi, \quad v = -\partial_u \varphi,$$

and define

$$\mathfrak{x} : \mathcal{U} \rightarrow \mathcal{V}$$

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow w = \begin{pmatrix} u \\ v \end{pmatrix}$$

in neighborhood $\mathcal{U} \ni (x_0, \partial_x \varphi(x_0, u_0))$.

Implicit function theorem $\Rightarrow \mathfrak{x}$ smooth.

Thm 6.2 $\mathfrak{x}$ is symplectic.

Proof : (a) $du \wedge dv = du \wedge d(-\partial_u \varphi)$
 $= du \wedge (-\partial_u^2 \varphi du - \partial_{xu}^2 \varphi dx)$
 $= du \wedge (-\partial_{xu}^2 \varphi) dx$

(b) $dx \wedge dy = dx \wedge d(\partial_x \varphi)$
 $= dx \wedge (\partial_x^2 \varphi dx + \partial_{ux}^2 \varphi du)$
 $= dx \wedge (\partial_{ux}^2 \varphi) du$
 $= du \wedge dv$



Hamiltonian dynamics

Describes the time evolution of a point in "phase space" $\mathbb{R}^{2n}$

$$\mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}^{2n}$$

$$t \longmapsto (x(t), y(t))$$

Convenient formalism for "classical" Newtonian mechanics.

Hamilton's equations of motion:

$$\left\{ \begin{array}{l} \dot{x} = \frac{\partial}{\partial y} H(x, y) \\ \dot{y} = -\frac{\partial}{\partial x} H(x, y) \\ x(0) = x_0, \quad y(0) = y_0 \end{array} \right. \quad \begin{array}{l} \textcircled{F=ma} \\ (6.2) \end{array}$$

↑
initial data

$$\dot{f} = \frac{d}{dt} f, \quad H \in C^1(\mathbb{R}^{2n} \rightarrow \mathbb{R})$$

"Hamilton function"
(may depend on t)

In symplectic notation

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \quad J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$$

$$\begin{cases} \dot{z} = J \frac{\partial}{\partial z} H(z) \\ z(0) = z_0 \end{cases} \quad (6.3)$$

$$\varphi_t : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n} \quad \text{"Hamiltonian flow"} \\ z_0 \longmapsto z(t)$$

Thm 6.3 $\varphi_t^* \omega = \omega \quad \forall t$

That is Hamiltonian flow preserves ω .

Proof:

$$\frac{d}{dt} (dx \wedge dy)$$

$$= d\dot{x} \wedge dy + dx \wedge d\dot{y}$$

$$= d(\partial_y H) \wedge dy - dx \wedge d(\partial_x H)$$

$$= ((\partial_{xy}^2 H) dx + (\partial_{yy}^2 H) dy) \wedge dy$$

$$- dx \wedge ((\partial_{xx}^2 H) dx + \partial_{yx}^2 H dy)$$

$$= 0$$

QED

Examples

(i) Free motion $H(x, y) = \frac{1}{2} \|y\|^2$

$$\begin{cases} \dot{x} = y \\ \dot{y} = 0 \end{cases} \Rightarrow \begin{aligned} x(t) &= x_0 + t y_0 \\ y(t) &= y_0 \end{aligned}$$

(ii) Isotropic harmonic oscillator

$$H(x, y) = \frac{1}{2} \|y\|^2 + \frac{1}{2} \|x\|^2$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases} \quad \text{i.e. } \dot{z} = J z$$

define exp by matrix Taylor series

$$\Rightarrow \varphi_t(z_0) = e^{Jt} z_0$$

In dim $n=1$ $J \cong i$ and $z(t) = e^{it} z_0$

(iii) General quadratic Hamiltonian functions

$$H(z) = \frac{1}{2} z \cdot Q z \quad Q \in M_{2n \times 2n}(\mathbb{R}) \text{ symm}$$

$$\dot{z} = J Q z \Rightarrow$$

$$\varphi_t(z_0) = e^{J Q t} z_0 \quad (6.4)$$

Note: By Thm 6.3 $e^{J Q t} \in Sp(n, \mathbb{R})$.

Symplectic Lie algebra

$A = JQ$ is a "Hamiltonian matrix"

i.e. $A \in M_{2n \times 2n}(\mathbb{R})$ s.t. ${}^t(JA) = JA$

$$\Leftrightarrow {}^tA J + J A = 0$$

Explicitly, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Hamiltonian matrix
iff ${}^ta = -d$, ${}^tb = b$, ${}^tc = c$.

The Hamiltonian matrices are closed under addition and the Lie bracket

$$[A_1, A_2] = A_1 A_2 - A_2 A_1$$

and form the Lie algebra $\mathfrak{sp}(n, \mathbb{R})$ of $Sp(n, \mathbb{R})$.

Exponential map

$$\begin{array}{ccc} \exp: & \mathfrak{sp}(n, \mathbb{R}) & \longrightarrow Sp(n, \mathbb{R}) \\ & A & \longmapsto e^A \end{array}$$

not surjective

Hamiltonian vector fields

can be used to generalise (6.4) to non-quadratic Hamiltonians:

Given a smooth Hamiltonian function H

let

$$X_H = \partial_y H \cdot \partial_x - \partial_x H \partial_y$$

$$= J \partial_z H \cdot \partial_z$$

and define the operator $\mathcal{L}^t$ on $C_c^\infty(\mathbb{R}^{2n})$

by
$$\mathcal{L}^t a(z) = a(\varphi_t z).$$

Then 6.4

$$\mathcal{L}^t = e^{X_H t}$$

Proof:

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{L}^t a(z) &= \partial_z a(\varphi_t(z)) \cdot \underbrace{\dot{z}(t)}_{J \partial_z H} \\ &= X_H \mathcal{L}^t a(z) \end{aligned}$$



Lecture 4: Heisenberg group, Schrödinger representation and Stone-von Neumann theorem

§ 7. Heisenberg group and Schrödinger representation

Recall $z = \begin{pmatrix} x \\ y \end{pmatrix} \in V \cong \mathbb{R}^{2n}$

$\sigma(z_1, z_2) = x_1 y_2 - y_1 x_2$ symplectic form

Extend V to Heisenberg group

$\mathbb{H} := \mathbb{H}(V, \sigma) := V \oplus \mathbb{R}e_0 \cong \mathbb{R}^{2n} \times \mathbb{R} =: \mathbb{H}_n(\mathbb{R})$

with multiplication law

$$(z_1, t_1)(z_2, t_2) = (z_1 + z_2, t_1 + t_2 + \frac{1}{2}\sigma(z_1, z_2))$$

Exercise:

(i) Check it satisfies the group axioms.

(ii) Show that the map

$$M: \mathbb{H} \longrightarrow GL(n+2, \mathbb{R})$$

$$(z, t) \mapsto \begin{pmatrix} 1 & t x & t + \frac{1}{2} x \cdot y \\ 0 & \mathbb{1}_n & y \\ 0 & 0 & 1 \end{pmatrix} =: M(z, t)$$

is an injective group homomorphism.

The "matrix representation" of $\mathbb{H}$.

The center $\mathcal{Z}$ of $\mathbb{H}$ is defined as

$$\mathcal{Z} = \{ h \in \mathbb{H} : hg = gh \quad \forall g \in \mathbb{H} \}$$

Thm 7.1

(i) $\mathcal{Z} = \{ (0, t) : t \in \mathbb{R} \}$ and $\mathbb{H}/\mathcal{Z} = \mathbb{R}^{2n}$

(Recall $G_0 \subset G$ subgroup $G/G_0 = \{ gG_0 : g \in G \}$)

(ii) If L is Lagrangian then

$$\mathbb{L} = \{ (z, t) : z \in L, t \in \mathbb{R} \} \supset \mathbb{L}_0 = \{ (z, 0) : z \in L \}$$

are abelian subgroups of $\mathbb{H}$, and

$$\tilde{\mathbb{L}} = \mathbb{H}/\mathbb{L}_0 \quad \tilde{\mathbb{L}}_0 = \mathbb{H}/\mathbb{L}$$

with $\tilde{\mathbb{L}}$ the transverse Lagrangian of L

Proof: Exercise.

For (ii) note that $g \in \mathbb{H}$ can be uniquely written as $g = \tilde{h}_0 h$, $\tilde{h}_0 \in \tilde{\mathbb{L}}_0$, $h \in \mathbb{L}$

Haar measure

A group G has a left action on itself

by $\ell_h: g \mapsto hg$ and a right action

$r_h: g \mapsto gh^{-1}$. The left/right Haar

measure is the (up to normalisation

unique) left/right invariant measure

on G .

Thm 7.2

The (left and right) Haar measure μ of $HT_n(\mathbb{R})$ is the Lebesgue measure $\text{Leb}_{\mathbb{R}^{2n} \times \mathbb{R}}$ on $\mathbb{R}^{2n} \times \mathbb{R}$.

Proof: Suppose $h_n * \mu = \mu \quad \forall h \in HT$,
i.e. $\mu(h^{-1}B) = \mu(B) \quad \forall$ Borel sets B .

Now, HT is generated by the subgroups L , $\tilde{L}$ and $\mathbb{Z}$. For $B = B_0 \times [a, b]$ the $\mathbb{Z}$ -invariance implies that

$$\mu(B) = \mu_0(B_0) \cdot (b - a)$$

For some measure μ_0 on $\mathbb{R}^{2n} \cong HT/\mathbb{Z}$

$L, \tilde{L}$ acts on $HT/\mathbb{Z}$ by translations, so $\mu_0 = \text{Lebesgue on } \mathbb{R}^n$.

Same argument for the right action.



Similarly, the Haar measures ν_0, ν on IL_0, IL are the Lebesgue measures on $L, L \oplus \mathbb{R}e_0$.

The Lie algebra of IH is the Heisenberg algebra

$$\mathfrak{h} = V \oplus \mathbb{R}e_0$$

with Lie bracket defined by

$$(i) \quad [z_1, z_2] = \sigma(z_1, z_2)e_0 \quad \forall z_1, z_2 \in V$$

$$(ii) \quad [z, e_0] = 0 \quad \forall z \in V.$$

So in particular $[h_1, [h_2, h_3]] = 0 \quad \forall h_j \in \mathfrak{h}$
"2-step nilpotent"

Note: If we choose a standard symplectic basis $b_1, \dots, b_n, \tilde{b}_1, \dots, \tilde{b}_n$, then the Lie brackets satisfy exactly Heisenberg's commutation relations (1.2), where

$$b_j = D_j, \quad \tilde{b}_j = M_j, \quad e_0 = \frac{1}{2\pi i} \text{Id}$$

↑
differentiation
operators

↑
multiplication
operator

↑
identity
operator

Matrix representation of $\mathfrak{lh}$ (as Lie algebra)

$$u: \mathfrak{lh} \longrightarrow M_{n+2 \times n+2}(\mathbb{R})$$

$$(z, t) \longmapsto \begin{pmatrix} 0 & t \times & t \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} =: u(z, t)$$

$z = \begin{pmatrix} x \\ y \end{pmatrix}$

Proof: $u(z_1, t_1) u(z_2, t_2) = u(0, x_1 \cdot y_2)$

and hence

$$[u(z_1, t_1), u(z_2, t_2)] = u(0, \mathfrak{S}(z_1, z_2))$$



Note: $u(z, t)^k = 0$ if $k \geq 3$ and

hence

$$\begin{aligned} \exp(u(z, t)) &= 1 + u(z, t) + \frac{1}{2} u(0, x \cdot y) \\ &= M(z, t). \end{aligned}$$

$\Rightarrow \mathfrak{lh}$ is Lie algebra of H .

(There is also a direct way of proving this without going via the matrix rep.)

Representations

A unitary representation $\mathcal{J}$ of a topological group G on a Hilbert space $\mathcal{H}$ is a homomorphism

$$\mathcal{J}(g_1)\mathcal{J}(g_2) = \mathcal{J}(g_1 g_2)$$
$$G \longrightarrow \mathcal{U}(\mathcal{H})$$

group of unitary operators on $\mathcal{H}$

$$g \mapsto \mathcal{J}(g)$$

such that the map

$$G \longrightarrow \mathcal{H} \quad g \longmapsto \mathcal{J}(g)f$$

is continuous for every given $f \in \mathcal{H}$.

Example: Left-regular representation $\mathcal{L}$

$$G = \mathbb{H}, \quad \mathcal{H} = L^2(\mathbb{H}, \mu)$$

$\mathbb{R}$ topology induced by L^2 metric

$$\mathcal{L}(h)f(g) = f(h^{-1}g)$$

Check:

$$\begin{aligned} \mathcal{L}(h_1)\mathcal{L}(h_2)f(g) &= \mathcal{L}(h_2)f(h_1^{-1}g) \\ &= f(h_2^{-1}h_1^{-1}g) = f((h_1 h_2)^{-1}g) \\ &= \mathcal{L}(h_1 h_2)f(g) \quad \checkmark \end{aligned}$$

Unitarity follows from the $\mathbb{H}$ -invariance of Haar measure μ .

Schrödinger representation

Given $\alpha \in \mathbb{R}$, define the function

$$\begin{aligned} \chi_\alpha: \mathbb{H} &\longrightarrow S^1 = \{w \in \mathbb{C} : |w|=1\} \\ (z, t) &\longmapsto e^\alpha(t) = e(\alpha t) = e^{2\pi i \alpha t} \end{aligned}$$

(check that this is well defined)

If L is a Lagrangian subspace, then χ_α defines a unitary character (= unitary representation on $\mathbb{C}$) of the abelian subgroup $\mathbb{L} \cong L \oplus \mathbb{R}e_0$. That is

$$\chi_\alpha(h_1) \chi_\alpha(h_2) = \chi_\alpha(h_1 + h_2)$$

$$\forall h_1, h_2 \in \mathbb{L}$$

Note that it is not a character of $\mathbb{H}$ for $\alpha \neq 0$, since

$$\chi_\alpha((z_1, 0)) \chi_\alpha((z_2, 0)) = 1$$

$$\neq \chi_\alpha((z_1, 0) + (z_2, 0)) = e\left(\frac{\alpha}{2} G(z_1, z_2)\right)$$

Key construction: given Lagrangian subspace L , consider functions $F: \mathbb{H} \rightarrow \mathbb{C}$ such that

$$F(g h) = \chi_\alpha(h)^{-1} F(g) \quad (7.1)$$

$$\forall g \in \mathbb{H}, h \in L$$

This means $|F(g h)| = |F(g)|$ and hence $|F(g)|$ can be identified with a function on $\mathbb{H}/L \cong \tilde{\mathbb{L}}_0 \cong \tilde{\mathbb{L}}$ (Thm 7.2 (ii))

If F_1, F_2 satisfy (7.1) we can define the inner product

$$\langle F_1, F_2 \rangle_L = \int_{\tilde{\mathbb{L}}_0} \bar{F}_1(g) F_2(g) d\nu_0(g) \quad (7.2)$$

since again

$$\bar{F}_1(g h) F_2(g h) = \bar{F}_1(g) F_2(g) \quad \forall h \in L$$

and norm $\|F\|_L = \sqrt{\langle F, F \rangle_L}$.

$\mathcal{H}_{\alpha, L} :=$ Hilbert space of F satisfying (7.1)

and $\|F\|_L < \infty$, together with the inner product (7.2)

The Schrödinger representation $W_{\alpha, L}$ is now defined as the left-regular representation on $\mathcal{X}_{\alpha, L}$ for $\alpha \neq 0$:

$$W_{\alpha, L}(h) F(g) = F(h^{-1}g)$$

Let us specialise to

$$L = \bigoplus_{i=n+1}^{2n} \mathbb{R} e_i \cong \{0\} \times \mathbb{R}^n$$

$$\tilde{L} = \bigoplus_{i=1}^n \mathbb{R} e_i \cong \mathbb{R}^n \times \{0\}$$

We have the unique decomposition

$$g = \underbrace{\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, 0 \right)}_{\tilde{L}} \underbrace{\left(\begin{pmatrix} 0 \\ y \end{pmatrix}, t \right)}_{L} \quad (7.3)$$

and a bijection

$$\mathcal{R}_L: \mathcal{X}_{\alpha, L} \rightarrow L^2(\mathbb{R}^n) \quad (7.4)$$

$$F \mapsto f$$

defined by $f(x) = F\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, 0\right)$.

Note that the inverse $\mathcal{R}_L^{-1}$ is given by

$$\begin{aligned} F(g) &= F\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, 0\right) \mathcal{X}_{\alpha}^{-1}\left(\begin{pmatrix} 0 \\ y \end{pmatrix}, t\right) \\ &= f(x) e(-\alpha t) \end{aligned}$$

for g as in (7.3).

Thm 7.3

Let $\tilde{W}_q = R_L W_{q,L} R_L^{-1}$, $L = \{0\} \times \mathbb{R}^n$. Then

$$[\tilde{W}_q((\begin{smallmatrix} q \\ p \end{smallmatrix}), t) f](x) = f(x - q) e^{\sigma}(-p \cdot x + t + \frac{1}{2} p \cdot q)$$
$$e^{\sigma}(x) = e(\sigma x)$$

Proof:

For $h = ((\begin{smallmatrix} q \\ p \end{smallmatrix}), t) (\Rightarrow h^{-1} = (-\begin{smallmatrix} q \\ p \end{smallmatrix}), -t)$ we have

$$\begin{aligned} W_{q,L}(h) F\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, 0\right) &= F\left(-\begin{pmatrix} q \\ p \end{pmatrix}, -t\right) F\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, 0\right) \\ &= F\left(\begin{pmatrix} x - q \\ -p \end{pmatrix}, -t - \underbrace{\frac{1}{2} G\left(\begin{pmatrix} q \\ p \end{pmatrix}, \begin{pmatrix} x \\ 0 \end{pmatrix}\right)}_{= \frac{1}{2} p \cdot x}\right) \\ &= F\left(\begin{pmatrix} x - q \\ 0 \end{pmatrix}, 0\right) F\left(\begin{pmatrix} 0 \\ -p \end{pmatrix}, -t + p \cdot x - \frac{1}{2} q \cdot p\right) \\ &= F\left(\begin{pmatrix} x - q \\ 0 \end{pmatrix}, 0\right) e^{\sigma}(-p \cdot x + t + \frac{1}{2} q \cdot p) \end{aligned}$$

Corollary (set $\sigma = 1$ and recall (2.2)):

$$\begin{aligned} \tilde{W}_1\left(\begin{pmatrix} q \\ 0 \end{pmatrix}, 0\right) &= S^{-q} \\ \tilde{W}_1\left(\begin{pmatrix} 0 \\ p \end{pmatrix}, 0\right) &= T^{-p} \end{aligned} \quad (7.5)$$

7-10 $\sigma = h$ would give Heisenberg's commutation relations with explicit h -dependence

and the Weyl quantisation (§ 2)

$$Op(a) = \int_{\mathbb{R}^{2n}} \hat{a}(q, p) e\left(\frac{q \cdot p}{2}\right) T^p S^q \, dp \, dq$$

can be written as

$$Op(a) = \int_{\mathbb{R}^{2n}} \hat{a}(q, p) \tilde{W}_1\left(-\begin{pmatrix} q \\ p \end{pmatrix}, 0\right) dp dq \quad (7.6)$$

$$\begin{aligned} \text{since } e\left(\frac{q \cdot p}{2}\right) \tilde{W}_1\left(-\begin{pmatrix} 0 \\ p \end{pmatrix}, 0\right) \tilde{W}_1\left(-\begin{pmatrix} q \\ 0 \end{pmatrix}, 0\right) \\ = e\left(\frac{q \cdot p}{2}\right) \tilde{W}_1\left(-\begin{pmatrix} q \\ p \end{pmatrix}, -\frac{q \cdot p}{2}\right) = \tilde{W}_1\left(-\begin{pmatrix} q \\ p \end{pmatrix}, 0\right) \end{aligned}$$

$\tilde{W}_\alpha$ is an example of an χ_α -induced representation,

$$\tilde{W}_\alpha = \text{Ind}_{\mathbb{L}}^{\mathbb{H}} \chi_\alpha.$$

The corresponding "infinitesimal" representation of the Lie algebra $\mathfrak{h}$ gives the algebra generated by D_i and M_i (§ 1).

Thm 7.3 basis-independent:

$$V = L \oplus \tilde{L}, \quad x \in \tilde{L}, \quad u = \begin{matrix} p+q \\ \uparrow \quad \uparrow \\ L \quad \tilde{L} \end{matrix}$$

Thm 7.3^R

Let $\tilde{W}_q = R_L W_{q,L} R_L^{-1}$. Then

$$[\tilde{W}_q(u, t) f](x) = f(x - q) e^{\tau(-\phi(x, p) + t + \frac{1}{2}\phi(q, p))}$$

Proof: Let $h = (u, t)$. Then

$$W_{q,L}(h) F(x, 0)$$

$$= F((-u, -t)(x, 0))$$

$$= F((x - u, -t - \frac{1}{2}\phi(u, x)))$$

$$\underbrace{\phi(p, x)}$$

$$= F((x - q, 0) \underbrace{(-p, -t + \frac{1}{2}\phi(x - q, p) - \frac{1}{2}\phi(p, x))}_{\phi(x, p) - \frac{1}{2}\phi(q, p)})$$

$$= F((x - q, 0)) e^{\tau(-\phi(x, p) + t + \frac{1}{2}\phi(q, p))}$$



§ 8 Stone - von Neumann Theorem

W.L.O.G.
 $q=1/2$

Define the Fourier - Wigner transform

$$\begin{aligned}\mathcal{F}_w(f_1, f_2)(q, p) &= \langle f_1, \tilde{w}_1((q, p), 0) f_2 \rangle \\ &= \int_{\mathbb{R}^n} \overline{f_1}(x) f_2(x - q) e(-p \cdot x + \frac{1}{2} p \cdot q) dx \\ &= \int_{\mathbb{R}^n} \overline{f_1}(x + \frac{q}{2}) f_2(x - \frac{q}{2}) e(-p \cdot x) dx\end{aligned}$$

Thm 8.1

(i) If $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$ then $\mathcal{F}_w(f_1, f_2) \in \mathcal{S}(\mathbb{R}^{2n})$

(ii) $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathbb{C}$

(iii) $\langle \mathcal{F}_w(f_1, f_2), \mathcal{F}_w(g_1, g_2) \rangle = \overline{\langle f_1, g_1 \rangle} \langle f_2, g_2 \rangle$

Proof: Extend to linear map on $L^2(\mathbb{R}^{2n})$

$$\mathcal{F}_w F(q, p) = \int_{\mathbb{R}^n} F(x + \frac{q}{2}, x - \frac{q}{2}) e(-p \cdot x) dx$$

We have $\mathcal{F}_w = \mathcal{F}^{(1,0)} S(-\frac{1}{2}, \frac{1}{2})$ where

$$\mathcal{F}^{(1,0)} F(x, y) = [\mathcal{F} F(\cdot, y)](x)$$

$$S(q, p) F(x, y) = F(x - q, y - p)$$

Thus $\mathcal{F}_w : L^2(\mathbb{R}^{2n}) \rightarrow L^2(\mathbb{R}^{2n})$ is unitary,
maps $\mathcal{S}(\mathbb{R}^{2n}) \rightarrow \mathcal{S}(\mathbb{R}^{2n})$ and extends

to a map $S'(\mathbb{R}^{2n}) \rightarrow S'(\mathbb{R}^{2n})$.

Choosing $F(x, y) = f_1(x) f_2(y)$ proves the claim. 

A representation $\mathcal{T}$ of a group G on a Hilbert space $\mathcal{H}$ is irreducible if there is no non-trivial closed subspace $\mathcal{H}_0 \subsetneq \mathcal{H}$ so that $\mathcal{T}(g)(\mathcal{H}_0) \subseteq \mathcal{H}_0 \quad \forall g \in G$.

Thm 8.2 ($\gamma \neq 0$)

$\tilde{W}_\gamma$ is irreducible.

$\mathcal{H}_0 \neq \{0\}$

Proof: Assume w.l.o.g. $\gamma = 1$.

Assume $\exists \mathcal{H}_0 \subsetneq L^2(\mathbb{R}^n)$ such that $\tilde{W}_1(h)f \in \mathcal{H}_0 \quad \forall f \in \mathcal{H}_0, h \in H$.

If $f_0 \in \mathcal{H}_0^\perp$ then

$$\langle f_0, \tilde{W}_1(h)f \rangle = 0 \quad \forall h \in H$$

$$\Rightarrow \mathcal{F}_w(f_0, f) = 0$$

Thm 8.1 (iii)

$$\Rightarrow \|\mathcal{F}_w(f_0, f)\| = \|f_0\| \|f\| = 0$$

$$\Rightarrow f_0 = 0 \quad \text{or} \quad f = 0.$$



suggested by proved by (1930)

Thm 8.3 (Stone-von Neumann Theorem)

Let $\mathcal{T}$ be a unitary representation of $\mathbb{H} = \mathbb{H}_n(\mathbb{R})$ on a Hilbert space $\mathcal{H}$, such that $\mathcal{T}(0, t) = e(\sigma t) \text{Id}$ for some $\sigma \neq 0$. Then $\mathcal{H} = \oplus_i \mathcal{H}_i$ with $\mathcal{T} \mathcal{H}_i = \mathcal{H}_i$ and $\mathcal{T}|_{\mathcal{H}_i}$ unitarily equivalent to $\tilde{W}_\sigma$.

Proof: Assume w.l.o.g. $\sigma = 1$.

Use shorthand $\mathcal{T}(q, p) = \mathcal{T}\left(\begin{pmatrix} q \\ p \end{pmatrix}, 0\right)$, and define the linear operator

$$O_{p, \sigma}(a) = \int a(q, p) \mathcal{T}(q, p) dp dq$$

with $a \in L^1(\mathbb{R}^{2n})$.

More precisely, $O_{p, \sigma}(a)$ is defined by $\langle f, O_{p, \sigma}(a)g \rangle = \int a(q, p) \langle f, \mathcal{T}(q, p)g \rangle dp dq$ which is absolutely convergent since

$$|\langle f, \mathcal{T}(q, p)g \rangle|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}} \underbrace{\|\mathcal{T}(q, p)g\|_{\mathcal{H}}}_{\leq \|g\|_{\mathcal{H}}}$$

We have

$$\mathcal{F}(x, y) \mathcal{O}_{p\mathcal{F}}(a)$$

$$= \int a(q, p) \mathcal{F}(x+q, y+p) e\left(\frac{i}{2}(xp - yq)\right) dp dq$$

$$= \int a(q-x, p-y) \mathcal{F}(q, p) e\left(\frac{i}{2}(xp - yq)\right) dp dq$$

$$= \mathcal{O}_{p\mathcal{F}}(b) \quad (8.1)$$

$$b(q, p) = a(q-x, p-y) e\left(\frac{i}{2}(xp - yq)\right)$$

and (by the same calculation)

$$\mathcal{O}_{p\mathcal{F}}(a) \mathcal{F}(x, y) = \mathcal{O}_{p\mathcal{F}}(b) \quad (8.2)$$

$$b(q, p) = a(q-x, p-y) e\left(-\frac{i}{2}(xp - yq)\right)$$

Furthermore

$$\mathcal{O}_{p\mathcal{F}}(a_1) \mathcal{O}_{p\mathcal{F}}(a_2)$$

$$= \iint a_1(q_1, p_1) a_2(q_2, p_2) \mathcal{F}(q_1+q_2, p_1+p_2) e\left(\frac{i}{2}(q_1 p_2 - p_1 q_2)\right) dp_1 dq_1 dp_2 dq_2$$

$$= \iint a_1(q-q_2, p-p_2) a_2(q_2, p_2) \mathcal{F}(q, p) e\left(\frac{i}{2}(p q_2 - q p_2)\right) dp_2 dq_2 dp dq$$

$$= \mathcal{O}_{p\mathcal{F}}(a_1 \hat{*} a_2) \quad (8.3)$$

with $a_1 \hat{*} a_2(p, q)$

$$= \int a_1(q - q', p - p') a_2(q', p') e(\frac{1}{2}(qp' - q'p)) dp' dq'$$

Claim 1:

If $O_{p_T}(a) f = 0 \quad \forall f \in \mathcal{H}$ then $a = 0$ a.e.

Proof: $\forall f, g \in \mathcal{H}$, by assumption

$$0 = \langle \mathcal{T}(x, y) f, O_{p_T}(a) \mathcal{T}(x, y) g \rangle$$

$$= \langle f, \mathcal{T}(-x, -y) O_{p_T}(a) \mathcal{T}(x, y) g \rangle$$

$$= \int e(\frac{1}{2}(xp - yq)) a(q, p) \langle f, \mathcal{T}(q, p) g \rangle dp dq$$

Fourier inversion

$$\Rightarrow a(q, p) \langle f, \mathcal{T}(q, p) g \rangle = 0 \text{ for a.e. } p, q.$$

$$\Rightarrow a = 0 \text{ a.e.}$$



Claim 2:

If a is even, $O_{p_T}(a)^+ = O_{p_T}(\bar{a})$

$$\begin{aligned} \text{Proof: } O_{p_T}(a)^+ &= \int \overline{a(q, p)} \underbrace{\mathcal{T}(q, p)^+}_{= \mathcal{T}(q, p)^{-1}} dp dq \\ &= \mathcal{T}(-q, -p) \end{aligned}$$



Claim 3: If $a(x, y) = 2^{-n} e^{-\frac{\pi}{2}(\|x\|^2 + \|y\|^2)}$ then

$$Op_J(a) \mathcal{T}(x, y) Op_J(a) = 2^n a(x, y) Op_J(a) \quad (8.4)$$

Proof: Left hand side of (8.4) equals

$$Op(a) Op(b) = Op(a \hat{*} b)$$

with b as in (8.1) and

$$a \hat{*} b(q, p)$$

$$\begin{aligned} &= \int a(q - q', p - p') b(q', p') e(\frac{1}{2}(qp' - q'p)) dp' dq' \\ &= \int a(q - q', p - p') a(q' - x, p' - y) e(\frac{1}{2}(xp' - yq')) * \\ &\quad * e(\frac{1}{2}(qp' - p'q)) dp' dq' \\ &= 2^{-n} \int e^{-\frac{\pi}{2}(\|q - q'\|^2 + \|p - p'\|^2 + \|q' - x\|^2 + \|p' - y\|^2 - 2i(xp' - yq' + qp' - p'q))} dp' dq' \\ &= 2^{-n} e^{-\frac{\pi}{2}(\|q\|^2 + \|p\|^2 + \|x\|^2 + \|y\|^2)} * \quad \checkmark \text{ complete the square} \\ &\quad * \int e^{-\frac{\pi}{2}(\|q'\|^2 + \|p'\|^2 - 2q' \cdot (p + x - iy - ip) - 2p'(p + y + ix + iq))} dp' dq' \\ &\quad e^{-\frac{\pi}{2}(\|q\|^2 + \|p\|^2 + \|x\|^2 + \|y\|^2)} * \\ &\quad = 2^{2n} a(q, p) a(x, y) \\ &\quad * \underbrace{e^{\frac{\pi}{2}(\|q + x - iy - ip\|^2 + \|p + y + ix + iq\|^2)}}_{= 1} \underbrace{\int e^{-\frac{\pi}{2}(\|q'\|^2 + \|p'\|^2)} dp' dq'}_{= 2^n} \end{aligned}$$



In the following fix a as in Claim 3.

Claim 1

$$\Rightarrow O_p(a) \neq 0$$

Claim 2

$$\Rightarrow O_p(a) \text{ is self-adjoint}$$

Claim 3

$$\Rightarrow O_p(a)^2 = O_p(a)$$

Thus, $O_p(a)$ is an orthogonal projection on $\mathcal{H}$.

Set $\mathcal{R} = O_p(a) \mathcal{H}$ (the range of $O_p(a)$)

Then for $f, g \in \mathcal{R}$

$$\langle \mathcal{T}(x, y) f, \mathcal{T}(u, v) g \rangle$$

$$= \langle f, \mathcal{T}(-x, -y) \mathcal{T}(u, v) g \rangle$$

$$= e(\frac{1}{2}(uy - xv)) \underbrace{\langle f, \mathcal{T}(u-x, v-y) g \rangle}_{\langle O_p(a)f, \mathcal{T}(u-x, v-y) O_p(a)g \rangle}$$

$$\langle O_p(a)f, \mathcal{T}(u-x, v-y) O_p(a)g \rangle$$

$$= \langle f, O_p(a) \mathcal{T}(u-x, v-y) O_p(a)g \rangle$$

$$= e(\frac{1}{2}(uy - xv)) 2^n a(u-x, v-y) \langle f, g \rangle \quad (8.5)$$

Let $\{\varphi_i\}_i$ be an orthonormal basis*

of $\mathcal{R}$ and define the closed linear span

$$\mathcal{H}_i = \text{span} \{ \mathcal{T}(x, y) \varphi_i : x, y \in \mathbb{R}^n \}$$

8-7 *countable if $\mathcal{H}$ separable

$$(8.5) \Rightarrow \mathcal{H}_i \perp \mathcal{H}_j \quad \text{if } i \neq j$$

Note that if $N = (\bigoplus_i \mathcal{H}_i)^\perp$ then $\mathcal{I}N \subseteq N$,
and $Op_{\mathcal{I}}(a)$ is orthogonal projection $\mathcal{H} \rightarrow N$.

By above construction we must have

$$Op_{\mathcal{I}}(a)N = \{0\} \Rightarrow N = \{0\} \Rightarrow \boxed{\mathcal{H} = \bigoplus_i \mathcal{H}_i}$$

It remains to be shown that $\mathcal{I}|_{\mathcal{H}_j}$

is unitarily equivalent to $\tilde{W}_j$:

Define functions $f^{x,y} \in \mathcal{H}_j$

$$f^{x,y} = \mathcal{I}(x,y) \varphi_j \quad (j \text{ fixed})$$

Then

$$\langle f^{x,y}, f^{u,v} \rangle \stackrel{(8.5)}{=} e(\frac{i}{2}(uy - xv)) 2^n a(u-x, v-y)$$

Similarly define $g^{x,y} \in L^2(\mathbb{R}^n)$ by

$$g^{x,y} = w_1\left(\begin{pmatrix} x \\ y \end{pmatrix}, 0\right) g_0, \quad g_0(q) = 2^{-n/2} e^{-\pi/2 \|q\|^2}$$

By calculation similar to the one leading to (8.5),

$$\text{we have } \langle g^{x,y}, g^{u,v} \rangle_{L^2} = \langle f^{x,y}, f^{u,v} \rangle_{\mathcal{H}}$$

So for finite linear combinations of the form

$$f_i = \sum_{k \in \mathbb{Z}} c_{ik} f^{x_k y_k}, \quad g_i = \sum_{k \in \mathbb{Z}} c_{ik} g^{x_k y_k}$$

we have $\langle f_1, f_2 \rangle_{\mathcal{H}} = \langle g_1, g_2 \rangle_{L^2}$.

By the density of such f in $\mathcal{H}_j$ and g in L^2 , the correspondence $f_i \rightarrow g_i$ extends by linearity & continuity to unitary map $\mathcal{H}_j \rightarrow L^2(\mathbb{R})$, which intertwines $\mathcal{J} \upharpoonright \mathcal{H}_j$ and W_1 .



The Stone-von Neumann Theorem and Schur's Lemma* give the following classification of unitary representations of $\mathcal{H}_n(\mathbb{R})$ ($\alpha=0$ leads to representation of $\mathbb{R}^{2n}$)

Thm 8.4 Every irreducible unitary representation of $\mathcal{H}_n(\mathbb{R})$ is unitarily equivalent to one (and only one) of:

(i) Schrödinger rep W_α with $\alpha \in \mathbb{R} \setminus \{0\}$

(ii) $\phi_\beta \left(\begin{pmatrix} x \\ y \end{pmatrix}, t \right) = e(i\beta \cdot \begin{pmatrix} x \\ y \end{pmatrix})$ with $\beta \in \mathbb{R}^{2n}$

Lecture 5: The Shale-Weil representation of the metaplectic group

So far:

- Quadratic Hamiltonian functions generate linear symplectic transformations on phase space $\mathbb{R}^{2n}$
- The Schrödinger representation of the Heisenberg group produces quantum mechanical operators

Now:

- Will construct action of symplectic group on Heisenberg group, which yields a representation that describes quantum transformations corresponding to quadratic Hamiltonians

§ 9. Projective representation of $Sp(n, \mathbb{R})$

(V, σ) symplectic vector space as before,
 $Sp(V)$, $H(V)$ symplectic & Heisenberg group. The map

$$Sp(V) \times H(V) \longrightarrow H(V)$$

$$(g, h) = (g, (z, t)) \longmapsto g \cdot h := (g z, t)$$

$$\text{with } g z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

defines a left action of $Sp(V)$ on $H(V)$.

(Key is that $Sp(V)$ preserves ε .)

Warning: Note that $g \cdot (h_1, h_2) = (g \cdot h_1)(g \cdot h_2)$
(cf. action of $Sp(V)$ on V)

Recall from § 7:

W_L is the left-regular representation of H

$$W_L(h) F(h_0) = F(h^{-1} h_0)$$

on the Hilbert space $\mathcal{H}_L$ of functions

$$F(h l) = \chi(l)^{-1} F(h) \quad \forall l \in L$$

with $\chi(z, t) = e(t)$ ($\sigma = 1$ throughout)

$L = \{0\} \times \mathbb{R}^n$ gave the classical Schrödinger representation in "position space." It will be more convenient to work with $L = \mathbb{R}^n \times \{0\}$

which yields the (unitarily equivalent) Schrödinger representation in "momentum space".

$$\text{Fix } L = \mathbb{R}^n \times \{0\}, \quad \tilde{L} = \{0\} \times \mathbb{R}^n.$$

We now use the decomposition

$$g = \underbrace{\left(\begin{pmatrix} 0 \\ y \end{pmatrix}, 0 \right)}_{\tilde{L}} \underbrace{\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, t \right)}_{L} \quad (9.1)$$

and a bijection

$$\mathcal{R}_L: \mathcal{H}_{\sigma, L} \rightarrow L^2(\mathbb{R}^n) \quad (9.2)$$

$$F \mapsto f$$

defined by $f(y) = F\left(\left(\begin{pmatrix} 0 \\ y \end{pmatrix}, 0\right)\right)$.

Note that the inverse $\mathcal{R}_L^{-1}$ is given by, for g as in (9.1),

$$\begin{aligned} F(g) &= F\left(\left(\begin{pmatrix} 0 \\ y \end{pmatrix}, 0\right) \chi_{\sigma}^{-1}\left(\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, t\right)\right)\right) \\ &= f(y) e(-\sigma t). \end{aligned}$$

Thm 7.3

Let $\tilde{W}_{\sigma} = \mathcal{R}_L W_{\sigma, L} \mathcal{R}_L^{-1}$, $L = \{0\} \times \mathbb{R}^n$. Then

$$[\tilde{W}_{\sigma}\left(\begin{pmatrix} q \\ p \end{pmatrix}, t\right) f](x) = f(x - q) e^{\sigma(-p \cdot x + t + \frac{1}{2} p \cdot q)}$$

$$e^{\sigma}(x) = e(\sigma x)$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix}$$

9-3

Thm 9.1 (Thm 7.3, cf. also basis indep. version)

Let $\tilde{W}_{q,L} = R_L W_{q,L} R_L^{-1}$, $L = \mathbb{R}^n \times \{0\}$. Then

$$[\tilde{W}_{q,L}((q)_p, t) f](\gamma) = f(\gamma - p) e^{\sigma(q \cdot \gamma + t - \frac{1}{2} p \cdot q)}$$

$$e^{\sigma}(x) = e(\sigma x)$$

Proof:

For $h = ((q)_p, t) (\Rightarrow h^{-1} = (- (q)_p, -t))$ we have

$$W_{q,L}(h) F\left(\begin{pmatrix} 0 \\ \gamma \end{pmatrix}, 0\right)$$

$$= F\left(-\begin{pmatrix} q \\ p \end{pmatrix}, -t\right) \left(\begin{pmatrix} 0 \\ \gamma \end{pmatrix}, 0\right)$$

$$= F\left(\begin{pmatrix} -q \\ \gamma - p \end{pmatrix}, -t - \underbrace{\frac{1}{2} G\left(\begin{pmatrix} q \\ p \end{pmatrix}, \begin{pmatrix} 0 \\ \gamma \end{pmatrix}\right)}_{= -\frac{1}{2} q \cdot \gamma}\right)$$

$$= F\left(\left(\begin{pmatrix} 0 \\ \gamma - p \end{pmatrix}, 0\right) \left(\begin{pmatrix} -q \\ 0 \end{pmatrix}, -t - q \cdot \gamma + \frac{1}{2} q \cdot p\right)\right)$$

$$= F\left(\left(\begin{pmatrix} 0 \\ \gamma - p \end{pmatrix}, 0\right)\right) e^{\sigma(q \cdot \gamma + t - \frac{1}{2} q \cdot p)}$$



Corollary (set $\sigma = 1$ and recall (2.2)):

$$\tilde{W}_L\left(\begin{pmatrix} q \\ 0 \end{pmatrix}, 0\right) = T^q$$

$$\tilde{W}_L\left(\begin{pmatrix} 0 \\ p \end{pmatrix}, 0\right) = S^{-p}$$

(9.3)

Key idea:

If $\omega = \omega_{\tau=1, L}$ we obtain new representation ω^g on $\mathcal{H} = \mathcal{H}_{\tau=1, L} \quad \forall g \in \text{Sp}(V)$ by setting

$$\omega^g(h) = \omega(g \cdot h)$$

Stone-von Neumann Theorem $\Rightarrow \omega^g$ is irreducible and $\exists$ unitary $R(g)$ on $\mathcal{H}$ such that $\forall h \in \mathcal{H}(V)$

$$\omega^g(h) = R(g) \omega(h) R(g)^{-1} \quad (9.5)$$

Schur's Lemma: If $\mathcal{S}$ is a unitary representation of G on $\mathcal{H}$, then $\mathcal{S}$ is irreducible iff $[R\mathcal{S} = \mathcal{S}R \text{ for } R: \mathcal{H} \rightarrow \mathcal{H} \Rightarrow R = c \text{Id}, c \in \mathbb{C}]$.

Thm 9.2 $\forall g_1, g_2 \in \text{Sp}(V)$

$$R(g_1 g_2) = c(g_1, g_2) R(g_1) R(g_2)$$

with $c(g_1, g_2) \in \mathbb{C}$, $|c(g_1, g_2)| = 1$.

Proof: The operator $R(g_1)R(g_2)R(g_1g_2)^{-1}$ commutes with the irreducible representation ω ,

$$\begin{aligned} & R(g_1)R(g_2)R(g_1g_2)^{-1}\omega(h)R(g_1g_2)R(g_2)^{-1}R(g_1)^{-1} \\ & \quad \underbrace{\hspace{10em}}_{\omega(g_1g_2)^{-1}h = \omega(g_2^{-1}g_1^{-1}h)} \\ & \quad \underbrace{\hspace{10em}}_{= \omega(g_1^{-1}h)} \\ & \quad \underbrace{\hspace{10em}}_{= \omega(h)} \end{aligned}$$

hence by Schur's Lemma

$$R(g_1)R(g_2)R(g_1g_2)^{-1} = c(g_1, g_2) \text{Id}.$$

$|c(g_1, g_2)| = 1$ by unitarity. □

Since $c(g_1, g_2) \neq 1$ in general (as we will see) $R(g)$ is only a projective representation of $Sp(V)$.

Note: c is a cocycle, i.e.

$$c(g_1, g_2g_3)c(g_2, g_3) = c(g_1, g_2)c(g, g_2, g_3)$$

(9.6)

Proof:

$$\begin{aligned} \text{(i)} \quad R(g_1)R(g_2)R(g_3) &= c(g_2, g_3)R(g_1)R(g_2g_3) \\ &= c(g_1, g_2g_3)c(g_2, g_3)R(g_1g_2g_3) \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad R(g_1)R(g_2)R(g_3) &= c(g_1, g_2)R(g_1g_2)R(g_3) \\ &= c(g_1, g_2)c(g_1g_2, g_3)R(g_1g_2g_3). \quad \blacksquare \end{aligned}$$

The Mackey obstruction group turns every projective representation into a proper one.

Mackey obstruction group

Set $G_c = G \times S^1 \leftarrow$ unit circle in $\mathbb{C}$

$$\text{with } (g_1, t_1)(g_2, t_2) = (g_1 g_2, t_1 t_2 c(g_1, g_2)^{-1})$$

Then $R(g, t) = t R(g)$ is a unitary (proper) representation of G_c .

We will later see that there is a choice of $R(g)$ such that $c(g_1, g_2) = \pm 1$ and $R(g)$ can be realised as a proper rep of a double cover of $Sp(n, \mathbb{R})$ - the metaplectic group.

Explicit formulas for $R(g)$ (the easy case)

Consider subgroup

$$\begin{pmatrix} a & b \\ 0 & {}^t a^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -a^{-1} b {}^t a \\ 0 & {}^t a \end{pmatrix}$$

$$\mathcal{P} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in Sp(V), \det a \neq 0 \right\}$$

So in particular $d = {}^t a^{-1}$, ${}^t b d = {}^t d b$

and $g L = L \quad \forall g \in \mathcal{P}, \quad \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} a x \\ 0 \end{pmatrix} \in L$

As in the case of the Schrödinger representation, the idea is to use the left-regular representation of $\mathcal{S}p(V)$ on the Hilbert space $\mathcal{H}_L$ (§7).

For $F \in \mathcal{H}_L$, define $R(g)$ with $g \in \mathcal{S}$ and some normalisation constant $\alpha(g)$

$$R(g) F(h) = \alpha(g) F(g^{-1}h) \quad (9.7)$$

We will see that α is 1-dim rep. of $\mathcal{S}$ in $\mathbb{C}^\times$ to make $R(g)$ unitary on $\mathcal{H}_L$; we will see below that $\alpha(g) = (\det a)^{1/2}$ is a convenient choice (rather than the more obvious $|\det a|^{1/2}$).

Thm 9.3 $\det(a_1, a_2)^{1/2} = \pm \det(a_1)^{1/2} \det(a_2)^{1/2}$

(i) $R(g)$ maps $\mathcal{H}_L$ to $\mathcal{H}_L$

(ii) $R(g)$ satisfies (9.5) with

$$c(g_1, g_2) = \frac{\alpha(g_1 g_2)}{\alpha(g_1) \alpha(g_2)} = \pm 1$$

Proof (i) holds since $gL = L$. As to (ii),

$$\begin{aligned}
 \underbrace{W^g(h_0)}_{W(g \cdot h_0)} R(g) F(h) &= R(g) F(\underbrace{(g \cdot h_0)^{-1} h}_{= g \cdot h_0^{-1}}) \\
 &= \kappa(g) F(g^{-1}((g \cdot h_0^{-1}) h)) \\
 &= \kappa(g) F(h_0^{-1}(g^{-1} \cdot h)) \\
 &= \kappa(g) W(h_0) F(g^{-1} \cdot h) \\
 &= R(g) W(h_0) F(h)
 \end{aligned}$$

which yields (9.5). As to (ii),

$$\begin{aligned}
 R(g_1 g_2) &= \underbrace{\kappa(g_1 g_2) \kappa(g_1)^{-1} \kappa(g_2)^{-1}}_{= c(g_1, g_2)} R(g_1) R(g_2).
 \end{aligned}$$



Explicit formulas:

Since $F \in \mathcal{H}_L$, it suffices to evaluate (9.7) at $h = \left(\begin{pmatrix} 0 \\ y \end{pmatrix}, 0 \right)$, $y \in \mathbb{R}^n$.

Thm 9.4 $F \in \mathcal{H}_L$, $g = \begin{pmatrix} a & b \\ 0 & {}^t a^{-1} \end{pmatrix}$, $\det a \neq 0$.

$$\begin{aligned}
 R(g) F\left(\begin{pmatrix} 0 \\ y \end{pmatrix}, 0\right) \\
 = (\det a)^{1/2} F\left(\begin{pmatrix} 0 \\ {}^t a y \end{pmatrix}, 0\right) e^{\alpha\left(\frac{1}{2} y \cdot b {}^t a y\right)}
 \end{aligned}$$

Proof: $g^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}b^t a \\ 0 & \pm a \end{pmatrix}$ and hence

$$\begin{aligned} R(g) F\left(\begin{pmatrix} 0 \\ y \end{pmatrix}, 0\right) &= \kappa(g) F\left(\begin{pmatrix} -a^{-1}b^t a y \\ \pm a y \end{pmatrix}, 0\right) \\ &= \left(\begin{pmatrix} 0 \\ \pm a y \end{pmatrix}, 0\right) \left(\begin{pmatrix} -a^{-1}b^t a y \\ 0 \end{pmatrix}, \underbrace{-\frac{1}{2} \pm a y \cdot a^{-1} b^t a y}_{= y \cdot b^t a y}\right) \\ &= \kappa(g) F\left(\begin{pmatrix} 0 \\ \pm a y \end{pmatrix}, 0\right) e^{i\left(\frac{1}{2} y \cdot b^t a y\right)} \end{aligned}$$

$$L^2 \text{ normalisation} \Rightarrow \kappa(g) = (\det a)^{1/2} \quad \square$$

Note that via the isomorphism (9.2)

$$\mathcal{R}_L: \mathcal{R}_L \rightarrow L^2(\mathbb{R}^n)$$

we obtain the unitary (projective) representation

$$\tilde{R}(g) = \mathcal{R}_L R(g) \mathcal{R}_L^{-1}$$

of $\mathcal{P}$ on $L^2(\mathbb{R}^n)$,

$$\tilde{R}(g) f(y) = (\det a)^{1/2} f\left(\begin{pmatrix} \pm a \\ 0 \end{pmatrix} y\right) e^{i\left(\frac{1}{2} y \cdot b^t a y\right)}$$

(9.8)

§ 10. Generalised Fourier transforms

We continue with the task of determining $R(g)$ [and the cocycle $c(g_1, g_2)$ in § 12]. As in the case of the subgroup $\mathcal{P}$, we will use left translation, cf. (9.7).

Problem:

This does not preserve $\mathcal{H}_L$ for $g \notin \mathcal{P}$:

Write $\mathcal{H}_L = \mathcal{H}_{L,v}$, $v^g(B) = v(gB)$

$L^g = gL$, $\mathbb{L}^g = g\mathbb{L}$. Note: $\chi(g h) = \chi(h)$.

Theorem 10.1

The operator $\mathcal{L}(g) F(h) = F(g^{-1}h)$ defines a unitary map

$$\mathcal{L}_L(g) : \mathcal{H}_{L,v,x} \longrightarrow \mathcal{H}_{L^g,v^g,x}$$

Proof:

Assume $F \in \mathcal{H}_L$, i.e. $F(h h_0) = \chi(h_0)^{-1} F(h)$

$\forall h_0 \in \mathbb{L}$.

Then for $h_1 \in \mathbb{L}^g = g \mathbb{L}$ we have

$$\begin{aligned} \mathfrak{L}(g) F(h_1) &= F(g^{-1}(h_1)) \\ &= F(g^{-1} h g^{-1} h_1) \\ &= \chi(g^{-1} h_1)^{-1} F(g^{-1} h) \\ &= \chi(h_1)^{-1} \mathfrak{L}(g) F(h). \end{aligned}$$

$$\langle \mathfrak{L}(g) F_1, \mathfrak{L}(g) F_2 \rangle_{\mathcal{H}_{L^2}} = \int_{\mathbb{H}/\mathbb{L}^g} F_1(g^{-1} h) F_2(g^{-1} h) d\nu^g(h)$$

$$h = g \tilde{h} + g \mathbb{L}$$

$$= \int_{\mathbb{H}/\mathbb{L}} F_1(\tilde{h}) F_2(\tilde{h}) d\nu(h)$$

$$= \langle F_1, F_2 \rangle_{\mathcal{H}_L}$$



Goal: Find unitary map

$$\mathfrak{F}_{L_1, L_2} : \mathcal{H}_{L_1} \rightarrow \mathcal{H}_{L_2}$$

so that

$$\boxed{R(g) = \mathfrak{F}_{L^g, L} \mathfrak{L}(g)} \quad (10.1)$$

satisfies (9.5).

Idea: Let $\mathbb{K}_{12} = \mathbb{K}_2 / \mathbb{K}_1 \cap \mathbb{K}_2$

and define $\mathfrak{F}_{\mathbb{K}_1, \mathbb{K}_2}$ as a twisted average

$$\mathfrak{F}_{\mathbb{K}_1, \mathbb{K}_2} F(h) = \int_{\mathbb{K}_{12}} F(h h_1) \chi(h_1) d\nu_{\mathbb{K}_{12}}(h_1)$$

where $\nu_{\mathbb{K}_{12}}$ is the Haar (Lebesgue) measure on $\mathbb{K}_{12}$ (to be normalised below). (10.2)

check:

(i) For $h_0 \in \mathbb{K}_1 \cap \mathbb{K}_2$

$$\begin{aligned} F(h h_1 h_0) \chi(h_1 h_0) &= F(h h_1) \chi(h_0)^{-1} \chi(h_1 h_0) \\ &= F(h h_1) \chi(h_1) \end{aligned}$$

$\Rightarrow$ integral well defined

(ii) For $h_2 \in \mathbb{K}_2$

$$\begin{aligned} \mathfrak{F}_{\mathbb{K}_1, \mathbb{K}_2} F(h h_2) &= \int_{\mathbb{K}_{12}} F(h h_1 h_2) \chi(h_1) d\nu_{\mathbb{K}_{12}}(h_1) \\ &= \int_{\mathbb{K}_{12}} F(h h_1) \chi(h_2^{-1} h_1) d\nu_{\mathbb{K}_{12}}(h_1) \\ &= \chi(h_2)^{-1} \mathfrak{F}_{\mathbb{K}_1, \mathbb{K}_2} F(h) \end{aligned}$$

$\Rightarrow \mathfrak{F}_{\mathbb{K}_1, \mathbb{K}_2} F \in \mathcal{H}_{\mathbb{K}_2}$



Thm 10.2 For $R(g)$ as in (10.1)

$$W^g(h) R(g) = R(g) W(h)$$

Proof

$$\begin{aligned} W^g(h) R(g) F(h_0) &= R(g) F((gh)^{-1} h_0) \\ &= \mathcal{F}_{L, L^g} \mathcal{L}(g) F((gh)^{-1} h_0) \\ &= \int \mathcal{L}(g) F(\underbrace{(gh)^{-1} h_0 h_1}_{gh^{-1}}) \chi(h_1) d\nu_{\mathcal{L}_{12}}(h_1) \\ &= \int F(h^{-1} g^{-1}(h_0 h_1)) \chi(h_1) d\nu_{\mathcal{L}_{12}}(h_1) \end{aligned}$$

$$\begin{aligned} R(g) W(h) F(h_0) &= \int \mathcal{L}(g) W(h) F(h_0 h_1) \chi(h_1) d\nu_{\mathcal{L}_{12}}(h_1) \\ &= \int F(h^{-1} g^{-1}(h_0 h_1)) \chi(h_1) d\nu_{\mathcal{L}_{12}}(h_1) \end{aligned}$$



This yields (9.5) provided $R(g)$ is invertible.

We will see that for the correct normalisation of $\nu_{\mathcal{L}_{12}}$ it is in fact unitary.

Let's work out (10.1) in the case

$$L_1 = L \cong \mathbb{R}^n \times \{0\}, \quad L_2 = \tilde{L} \cong \{0\} \times \mathbb{R}^n.$$

We have $IL_{12} = IL_1 / \mathbb{Z}$ and hence for $h = \left(\begin{pmatrix} x \\ 0 \end{pmatrix}, 0\right)$

$$\mathcal{F}_{L, \tilde{L}} F\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, 0\right)$$

$$= \int_{\mathbb{R}^n} F\left(\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, 0\right) \left(\begin{pmatrix} 0 \\ y \end{pmatrix}, 0\right)\right) dy$$

$$= \int_{\mathbb{R}^n} F\left(\left(\begin{pmatrix} 0 \\ y \end{pmatrix}, 0\right) \left(\begin{pmatrix} x \\ 0 \end{pmatrix}, x \cdot y\right)\right) dy$$

$$= \int_{\mathbb{R}^n} F\left(\begin{pmatrix} 0 \\ y \end{pmatrix}, 0\right) e(-x \cdot y) dy.$$

Note also that

$$\mathcal{F}_{\tilde{L}, L} F\left(\begin{pmatrix} 0 \\ y \end{pmatrix}, 0\right) = \int_{\mathbb{R}^n} F\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, 0\right) e(x \cdot y) dx$$

so

$$\boxed{\mathcal{F}_{\tilde{L}, L} = \mathcal{F}_{L, \tilde{L}}^{-1}} \quad (10.2)$$

If L_1, L_2 are not transversal, we can relate $\mathcal{G}_{L_1, L_2}$ to lower dimensional Fourier transforms. This follows from

Fact: For any pair of Lagrangian planes L_1, L_2 , there is a standard symplectic basis $\langle b_i, \tilde{b}_i \rangle$ of V such that

$$L_1 = \bigoplus_{i=1}^n \mathbb{R} b_i$$

$$L_2 = \bigoplus_{i=1}^k \mathbb{R} b_i \oplus \bigoplus_{i=k+1}^n \mathbb{R} \tilde{b}_i$$

$$\text{so } L_1 \cap L_2 = \bigoplus_{i=1}^k \mathbb{R} b_i$$

$$\tilde{L}_1 = \bigoplus_{i=1}^n \mathbb{R} \tilde{b}_i \quad \tilde{L}_2 = \bigoplus_{i=1}^k \mathbb{R} \tilde{b}_i \oplus \bigoplus_{i=k+1}^n \mathbb{R} b_i$$

Exercise: Prove this claim.

The above would be useful to obtain formulas for $R(g)$ with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ all a, b, c, d singular (i.e. $\det = 0$). We will not further explore this here.

Thm 10.3

For $g = J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$ we have

$$\tilde{R}(J) := i^n \mathcal{R}_L \mathcal{F}_{L,L} \mathcal{L}_L(J) \mathcal{R}_L^{-1} = i^n \mathcal{F}^{-1}$$

$$\tilde{R}(J^{-1}) := i^{-n} \mathcal{R}_L \mathcal{F}_{L,L} \mathcal{L}_L(J^{-1}) \mathcal{R}_L^{-1} = i^{-n} \mathcal{F}$$

Proof :

this choice will make $c(g_1, g_2) = \pm 1$

$$\mathcal{F}_{L,L} \mathcal{L}_L(g) F\left(\begin{pmatrix} 0 \\ y \end{pmatrix}, 0\right)$$

$$= \int \underbrace{L(g) F\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, 0\right)}_{F\left(\begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}, 0\right)} e(x \cdot y) dx$$

$$= \int F\left(\begin{pmatrix} 0 \\ x \end{pmatrix}, 0\right) e(x \cdot y) dx \quad \square$$

We have determined $\tilde{R}(g)$ for $g = J$ and $g \in \mathcal{P}$. The following theorem shows that this determines $\tilde{R}(g) \forall g \in \text{Sp}(n, \mathbb{R})$ up to phase factors giving $c(g_1, g_2)$.

Thm 10.4

$\mathcal{P} \cup \{J\}$ generates $Sp(n, \mathbb{R})$.

Proof:

If $\det a \neq 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\begin{pmatrix} 1_n & 0 \\ ca^{-1} & 1_n \end{pmatrix}}_{\substack{\downarrow \\ \approx \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}}} \begin{pmatrix} a & b \\ 0 & \underbrace{d - ca^{-1}b}_{= t_{a^{-1}}} \end{pmatrix} \quad (10.3)$$

$$\approx \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \begin{pmatrix} 1_n & -ca^{-1} \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$$

$$\in J \mathcal{P} J^{-1} \mathcal{P}$$

$$\text{Now } \mathcal{U}^o = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det a \neq 0 \right\}$$

is an open neighbourhood of 1_{2n} .

Hence the subgroup G_o generated by $\mathcal{U}_o$ is open. It's complement $G_o^c = \bigcup_{g \notin G_o} G_o g$ is open,

hence G_o is also closed. Thus $G_o = Sp(n, \mathbb{R})$

since $Sp(n, \mathbb{R})$ is connected.*



Thm 10.5

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det d \neq 0$.

Then

$$\tilde{R}(g)f(y) = \pm (\det d)^{-1/2} \int e(\varphi_0(u, y)) \hat{f}(u) du$$

with $\varphi_0(u, y) = -\frac{1}{2} u \cdot d^{-1} c u + u \cdot d^{-1} y + \frac{1}{2} y \cdot b d^{-1} y$

Proof:

g_1

$$a = \pm d^{-1} + b d^{-1} c$$

$$g = \begin{pmatrix} \pm d^{-1} & b \\ 0 & d \end{pmatrix} \underbrace{\begin{pmatrix} \mathbb{1}_n & 0 \\ d^{-1} c & \mathbb{1}_n \end{pmatrix}}_{g_2}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -d^{-1} c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So

$$\tilde{R}(g)f(y) = \tilde{R}(g_1 \circ g_2 \circ J^{-1})f(y)$$

$$= \pm (\det d)^{-1/2} J^{-1} \tilde{R}(g_2) \hat{f}(d^{-1} y) e\left(\frac{1}{2} y \cdot b d^{-1} y\right)$$

$$= \int \tilde{R}(g_2) \hat{f}(u) e(u \cdot d^{-1} y) du$$

$$= \int \hat{f}(u) e\left(-\frac{1}{2} u \cdot d^{-1} c u + u \cdot d^{-1} y\right) dx$$



Thm 10.6

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det c \neq 0$.

Then

$$\tilde{R}(g) f(y) = \pm i^{-n/2} (\det c)^{-1/2} \int e(\varphi(v, y)) f(v) dv$$

with $\varphi(v, y) = \frac{1}{2} v \cdot c^{-1} d v - v \cdot c^{-1} y + \frac{1}{2} y \cdot a c^{-1} y$

Proof:

$$g J^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} = g_1$$

compensated by $(-1)^{-n/2}$

$$\text{So } \tilde{R}(g) = \tilde{R}(g_1, J) = \pm \tilde{R}(g_1) i^{n/2} J^{-1}$$

and the claim follows from Thm 10.5. 

Note: φ generates the symplectic map g ,

$$\text{i.e. } \begin{pmatrix} \partial_y \varphi \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -\partial_v \varphi \\ v \end{pmatrix}$$

$$\text{Proof: } \begin{cases} \partial_y \varphi = a c^{-1} y - {}^t c^{-1} v \\ \partial_v \varphi = c^{-1} d v - c^{-1} y \end{cases}$$

$$\Rightarrow \begin{cases} y = -c \partial_v \varphi + d v \\ \partial_y \varphi = -a \partial_v \varphi + \underbrace{a c^{-1} d v - {}^t c^{-1} v}_{b v} \end{cases}$$

$$\underbrace{a c^{-1} d - {}^t c^{-1}}_{{}^t c^{-1} {}^t a} = {}^t c^{-1} ({}^t c b) = b$$

10-10

Heisenberg evolution

We call $\tilde{\mathcal{R}}(g)$ a quantisation of g .

To see how it relates to the Weyl quantisation, we re-define

$$\mathcal{O}_p(a) = \int_V \hat{a}(w) \tilde{W}_L(w, 0) dw \quad (10.4)$$

where $V = L \oplus \tilde{L}$

with

$$a(z) = \int_V \tilde{a}(w) e(\varsigma(w, z)) dw \quad (10.5)$$

which is now independent of the choice of basis.

In the standard identification $V = \mathbb{R}^n \times \mathbb{R}^n$ we have the modified Fourier transform

$$a(x, y) = \int_{\mathbb{R}^{2n}} \tilde{a}(q, p) e\left(\underbrace{\begin{pmatrix} q \\ p \end{pmatrix} \cdot \mathcal{J} \begin{pmatrix} x \\ y \end{pmatrix}}_{qy - px}\right) dp dq$$

$$\text{i.e.} \quad \tilde{a}(q, p) = \hat{a}(q, -p).$$

Thm 10.7 $\forall g \in Sp(n, \mathbb{R})$

$$\tilde{R}(g) O_p(a) \tilde{R}(g)^{-1} = O_p(a \circ g^{-1})$$

Proof:

$$\begin{aligned} \tilde{R}(g) O_p(a) \tilde{R}(g)^{-1} &= \int \tilde{a}(w) \tilde{\omega}_L^g(w, 0) dw \\ &= \int \tilde{a}(g^{-1}w) \tilde{\omega}_L(w, 0) dw \end{aligned}$$

$$\begin{aligned} a \circ g^{-1}(z) &= \int \tilde{a}(w) e(\underbrace{\epsilon(w, g^{-1}z)}_{\epsilon(w, z)}) dw \\ &= \int \tilde{a}(g^{-1}w) e(\underbrace{\epsilon(g^{-1}w, g^{-1}z)}_{\epsilon(w, z)}) dw \end{aligned}$$

$\Rightarrow$ same Fourier coefficients $\checkmark$. ◻

If $g = g_t = e^{Xt}$ generates a one-parameter subgroup - and thus the Hamiltonian flow $\varphi_t(z) = g_t z$ - then the family of unitary operators $U_t = x(t) \tilde{R}(g_t)$ satisfies

$$U_t \mathcal{O}_p(a) U_t^{-1} = \mathcal{O}_p(a \circ \varphi_t^{-1}) \quad (10.6)$$

"exact Egorov theorem"

Here the phase factors $\alpha(t) = \pm 1$ are chosen so that $\{U_t \mid t \in \mathbb{R}\}$ form a group ($\rightarrow$ metaplectic representation).

Differentiation of (10.6) w.r.t. t yields, with $U_t = e^{-iHt}$,

$$\frac{d}{dt} \mathcal{O}_p(a \circ \varphi_t^{-1}) = \frac{1}{i} [H, \mathcal{O}_p(a)]$$

Comparison with the Heisenberg evolution equation $\frac{d}{dt} A(t) = i [H, A(t)]$

for $A(t) = U_t^{-1} A_0 U_t$ yields

$$A(t) = \mathcal{O}_p(a \circ \varphi_t) \quad \text{for } A_0 = \mathcal{O}_p(a).$$

Lecture 6: Bargmann representation and Fock space

§ 11. Bargmann representation

as in Thm 9.1

Recall Schrödinger representation ($L = \mathbb{R}^n \times \{0\}$)

$$[\tilde{W}_L((q)_p, t) f](\gamma) = f(\gamma - p) e(q \cdot \gamma + t - \frac{1}{2} p \cdot q)$$

and the (redefined) Fourier-Wigner transform

$$\mathcal{F}_W(f_1, f_2)(q + ip) = \langle f_1, \tilde{W}_L(-(\frac{q}{p}), 0) f_2 \rangle$$

$$= \int_{\mathbb{R}^n} \overline{f_1}(\gamma) f_2(\gamma + p) e(-q \cdot \gamma - \frac{1}{2} p \cdot q) d\gamma$$

$$= \int_{\mathbb{R}^n} \overline{f_1}(\gamma - \frac{p}{2}) f_2(\gamma + \frac{p}{2}) e(-q \cdot \gamma) d\gamma$$

Set $\phi_0(\gamma) = 2^{n/4} e^{-\pi \|\gamma\|^2}$. By Thm 8.1 (iii),

$$L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$$

$$f \longmapsto \mathcal{F}_W(\Phi_0, f)$$

is an isometry.

Explicitly,

$$z = q + ip$$

$$\mathcal{F}_w(\Phi_0, f)(z) = \langle \tilde{\omega} \left(\begin{pmatrix} q \\ p \end{pmatrix}, 0 \right) \Phi_0, f \rangle$$

$$= \int_{\mathbb{R}^n} 2^{n/4} e^{-\pi \|y - p\|^2 - 2\pi i q \cdot y + \pi i p \cdot q} f(y) dy$$

$$= 2^{n/4} e^{-\pi \|p\|^2 + \pi i p \cdot q} \int_{\mathbb{R}^n} e^{-\pi \|y\|^2 - 2\pi i (q + ip) \cdot y} f(y) dy$$

$$= e^{-\frac{\pi}{2} \|q + ip\|^2 + \frac{\pi}{2} (q + ip)(q + ip)}$$

$$= 2^{n/4} e^{-\pi/2 \|z\|^2} \int_{\mathbb{R}^n} e^{-\pi \|y\|^2 - 2\pi i z \cdot y + \frac{\pi}{2} z \bar{z}} f(y) dy$$

Def: Bargmann transform ($z \in \mathbb{C}^n$)

$$Bf(z) = 2^{n/4} \int_{\mathbb{R}^n} e^{-\pi \|y\|^2 - 2\pi i z \cdot y + \frac{\pi}{2} z \bar{z}} f(y) dy$$

(cf. Folland $z \rightarrow iz$)

(11.1)

Def: Fock space

$$\mathcal{F}_n = \{ F \text{ entire on } \mathbb{C}^n \mid \|F\|_{\mathcal{F}} < \infty \}$$

with $\|F\|_{\mathcal{F}}^2 = \langle F, F \rangle_{\mathcal{F}}$ and

$$\langle F_1, F_2 \rangle_{\mathcal{F}} = \int_{\mathbb{C}^n} \overline{F_1(z)} F_2(z) e^{-\pi \|z\|^2} dV(z)$$

Set $d\mu_{\mathcal{F}}(z) = e^{-\pi \|z\|^2} dV(z)$. Lebesgue on $\mathbb{C}^n$

Useful properties

- (i) $f \in L^2(\mathbb{R}^n) \implies Bf \in \mathcal{F}_n$
and $\|f\|_{L^2} = \|Bf\|_{\mathcal{F}_n}$. ↖ α multi-index
- (ii) $\{J_\alpha(z) = \left(\frac{\pi^{|\alpha|}}{\alpha!}\right)^{1/2} z^\alpha \mid |\alpha| \geq 0\}$ is an ONB.
- (iii) The Taylor series of $F \in \mathcal{F}_n$ converges to F in the $\|\cdot\|_{\mathcal{F}}$ norm.
- (iv) $|F(z)| \leq e^{\pi/2 \|z\|^2} \|F\|_{\mathcal{F}} \quad \forall F \in \mathcal{F}_n, z \in \mathbb{C}^n$

(for proofs see Folland Chap 1 § 6)

By (iv), $\mathcal{F}_n \rightarrow \mathcal{F}_n$, $F \mapsto F(w)$ is
bld linear functional $\forall w$, so $\exists E_w$
such that $F(w) = \langle E_w, F \rangle_{\mathcal{F}}$.

Fact: $E_w(z) = e^{\pi \bar{w} \cdot z}$ (11.2)

Proof:

$$\begin{aligned} E_w(z) &= \sum_{|\alpha| \geq 0} \langle J_\alpha, E_w \rangle J_\alpha(z) \\ &= \sum_{|\alpha| \geq 0} \overline{J_\alpha(w)} J_\alpha(z) \\ &= \sum_{|\alpha| \geq 0} \frac{\pi^{|\alpha|}}{\alpha!} \bar{w}^\alpha z^\alpha = e^{\pi \bar{w} \cdot z} \end{aligned}$$



Thus for $K_1(z, \bar{w}) = e^{\pi z \bar{w}}$

$$F(z) = \int K_1(z, \bar{w}) F(w) d\mu_{\mathcal{F}}(w)$$

"reproducing kernel"

Thm 11.1

Let $T: \mathcal{F}_n \rightarrow \mathcal{F}_n$ be bounded, and set

$K_T(z, \bar{w}) = T E_w(z)$. Then K_T is an entire function on $\mathbb{C}^{2n}$ with

(i) $K_T(\cdot, w) \in \mathcal{F}_n$, $K_T(z, \cdot) \in \mathcal{F}_n$.

(ii) $|K_T(z, \bar{w})| \leq e^{\frac{\pi}{2}(\|z\|^2 + \|w\|^2)} \|T\|$

(iii) $TF(z) = \int K_T(z, \bar{w}) F(w) d\mu_{\mathcal{F}}(w)$

Proof:

$$TF(z) = \langle E_z, TF \rangle_{\mathcal{F}} = \langle T^* E_z, F \rangle_{\mathcal{F}}$$

$$= \int \overline{T^* E_z(w)} F(w) d\mu_{\mathcal{F}}(w)$$

where $\overline{T^* E_z(w)} = \overline{\langle E_w, T^* E_z \rangle_{\mathcal{F}}}$

$$= \langle E_z, T E_w \rangle_{\mathcal{F}} \quad (11.3)$$

$$= T E_w(z) = K_T(z, \bar{w})$$

$\Rightarrow K_T(z, \bar{w})$ holomorphic in z by (11.3)
and holomorphic in $\bar{w}$; (i) & (ii) follow.

(ii) follows from

$$\begin{aligned} |K_T(z, \bar{w})| &= |\langle E_z, T E_w \rangle| \\ &\leq \|E_z\| \|T E_w\| \\ &\leq \|T\| \|E_z\| \|E_w\| \\ &= \|T\| e^{\pi/2 (\|z\|^2 + \|w\|^2)} \quad \square \end{aligned}$$

Theorem 11.2

If $T : \mathcal{F}_n \rightarrow \mathcal{F}_n$ is bounded, then it is uniquely determined by $K_T(z, \bar{z})$.

Proof: $K_T(z, \bar{w})$ is entire in $z, \bar{w}$.

Let $u = \frac{1}{2}(z + \bar{w})$, $v = -\frac{1}{2}i(z - \bar{w})$.

Then $G(u, v) := K_T(z, \bar{w})$ is entire in u, v . G is determined by u, v

real (via Taylor series), i.e. when

$$\operatorname{Im}(z + \bar{w}) = 0 = \operatorname{Re}(z - \bar{w}) \iff z = w.$$

Identify $\mathbb{R}^{2n} \xrightarrow{\sim} \mathbb{C}^n$, $\begin{pmatrix} q \\ p \end{pmatrix} \mapsto z = q + i p$

So that the multiplication law of H_c is

$$(z, t)(z', t') = (z + z', t + t' + \frac{1}{2} \operatorname{Im}(\bar{z} z'))$$

Inverse Bargmann transform

By unitarity, for $F \in \mathcal{F}_n$, $f \in L^2(\mathbb{R}^n)$

$$\langle f, B^{-1} F \rangle_{L^2} = \langle B f, F \rangle_{\mathcal{F}}$$

$$= 2^{n/4} \int_{\mathbb{C}^n} \int_{\mathbb{R}^n} e^{-\pi \|y\|^2 + 2\pi i \bar{z} \cdot y + \frac{\pi}{2} \bar{z} \cdot z} \overline{f(y)} F(z) dy d\mu_{\mathcal{F}}(z)$$

so

$$B^{-1} F(y)$$

$$= 2^{n/2} \int_{\mathbb{C}^n} e^{-\pi \|y\|^2 + 2\pi i \bar{z} \cdot y + \frac{\pi}{2} \bar{z} \cdot z} F(z) d\mu_{\mathcal{F}}(z)$$

Bargmann Kernel

$$B(z, y) = 2^{n/4} e^{-\pi \|y\|^2 - 2\pi i z \cdot y + \frac{\pi}{2} z \cdot z}$$

Then

$$B f(z) = \int_{\mathbb{R}^n} B(z, y) f(y) dy$$

$$B^{-1} F(y) = \int_{\mathbb{C}^n} B(-\bar{z}, y) F(z) d\mu_{\mathbb{F}}(z)$$

Bargmann representation of $\mathcal{H}$:

$$\mathcal{V}(q+i\rho, t) := B \tilde{\omega}_L\left(\begin{pmatrix} q \\ \rho \end{pmatrix}, t\right) B^{-1}$$

So in particular: (now use $z = x+iy$)

$$\mathcal{V}(q+i\rho, 0) B f(z)$$

$$\omega = q+i\rho$$

$$= B \tilde{\omega}\left(\begin{pmatrix} q \\ \rho \end{pmatrix}, 0\right) f(z)$$

$$\begin{aligned} & -x\rho + yq \\ & = \operatorname{Im} (z+i\rho)(\overline{q+i\rho}) \end{aligned}$$

$$= e^{\pi/2 \|z\|^2} \langle \Phi_0, \tilde{\omega}\left(-\begin{pmatrix} x \\ y \end{pmatrix}, 0\right) \tilde{\omega}\left(\begin{pmatrix} q \\ \rho \end{pmatrix}, 0\right) f \rangle$$

$$= e^{\pi/2 \|z\|^2} e^{\pi i \operatorname{Im} z \bar{\omega}} \langle \Phi_0, \tilde{\omega}\left(-\begin{pmatrix} x-q \\ y-\rho \end{pmatrix}, 0\right) f \rangle$$

$$\underbrace{e^{\pi/2 \|z-\omega\|^2 + \pi \operatorname{Re} z \bar{\omega} - \frac{\pi}{2} \|\omega\|^2}}$$

$$= e^{-\frac{\pi}{2} \|\omega\|^2 + \pi z \cdot \bar{\omega}} B f(z-\omega)$$

We conclude :

$$\mathcal{V}(\omega, t) F(z) = e^{\pi z \cdot \bar{\omega} + 2\pi i t - \frac{\pi}{2} \|\omega\|^2} F(z - \omega)$$

$$= e^{\pi \omega \cdot \bar{z} + 2\pi i t - \frac{\pi}{2} \omega \cdot \bar{\omega}}$$

Compare with $\tilde{\omega}_L\left(\begin{pmatrix} q \\ p \end{pmatrix}, t\right)$:

$$\tilde{\omega}_L\left(\begin{pmatrix} q \\ p \end{pmatrix}, t\right) f(\gamma) = f(\gamma - p) e^{(q \cdot \gamma + t - \frac{1}{2} p \cdot q)}$$

We have formally

$$\tilde{\omega}_L\left(\begin{pmatrix} \frac{\omega}{\sqrt{2}} \\ -i \frac{\bar{\omega}}{\sqrt{2}} \end{pmatrix}, t\right) f\left(-i \frac{\bar{z}}{\sqrt{2}}\right) = f\left(-i \frac{\bar{z} - \bar{\omega}}{\sqrt{2}}\right) e\left(-i \frac{\omega \cdot \bar{z}}{\sqrt{2}} + t - \frac{i}{2} \frac{\omega \cdot \bar{\omega}}{\sqrt{2}}\right)$$

and thus

$$\mathcal{V}(\omega, t) F(z) = \tilde{\omega}_L\left(\begin{pmatrix} \frac{\omega}{\sqrt{2}} \\ -i \frac{\bar{\omega}}{\sqrt{2}} \end{pmatrix}, t\right) f\left(-i \frac{\bar{z}}{\sqrt{2}}\right)$$

with $F(z) = f\left(-i \frac{\bar{z}}{\sqrt{2}}\right)$

Also note

$$e^{-\pi \|\omega\|^2} = e^{-\pi \omega \cdot \bar{\omega}} = e\left(-i \frac{\omega \cdot \bar{\omega}}{\sqrt{2}}\right)$$

$$\omega = q + ip \Rightarrow \begin{pmatrix} \frac{\omega}{\sqrt{2}} \\ -i \frac{\bar{\omega}}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} q + ip \\ p - iq \end{pmatrix}$$

§ 12. Metaplectic representation on Fock space

Complex representation of $Sp(n, \mathbb{R})$

Want to conjugate $Sp(n, \mathbb{R})$ so that

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow J_c = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\text{So that } J \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix} \rightarrow J_c \begin{pmatrix} q+ip \\ q-ip \end{pmatrix} = \begin{pmatrix} p-iq \\ p+iq \end{pmatrix}$$

This is achieved by the unitary matrix

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \in U(2n), \quad C^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

$$C J C^{-1} = \frac{1}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}}_{\begin{pmatrix} -i & i \\ -1 & -1 \end{pmatrix}} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = J_c$$

Set $g_c = C g C^{-1}$ and note

$$\frac{1}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} r & s \\ \bar{s} & \bar{r} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} a-ib & a+ib \\ c-id & c+id \end{pmatrix}}$$

$$2r = a+d+i(-b+c)$$

$$2s = a-d+i(b+c)$$

In particular:

$$g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \Rightarrow g_c = \begin{pmatrix} 1 - \frac{i}{2}b & \frac{i}{2}b \\ -\frac{i}{2}b & 1 + \frac{i}{2}b \end{pmatrix}$$

$b = b$

$$g = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \Rightarrow g_c = \begin{pmatrix} \frac{1}{2}(a + {}^t \bar{a}^{-1}) & \frac{1}{2}(a - {}^t \bar{a}^{-1}) \\ \frac{1}{2}(a - {}^t \bar{a}^{-1}) & \frac{1}{2}(a + {}^t \bar{a}^{-1}) \end{pmatrix}$$

$\det a \neq 0$

$$g = \begin{pmatrix} \operatorname{Re} u & -\operatorname{Im} u \\ \operatorname{Im} u & \operatorname{Re} u \end{pmatrix} \Rightarrow g_c = \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix}$$

${}^t \bar{u} u = 1$

Furthermore

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} s & r \\ \bar{r} & \bar{s} \end{pmatrix} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} = \begin{pmatrix} \operatorname{Re}(s+r) & \operatorname{Im}(-s+r) \\ \operatorname{Im}(s+r) & \operatorname{Re}(s-r) \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} s+r & i(s-r) \\ \bar{s}+\bar{r} & i(\bar{r}-\bar{s}) \end{pmatrix}}_{\begin{pmatrix} \frac{1}{2}(s+\bar{s}+r+\bar{r}) & \frac{i}{2}(s-\bar{s}-r+\bar{r}) \\ \frac{i}{2}(-s+\bar{s}-r+\bar{r}) & \frac{1}{2}(s+\bar{s}-r-\bar{r}) \end{pmatrix}}$$

Def: $Sp_c(V) = C Sp(V) C^{-1}$

subgroup of $GL(2n, \mathbb{C})$

Thm 5.2

(i) $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n, \mathbb{R})$ iff $\begin{cases} {}^t c a = {}^t a c \\ {}^t a d - {}^t c b = \mathbb{1}_n \\ {}^t d b = {}^t b d \end{cases}$

(ii) $g^{-1} = \begin{pmatrix} {}^t d & -{}^t b \\ -{}^t c & {}^t a \end{pmatrix}$

Thm 12.1

(i) $g = \begin{pmatrix} r & s \\ \bar{s} & \bar{r} \end{pmatrix} \in Sp_c$ iff $\begin{cases} {}^t \bar{r} r - {}^t s \bar{s} = 1 \\ {}^t \bar{r} s = {}^t s \bar{r} \end{cases}$

(ii) $g^{-1} = \begin{pmatrix} {}^t \bar{r} & -{}^t s \\ -{}^t \bar{s} & {}^t r \end{pmatrix}$

Proof: ${}^t (C^{-1} g C) J C^{-1} g C = J$

$\Leftrightarrow {}^t g \underbrace{{}^t C^{-1} J C^{-1}} = {}^t C^{-1} J C^{-1}$

$\frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

$\Leftrightarrow {}^t g J g = J$

$\Rightarrow g$ satisfies same relations as $Sp(n, \mathbb{R})$, Thm 5.2.

□

Note (i) is equivalent to (use $g g^{-1} = 1$)

$\begin{cases} r {}^t \bar{r} - s {}^t \bar{s} = 1 \\ s {}^t r = r {}^t s \end{cases} \quad (12.1)$

Corollary of Thm 12.1

(i) $\det r \neq 0$, $\|r\| \geq 1$.

(ii) $\|r^{-1}s\| = \|\bar{s} r^{-1}\| = 1 - \|r\|^{-2}$

(iii) $r^{-1}s = {}^t(r^{-1}s)$ $\bar{s} r^{-1} = {}^t(\bar{s} r^{-1})$

Proof:

(i) $\|ru\|^2 = \langle ru, ru \rangle = \langle u, \underbrace{{}^t \bar{r} r}_{1 + {}^t s \bar{s}} u \rangle$
 $= \|u\|^2 + \|su\|^2.$

(ii) Set $u = r^{-1}v$ in above.

(iii) Easy restatement of $s^t r = r^t s$ & ${}^t \bar{r} s = {}^t s \bar{r}.$

The key point here is that $\det r \neq 0$ 
indicates we may be able to construct
a projective unitary representation
 $Q(g)$ via an integral formula that
simultaneously works for all g .

$Q(g)$ is induced from $\mathcal{U}(h)$ via the
relation

$$\mathcal{U}(gh) = Q(g) \mathcal{U}(h) Q(g^{-1}) \quad (12.2)$$

By Thm 11.1, there is a kernel $E_w(z) = e^{\pi \bar{w} \cdot z}$

$$K_g(z, \bar{w}) = \langle E_z, Q(g) E_w \rangle$$

such that

$$Q(g) F(z) = \int_{\mathbb{C}^n} K_g(z, \bar{w}) F(w) d\mu_g(w)$$

Thm 12.2 If $g = \begin{pmatrix} r & s \\ \bar{s} & r \end{pmatrix} \in Sp_c(V)$ and $Q(g)$ a unitary operator satisfying (12.2), then

$$K_g(z, \bar{w}) = C_g e^{\frac{\pi}{2} (z \cdot \bar{s} r^{-1} z + 2 \bar{w} \cdot r^{-1} z - \bar{w} \cdot r^{-1} s \bar{w})} \quad (12.3)$$

for some constant C_g .

Proof: We have

$$E_w(z) = e^{\pi z \bar{w}} = e^{\frac{\pi}{2} \|w\|^2} \underbrace{e^{\pi z \bar{w} - \frac{\pi}{2} \|w\|^2}}_{e^{\pi z \bar{w} - \frac{\pi}{2} \|w\|^2} E_0(z-w)} E_0(z)$$

$$\text{so } K_g(z, \bar{w}) = C_g e^{\frac{\pi}{2} (\|z\|^2 + \|w\|^2)} * \underbrace{1}_{=1}$$

$$* \langle \mathcal{V}(z, 0) E_0, Q(g) \mathcal{V}(w, 0) E_0 \rangle_g$$

Step 1: Assume $z = gw$, i.e. $\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = g \begin{pmatrix} w \\ \bar{w} \end{pmatrix}$. Then

$$z = rw + s\bar{w} \quad (12.4)$$

and (Thm 2.1 (ii))

$$\bar{w} = {}^t r \bar{z} - {}^t \bar{s} z \quad (12.5)$$

$$\Rightarrow w = r^{-1} z - r^{-1} s \bar{w}, \Rightarrow \bar{z} = {}^t r^{-1} \bar{w} + \underbrace{{}^t r^{-1} {}^t \bar{s}}_{\bar{s} r^{-1}} z$$

$$\text{So } \|w\|^2 + \|z\|^2 = \bar{w} \cdot w + z \cdot \bar{z}$$

$$= \bar{w} \cdot r^{-1} z - \bar{w} \cdot r^{-1} s \bar{w} + z \cdot {}^t r^{-1} \bar{w} + z \cdot \bar{s} r^{-1} z$$

$$= z \cdot \bar{s} r^{-1} z + 2 \bar{w} \cdot r^{-1} z - \bar{w} \cdot r^{-1} s \bar{w}$$

By assumption (12.2)

$$\langle \mathcal{V}(z, 0) E_0, Q(g) \mathcal{V}(w, 0) E_0 \rangle_{\mathfrak{F}}$$

$$= \langle Q(g) \underbrace{\mathcal{V}(\bar{g}^{-1} z, 0)}_{\bar{w}} Q(g)^{-1} E_0, Q(g) \mathcal{V}(w, 0) E_0 \rangle_{\mathfrak{F}}$$

$Q(g), \mathcal{V}(h)$ unitary

$$= \langle Q(g)^{-1} E_0, E_0 \rangle_{\mathfrak{F}} = \langle E_0, Q(g) E_0 \rangle_{\mathfrak{F}}$$

We conclude that the claim is correct

in the case $z = gw$ with the choice

$$c_g = \langle E_0, Q(g) E_0 \rangle_{\mathfrak{F}}.$$

Step 2

Define the complex variables $u, v \in \mathbb{C}$

$$\text{by } \begin{aligned} u &= \frac{1}{2} (\bar{w} + r^{-1} z - r^{-1} s \bar{w}) \\ v &= \frac{1}{2} i (\bar{w} - r^{-1} z + r^{-1} s \bar{w}) \end{aligned}$$

$$\text{so that } \begin{aligned} z &= r(u + iv) + s(u - iv) \\ \bar{w} &= u - iv \end{aligned}$$

Then $K_g(z, \bar{w})$ is entire in u, v and hence determined by its values for $u, v \in \mathbb{R}$. But this happens precisely when $z = gw$ as in step 1. So $K_g(z, \bar{w})$ is determined by the restriction $z = gw$. Since the RHS of (12.3) is entire, claim follows. □

Note that, by unitarity of $Q(g)$,

$$\begin{aligned} 1 &= \|E_0\|_g^2 = \|Q(g)E_0\|_g^2 \quad (Q(g)E_0(z) = K_g(z, 0)) \\ &= \int |K_g(z, 0)|^2 e^{-\pi \|z\|^2} dz \\ &= |C_g|^2 \int e^{\frac{\pi}{2}(z \bar{s} r^{-1} z + \bar{z} s \bar{r}^{-1} \bar{z} - 2\|z\|^2)} dz \\ &= |C_g|^2 \left(\det(1 - s \bar{r}^{-1} \bar{s} r^{-1}) \right)^{-1/2} \\ &\quad = |C_g|^2 \left(1 - s \bar{r}^{-1} \bar{s} r^{-1} \right)^{-1} = (r^* \bar{r})^{-1} \end{aligned}$$

$$\Rightarrow \boxed{|C_g| = |\det r|^{-1/2}}$$

For $g = \begin{pmatrix} r & s \\ \bar{s} & \bar{r} \end{pmatrix} \in Sp_c(V)$ define

$$Q(g): \mathcal{F}_n \rightarrow \mathcal{F}_n$$

by

$$Q(g) F(z) = \int_{\mathbb{C}^n} K_g(z, \bar{w}) F(w) d\mu_g(w)$$

with

$$K_g(z, \bar{w}) = (\det r)^{-1/2} *$$

$$* e^{\frac{i}{2} (z \cdot \bar{s} r^{-1} z + 2 \bar{w} \cdot r^{-1} z - \bar{w} \cdot r^{-1} s \bar{w})} \quad (12.6)$$

Thm 12.3

(i) $Q(g)$ is unitary

$$(ii) Q(g) \mathcal{V}(h) Q(g)^{-1} = \mathcal{V}(gh)$$

$$(iii) Q(g_1) Q(g_2) = c(g_1, g_2) Q(g_1 g_2)$$

with $c(g_1, g_2) \in \{\pm 1\}$.

Proof: $\mathcal{V}$ is unitarily equivalent to ω_c , hence irreducible. Stone-von Neumann implies that there is unitary $Q(g)$ satisfying (i) and (ii). As in the discussion

of $\mathcal{R}(g)$ (Thm 9.2), we conclude that

$$Q(g_1) Q(g_2) = c(g_1, g_2) Q(g_1 g_2)$$

for some complex cocycle with $|c(g_1, g_2)| = 1$.

To work out $c(g_1, g_2)$, set $g_i = \begin{pmatrix} r_i & s_i \\ \bar{s}_i & \bar{r}_i \end{pmatrix}$.

Then

$$\begin{aligned} c(g_1, g_2) & \langle E_0, Q(g_1 g_2) E_0 \rangle_{\mathfrak{F}} \\ &= \langle E_0, Q(g_1) Q(g_2) E_0 \rangle_{\mathfrak{F}} \\ &= \langle Q(g_1)^{\dagger} E_0, Q(g_2) E_0 \rangle_{\mathfrak{F}} \\ &= \int K_{g_1}(0, \bar{z}) K_{g_2}(z, 0) e^{-\pi \|z\|^2} dz \\ &= (\det r_1)^{-1/2} (\det r_2)^{-1/2} \int e^{\frac{\pi}{2} (z \bar{s}_1 r_1^{-1} z - \bar{z} r_1^{-1} s_1 \bar{z} - 2 z \cdot \bar{z})} dz \\ &= (\det r_1)^{-1/2} (\det r_2)^{-1/2} \left(\det \left(1 + \underbrace{r_1^{-1} s_1 \bar{s}_2 r_2^{-1}}_{r_1^{-1} (r_1 r_2 + s_1 \bar{s}_2) r_2^{-1}} \right) \right)^{-1/2} \\ &= \pm (\det (r_1 r_2 + s_1 \bar{s}_2))^{-1/2} \\ &= \pm \langle E_0, Q(g_1 g_2) E_0 \rangle_{\mathfrak{F}} \end{aligned}$$

since

$$g_1 g_2 = \begin{pmatrix} r_1 & s_1 \\ \bar{s}_1 & \bar{r}_1 \end{pmatrix} \begin{pmatrix} r_2 & s_2 \\ \bar{s}_2 & \bar{r}_2 \end{pmatrix} = \begin{pmatrix} r_1 r_2 + s_1 \bar{s}_2 & * \\ * & * \end{pmatrix}$$



Via the Mackey obstruction group construction, we thus obtain the metaplectic representation of the double cover of $Sp(V)$.

Generating functions:

$$\varphi_g(z, \bar{w}) = \frac{1}{2} z \cdot \bar{s} r^{-1} z + \bar{w} \cdot r^{-1} z - \frac{1}{2} \bar{w} \cdot r^{-1} s \bar{w}$$

$$\begin{pmatrix} z \\ \partial_z \varphi_g \end{pmatrix} = \begin{pmatrix} r & s \\ \bar{s} & \bar{r} \end{pmatrix} \begin{pmatrix} \partial_{\bar{w}} \varphi_g \\ \bar{w} \end{pmatrix}$$

Proof:

$$\begin{cases} \partial_z \varphi_g = \bar{s} r^{-1} z + \bar{w} \cdot r^{-1} \\ \partial_{\bar{w}} \varphi_g = r^{-1} z - r^{-1} s \bar{w} \end{cases}$$

$$\Rightarrow z = r \partial_{\bar{w}} \varphi_g + s \bar{w}$$

$$\partial_z \varphi_g = \bar{s} \partial_{\bar{w}} \varphi_g + \underbrace{(\bar{s} r^{-1} s + \bar{w} \cdot r^{-1})}_{\bar{r}} \cdot \bar{w}$$



$$\begin{aligned} \bar{r} r - s \bar{s} &= 1 \\ \bar{r} s &= s \bar{r} \end{aligned}$$

Exercise

Work out $\tilde{R}(g) = B^{-1} Q(g_e) B$
for $g \in \mathcal{P}$ and $g = J$.

Lecture 7: Poisson summation and theta functions

§ 13 Poisson summation

Thm 13.1 Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\sum_{m \in \mathbb{Z}^n} f(m) = \sum_{m \in \mathbb{Z}^n} \hat{f}(m) \quad (13.1)$$

Proof: Define $\varphi(x) = \sum_{m \in \mathbb{Z}^n} f(x+m)$.

Since $f \in \mathcal{S}(\mathbb{R}^n)$ this converges absolutely uniformly on compacta $C \subset \mathbb{R}^n$, and

$$\varphi(x+r) = \varphi(x) \quad \forall r \in \mathbb{Z}^n.$$

The Fourier series of φ is

$$\varphi(x) = \sum_{k \in \mathbb{Z}^n} \hat{\varphi}_k e(k \cdot x) \quad (13.2)$$

with

$$\hat{\varphi}_k = \int_{[0,1]^n} \varphi(x) e(-kx) dx$$

$$= \int_{[0,1]^n} \left(\sum_{m \in \mathbb{Z}^n} f(x+m) \right) e(-kx) dx$$

(series converges uniformly in $y \in [0, 1]$)

$$\begin{aligned}
 &= \sum_{m \in \mathbb{Z}^n} \int_{[0,1]^n} f(x+m) e(-kx) dx \\
 &\quad (k \text{ integer}) \\
 &= \sum_{m \in \mathbb{Z}^n} \int_{[0,1]^n + m} f(x) e(-kx) dx \\
 &\quad \underbrace{\hspace{10em}}_{\int_{\mathbb{R}^n}} \\
 &= \hat{f}(k).
 \end{aligned}$$

So the Fourier series (13.2) of f can be written

$$\sum_{m \in \mathbb{Z}^n} f(x+m) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e(k \cdot x) \quad (13.3)$$

Setting $x=0$ yields the theorem. 

We can re-write (13.1) as (cf. Thm 10.3)

$$\sum_{m \in \mathbb{Z}^n} f(m) = i^{-n} \sum_{m \in \mathbb{Z}^n} \tilde{\mathcal{F}}(J^{-1}) f(m)$$

$$\sum_{m \in \mathbb{Z}^n} f(m) = i^{-n} \sum_{m \in \mathbb{Z}^n} \tilde{\mathcal{F}}(J) f(m)$$

(use $\mathcal{F}^{-1}f$ for f)

13-2

(13.4)

The Poisson summation formula is often stated for more general lattices $\mathcal{L}$ of full rank in $\mathbb{R}^n$. Set $\text{covol } \mathcal{L} = \text{vol}(\mathbb{R}^n / \mathcal{L})$. Let $\mathcal{L}^* = \{ \gamma \in \mathbb{R}^n : x \cdot \gamma \in \mathbb{Z} \ \forall x \in \mathcal{L} \}$ denote the dual lattice of $\mathcal{L}$.

Thm 13.1' Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\sum_{x \in \mathcal{L}} f(x) = (\text{covol } \mathcal{L})^{-1} \sum_{\gamma \in \mathcal{L}^*} \hat{f}(\gamma) \quad (13.5)$$

Proof: $\exists a \in GL(n, \mathbb{R})$ s.t. $\mathcal{L} = a \mathbb{Z}^n$

(take $a = (b_1 \dots b_n)$ where b_i form a basis of $\mathcal{L}$), and $\text{covol } \mathcal{L} = |\det a|$. Then

$$\mathcal{L}^* = \{ \gamma \in \mathbb{R}^n : \underbrace{a m \cdot \gamma}_{m \cdot a^T \gamma} \in \mathbb{Z} \ \forall m \in \mathbb{Z}^n \}$$

$$= a^{-T} \{ \gamma \in \mathbb{R}^n : m \cdot \gamma \in \mathbb{Z} \ \forall m \in \mathbb{Z}^n \}$$

$$= a^{-T} \mathbb{Z}^n.$$

Now use $f(x) = f_0(ax)$ as test function in Theorem 13.1 and note that

$$\begin{aligned}\hat{f}(\gamma) &= \int f_0(ax) e(-x \cdot \gamma) dx \\ &= \frac{1}{|\det a|} \int f_0(x) e(-a^{-1}x \cdot \gamma) dx \\ &= \frac{1}{|\det a|} \hat{f}_0({}^t a^{-1} \gamma).\end{aligned}$$



Take $g = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}$, then (13.5) can be restated as

$$\sum_{u \in \mathbb{Z}^n} \tilde{\mathcal{R}}(g) f(u) = i^n \sum_{u \in \mathbb{Z}^n} \tilde{\mathcal{R}}({}^t g^{-1}) \tilde{\mathcal{R}}(J^{-1}) f(u)$$

(13.6)

with a quick proof: apply (13.4) with $f(x) = \tilde{\mathcal{R}}(g) f_0(x)$ and note that

$$\tilde{\mathcal{R}}(J^{-1}) \tilde{\mathcal{R}}(g) = \tilde{\mathcal{R}}(\underbrace{J^{-1} g J}_{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} {}^t a^{-1} & 0 \\ 0 & a \end{pmatrix}})$$

What about general g in (13.6)?

Representation - theoretic approach

(V, σ) symplectic vector space

L Lagrangian subspace

Λ Self-dual lattice^{*} in (V, σ) such
that $\text{span}(\Lambda \cap L) = L$

* The dual of Λ is

$\Lambda^* = \{ v \in V : \sigma(v, w) \in \mathbb{Z} \ \forall w \in \Lambda \}$
and Λ is self-dual if $\Lambda^* = \Lambda$.

Lemma:

$\exists$ symplectic basis $(b_1, \dots, b_n, \tilde{b}_1, \dots, \tilde{b}_n)$
such that $L = \bigoplus_{j=1}^n \mathbb{R} b_j \simeq \mathbb{R}^n \times \{0\}$
 $\Lambda = \bigoplus_{j=1}^n \mathbb{Z} b_j + \bigoplus_{j=1}^n \mathbb{Z} \tilde{b}_j \simeq \mathbb{Z}^n \times \mathbb{Z}^n$

Proof: Exercise.

Can assume in the following w.l.o.g.

$$L = \mathbb{R}^n \times \{0\}, \quad \Lambda = \mathbb{Z}^{2n}$$

We now construct a representation of $\mathrm{IH}(V)$ induced by a character χ of the subgroup

$$\Lambda = \Lambda \oplus \mathbb{R}e_0 \subset \mathrm{IH}(V)$$

(recall the construction via $\mathbb{L} = L \oplus \mathbb{R}e_0$)

We assume as before that $\chi(z, t) = e(t)$.

Better to work mod $\mathbb{1}$ in e_0 -component:

Let $\overline{\mathrm{IH}}(V) = \mathrm{IH}(V)/\mathfrak{Z}(\mathbb{Z})$, $\overline{\Lambda} = \Lambda/\mathfrak{Z}(\mathbb{Z})$,
with $\mathfrak{Z}(\mathbb{Z}) = \{0\} \oplus \mathbb{Z}e_0 = \{(0, t) : t \in \mathbb{Z}\}$.

Thm 13.2

(i) $\overline{\Lambda}$ is an abelian subgroup of $\overline{\mathrm{IH}}(V)$

(ii) $\mathrm{IH}(V)/\Lambda \cong V/\Lambda \cong \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ a torus

Proof:

(i) $(z_1, t_1)(z_2, t_2)$

$$= (z_2, t_2)(z_1, t_1) \underbrace{(0, \sigma(z_1, z_2))}_{\in \mathbb{Z}}$$

(ii) Clear.

since Λ self-dual 

Key construction: (cf. p. 7-8)

Given self-dual lattice and χ as above, consider functions $F: \mathbb{H} \rightarrow \mathbb{C}$ such that

$$F(gu) = \chi(u)^{-1} F(g) \quad \forall g \in \mathbb{H} \quad \forall u \in \Lambda \quad (13.7)$$

Note that, since $\chi(u) = 1$ for $u \in \mathfrak{u}(\mathbb{Z})$, F can be viewed as a function on $\overline{\mathbb{H}}$.

Define inner product

$$\langle F_1, F_2 \rangle_\Lambda = \int_{\mathbb{H}/\Lambda} \overline{F_1(g)} F_2(g) d\mu_{\mathbb{H}/\Lambda}(g)$$

where $\mu_{\mathbb{H}/\Lambda}$ is the Haar probability measure of $\mathbb{H}/\Lambda \cong \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. (13.8)

$\mathcal{H}_\Lambda :=$ Hilbert space of F satisfying (13.7)

and $\|F\|_\Lambda := \sqrt{\langle F, F \rangle_\Lambda} < \infty$.

Now define the "lattice" representation

ω_Λ of $\mathbb{H}(\nu)$ on $\mathcal{H}_\Lambda$ by

$$\omega_\Lambda(u) F(g) = F(u^{-1}g).$$

This is once more an induced representation,

$$\omega_\Lambda = \text{Ind}_{\Lambda}^{\mathbb{H}} \chi \quad (\text{cf. p. 7-9})$$

$\uparrow$
viewed as a character of Λ

Thm 13.3

ω_Λ is unitarily equivalent to ω_L .

Proof: The plan is to construct an isometry

$$\mathcal{F}_{L,\Lambda}: \mathcal{H}_L \rightarrow \mathcal{H}_\Lambda$$

which intertwines ω_L and ω_Λ .

For $F \in \mathcal{H}_L$ set

$$\mathcal{F}_{L,\Lambda} F(g) = \sum_{h \in \mathbb{H}/\Lambda \cap L} \chi(h) F(gh) \quad (13.8)$$

and note

$$\begin{aligned} \text{(i)} \quad \omega_\Lambda(h) \mathcal{F}_{L,\Lambda} F(g) &= \mathcal{F}_{L,\Lambda} F(h^{-1}g) \\ &= \sum_{h_0 \in \mathbb{H}/\Lambda \cap L} \chi(h_0) F(h^{-1}g h_0) \\ &= \mathcal{F}_{L,\Lambda} \omega_L(h) F(g) \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \mathcal{F}_{L,\Lambda} F(gh_0) &= \sum_{h \in \mathbb{H}/\Lambda \cap L} \chi(h) F(gh_0 h) \\ &= \sum_{h \in \mathbb{H}/\Lambda \cap L} \chi(h_0^{-1}h) F(gh) \\ &= \chi(h_0)^{-1} \mathcal{F}_{L,\Lambda} F(g) \end{aligned}$$

so (13.7) holds.

For $g = \left(\begin{pmatrix} x \\ y \end{pmatrix}, 0 \right)$ and $h = \left(\begin{pmatrix} 0 \\ u \end{pmatrix}, 0 \right)$

$$\mathcal{F}_{L,\lambda} F \left(\begin{pmatrix} x \\ y \end{pmatrix}, 0 \right) = \sum_{m \in \mathbb{Z}^n} F \left(\underbrace{\left(\begin{pmatrix} x \\ y \end{pmatrix}, 0 \right) \left(\begin{pmatrix} 0 \\ u \end{pmatrix}, 0 \right)}_{\left(\begin{pmatrix} 0 \\ y+u \end{pmatrix}, 0 \right) \left(\begin{pmatrix} x \\ 0 \end{pmatrix}, \frac{1}{2}xy + u \cdot x} \right)$$

$$= \sum_m e(-u \cdot x - \frac{1}{2}x \cdot y) F \left(\underbrace{\begin{pmatrix} 0 \\ y+u \end{pmatrix}, 0}_{=: f(y+u)} \right) \quad (13.9)$$

and therefore

$$\langle \mathcal{F}_{L,\lambda} F_1, \mathcal{F}_{L,\lambda} F_2 \rangle_\lambda$$

$$= \int_{\mathbb{H}/\Lambda} \sum_{h_1, h_2} \overline{\chi}(h_1) \chi(h_2) \overline{F_1}(gh_1) F_2(gh_2) d\mu(g)$$

$$= \int_{[0,1)^{2n}} \sum_{u_1, u_2} e(\underbrace{(u_1 - u_2) \cdot x}_{\Rightarrow u_1 = u_2}) \overline{f_1}(y + u_1) f(y + u_2) dx dy$$

$$= \int_{[0,1)^n} \sum_{u_1} \overline{f_1}(y + u_1) f_2(y + u_2) dy$$

$$= \int_{\mathbb{R}^n} \overline{f_1}(y) f_2(y) dy$$

$$= \langle F_1, F_2 \rangle_{\mathcal{H}_L}$$

This proves $\mathcal{F}_{L,\lambda}$ is an isometry.

To construct the inverse of $\mathcal{F}_{L,\Lambda}$ set $\chi(h)$ normalised as probability measure

$$\mathcal{F}_{\Lambda,L} F(g) = \int_{\mathbb{H}/L \cap \Lambda} \chi(h) F(gh) d\nu_L(h)$$

By construction

(13.10)

$$\mathcal{F}_{\Lambda,L} F(gh) = \chi(h)^{-1} \mathcal{F}_{\Lambda,L} F(g).$$

We leave the proof of

$$\mathcal{F}_{L,\Lambda} \circ \mathcal{F}_{\Lambda,L} = \text{id}$$

(13.11)

as an exercise.



Solution: For $F \in \mathcal{X}_\Lambda$, $g = ((\begin{smallmatrix} x \\ y \end{smallmatrix}), 0)$,

$$\mathcal{F}_{L,\Lambda} \circ \mathcal{F}_{\Lambda,L} F(g) = \sum_{h \in \mathbb{H}/\Lambda \cap L} \chi(h) \mathcal{F}_{\Lambda,L} F(gh)$$

$$h = ((\begin{smallmatrix} 0 \\ u \end{smallmatrix}), 0)$$

$$h_0 = ((\begin{smallmatrix} p \\ 0 \end{smallmatrix}), 0)$$

$$= \sum_{h \in \mathbb{H}/\Lambda \cap L} \int_{\mathbb{H}/L \cap \Lambda} \chi(h) \chi(h_0) F(ghh_0) d\nu_L(h_0)$$

$$= \sum_{u \in \mathbb{Z}^n} \int_{\mathbb{R}^n/\mathbb{Z}^n} \chi((\begin{smallmatrix} 0 \\ u \end{smallmatrix}), 0) \chi((\begin{smallmatrix} p \\ 0 \end{smallmatrix}), 0) F(\underbrace{((\begin{smallmatrix} x \\ y \end{smallmatrix}), 0)(\begin{smallmatrix} 0 \\ u \end{smallmatrix}), 0)(\begin{smallmatrix} p \\ 0 \end{smallmatrix}), 0)}_{((\begin{smallmatrix} x \\ y \end{smallmatrix}), 0)(\begin{smallmatrix} p \\ 0 \end{smallmatrix}), 0)(\begin{smallmatrix} 0 \\ u \end{smallmatrix}), -u \cdot p)} dp$$

$$= \sum_{u \in \mathbb{Z}^n} \int_{\mathbb{R}^n/\mathbb{Z}^n} e(u \cdot p) F(\underbrace{((\begin{smallmatrix} x \\ y \end{smallmatrix}), 0)(\begin{smallmatrix} p \\ 0 \end{smallmatrix}), 0)}_{=: a(p) \text{ p\u0119\u017ci\u0105\u0107 w i p}} dp$$

$$= \sum_{u \in \mathbb{Z}^n} \int_{\mathbb{R}^n/\mathbb{Z}^n} e(u \cdot p) a(p) dp$$

$$= a(0) = F((\begin{smallmatrix} x \\ y \end{smallmatrix}), 0).$$



Thm 13.4

Assume $(L, \tilde{L})$ are transversal Lagrangian planes ($L \cong \mathbb{R}^n \times \{0\}$, $\tilde{L} \cong \{0\} \times \mathbb{R}^n$)

Then $\mathcal{F}_{\Lambda, \tilde{L}} \circ \mathcal{F}_{L, \Lambda} = \mathcal{F}_{L, \tilde{L}}$.

In fact one can show $\mathcal{F}_{\Lambda, L_2} \circ \mathcal{F}_{L_1, \Lambda} = \mathcal{F}_{L_1, L_2}$.

Proof: $\forall F \in \mathcal{H}_L$

$$\mathcal{F}_{\Lambda, \tilde{L}} \circ \mathcal{F}_{L, \Lambda} F(g) = \int_{\tilde{L}/\tilde{L} \cap \Lambda} \chi(h) \mathcal{F}_{L, \Lambda} F(g h) dv_{\tilde{L}}(h)$$

$$g = \left(\begin{pmatrix} x \\ 0 \end{pmatrix}, 0 \right) \quad h = \left(\begin{pmatrix} 0 \\ p \end{pmatrix}, 0 \right) \\ h_0 = \left(\begin{pmatrix} 0 \\ m \end{pmatrix}, 0 \right)$$

$$= \int_{\tilde{L}/\tilde{L} \cap \Lambda} \sum_{h_0 \in \Lambda/\Lambda \cap \tilde{L}} \chi(h) \chi(h_0) F(g h h_0) dv_{\tilde{L}}(h)$$

$$= \int_{\mathbb{R}^n/\mathbb{Z}^n} \sum_{m \in \mathbb{Z}^n} F\left(\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, 0\right) \left(\begin{pmatrix} 0 \\ p+m \end{pmatrix}, 0\right)\right) dp$$

$$= \int_{\mathbb{R}^n} F\left(\left(\begin{pmatrix} x \\ 0 \end{pmatrix}, 0\right) \left(\begin{pmatrix} 0 \\ y \end{pmatrix}, 0\right)\right) dy$$

$$= \mathcal{F}_{L, \tilde{L}} F(g), \quad \text{cf. p. 10-5.}$$



Hence $\mathcal{F}_{L, \Lambda} = \mathcal{F}_{\tilde{L}, \Lambda} \cdot \mathcal{F}_{L, \tilde{L}}$ and the

The Poisson summation formula is equivalent to

$$\mathcal{F}_{L, \Lambda} F(c) = \mathcal{F}_{\tilde{L}, \Lambda} \cdot \mathcal{F}_{L, \tilde{L}} F(c)$$

§ 14. Theta functions

For $f \in \mathcal{S}(\mathbb{R}^n)$, define the theta function

$$\Theta_f : \mathrm{Sp}(n, \mathbb{R}) \times \mathbb{H}_n(\mathbb{R}) \rightarrow \mathbb{C}$$

by

$$\Theta_f(g, h) = \sum_{m \in \mathbb{Z}^n} \tilde{\omega}(h) \tilde{\rho}(g) f(m)$$

For the special choice $g = \begin{pmatrix} a & b \\ 0 & {}^t a^{-1} \end{pmatrix}$
and $h = \left(\begin{pmatrix} q \\ p \end{pmatrix}, t \right)$ (cf. Thm 9.3 and (9.8))

$$\Theta_f(g, h) = \sum_{m \in \mathbb{Z}^n} \tilde{\rho}(g) f(m-p) e(q \cdot m + t - \frac{1}{2} p \cdot q)$$

$$= (\det a)^{1/2} \sum_{m \in \mathbb{Z}^n} f({}^t a(m-p)) *$$

$$* e\left(\frac{1}{2} (m-p) \cdot b {}^t a(m-p) + q \cdot m + t - \frac{1}{2} p \cdot q\right)$$

$\Rightarrow$ a quadratic exponential sum

"theta function/series/sum"

"Gauss sum" (usually for rational argument)

Example:

$$\text{Take } f(y) = e^{-\pi \|y\|^2}$$

$$g = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix}$$

$$u, v \in M_{n \times n}^{\text{symm}}(\mathbb{R}), v \text{ pos. definite}$$

$$h = \left(\begin{pmatrix} x \\ 0 \end{pmatrix}, 0 \right)$$

Then (check!)

$$\tau = u + i v$$

$$\oplus_f(g, h)$$

$$= (\det v)^{1/4} \underbrace{\sum_{m \in \mathbb{Z}^n} e\left(\frac{1}{2} m \cdot \tau m + m \cdot x\right)}_{\theta(x|\tau)}$$

"Riemann theta function"

"Siegel theta series"

Transformation formulas

Thm 14.1

$$(i) \quad \oplus_f(\mathcal{J}h, \mathcal{J}g) = i^n c(\mathcal{J}, g)^{-1} \oplus_f(h, g)$$

$$(ii) \quad \text{For } \gamma = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \in M_{n \times n}^{\text{symm}} (2\mathbb{Z})$$

$$\oplus_f(\gamma h, \gamma g) = c(\gamma, g)^{-1} \oplus_f(g, h)$$

$$(iii) \quad \text{For } \eta = \left(\begin{pmatrix} r \\ s \end{pmatrix}, 0 \right) \in \Lambda$$

$$\oplus_f(\eta h, g) = e\left(\frac{1}{2} r \cdot s\right) \oplus_f(h, g)$$

Proof

$$(i) \quad \text{By (13.4)}$$

$$\sum_{u \in \mathbb{Z}^n} \tilde{\omega}(u) \tilde{R}(g) f(u)$$

$$= i^{-n} \sum_{u \in \mathbb{Z}^n} \tilde{R}(\mathcal{J}) \underbrace{\tilde{\omega}(u) \tilde{R}(g)}_{= \tilde{\omega}(\mathcal{J}u) \tilde{R}(\mathcal{J})} f(u)$$

$$= \tilde{\omega}(\mathcal{J}u) \tilde{R}(\mathcal{J})$$

$$= i^{-n} c(\mathcal{J}, g) \sum_{u \in \mathbb{Z}} \tilde{\omega}(\mathcal{J}u) \tilde{R}(\mathcal{J}g) f(u).$$

$$\begin{aligned}
 \text{(ii)} \quad \sum_{u \in \mathbb{Z}^n} \tilde{R}\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) f(u) \\
 = \sum_{u \in \mathbb{Z}^n} f(u) e\left(\frac{1}{2} u \cdot b u\right)
 \end{aligned}$$

$= 1$ for b even

Then proceed as in (i), using

$$\begin{aligned}
 \tilde{R}(r) \tilde{\omega}(u) \tilde{R}(g) &= \tilde{\omega}(ru) \tilde{R}(r) \tilde{R}(g) \\
 &= c(rg) \tilde{\omega}(ru) \tilde{R}(rg)
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad \sum_{u \in \mathbb{Z}^n} \tilde{\omega}\left(\begin{pmatrix} r \\ 0 \end{pmatrix}, 0\right) f(u) \\
 = \sum_{u \in \mathbb{Z}^n} f(u-s) e\left(\underbrace{u \cdot r}_{\in \mathbb{Z}} - \frac{1}{2} rs\right) \\
 = \sum_{u \in \mathbb{Z}^n} f(u) e\left(\frac{1}{2} rs\right)
 \end{aligned}$$

Now replace in this relation f by $\tilde{R}(g) f$ and the claim follows. ◻

For further reading, see the book by Lion & Vergne, and my papers with M. Welsh in Journal d'Analyse Math. (2023).