Proceedings of the XIIIth International Congress on Mathematical Physics, London 2000 (International Press, Boston, 2001), pp. 359-363

Level spacing statistics and integrable dynamics

Jens Marklof

Abstract. Level spacing statistics of quantum systems, which have a completely integrable classical limit, are expected to follow locally the statistics of a Poisson process, according to a conjecture of Berry and Tabor. I will report on a recent proof of this fact in the case of two-point statistics of a ring threaded by Aharonov-Bohm flux lines.

1. Introduction

The theory of quantum chaos is concerned with quantum systems which possess a classical limit. When the classical dynamics is chaotic, one finds the level spacing statistics typically follow those of suitable random matrix ensembles [**3**]. If, in contrast, the classical dynamics is completely integrable the statistics can in general be modeled by a Poisson process [**1**]. Although these observations are supported by overwhelming numerical evidence, only a few rigorous results are available, mostly in the integrable case. For recent reviews on the state-of-the-art, the reader is referred to [**12, 2, 8, 9**]. Here I will discuss a family of integrable systems previously studied in [**4, 5**] and [**10**], for which the connection to the Poisson model can be understood rigorously in the case of two-point statistics.

The eigenvalues studied in [**4, 5, 10**] are of the form $\lambda_j = (m-\alpha)^2 + (n-\beta)^2$, where α, β are constants and m, n run over the integers. Let us here focus on the special case when $\beta = 0$. The λ_j can then be interpreted as the energy eigenvalues (in suitable units) of a quantum particle constrained to a cylindrical ring with length π and radius one, which is threaded by an Aharonov-Bohm flux α. More precisely,

$$\lambda_j = (m-\alpha)^2 + n^2$$

where $m \in \mathbb{Z}$ and $n = 1, 2, 3, \ldots$, if we assume Dirichlet boundary conditions on the cylinder's rim.

2. Pair correlation

The mean density D of the sequence of λ_j is clearly

$$D := \lim_{\lambda\to\infty} \frac{1}{\lambda} \#\{j : \lambda_j \leq \lambda\} = \frac{\pi}{2}.$$

(Recall: the number of lattice points in a semicircle of radius $\sqrt{\lambda}$ is asymptotically $\pi\lambda/2$.)

The pair correlation function of the eigenvalue sequence λ_j is now defined as

$$R_2[a,b](\lambda) = \frac{1}{D\lambda} \#\{j \neq k : \lambda_j \leq \lambda,\ \lambda_k \leq \lambda,\ a \leq \lambda_k - \lambda_j \leq b\}.$$

It is well known that if the λ_j come from a Poisson process with mean density D, one has

$$\lim_{\lambda\to\infty} R_2[a,b](\lambda) = D(b-a)$$

almost surely. In the case, when the λ_j are the energy levels defined above, we have the following results, cf. [**10**].

We shall call α *diophantine* if there exist constants $\kappa, C > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^\kappa}$$

for all $p, q \in \mathbb{Z}$. The smallest possible value of κ is $\kappa = 2$. We will say α is of *type* κ.

Theorem 1 ([**10**], Theorem A.10). *Assume α is diophantine. Then*

$$\lim_{\lambda\to\infty} R_2[a,b](\lambda) = \frac{\pi}{2}(b-a).$$

This is clearly in accordance with the Poisson model. In the case of rational values of α the spectrum is highly degenerate. One has

$$R_2[-a,a](\lambda) \sim c_\alpha \log\lambda \qquad (\lambda\to\infty)$$

for any $a > 0$, and some constant c_α depending only on α. This in turn can be used to show that the previous theorem is in fact wrong for topologically generic α:

Theorem 2. *For any $a > 0$, there exists a set $C \subset [0,1]$ of second Baire category, for which the following holds.*[1]

(i) *For $\alpha \in C$, we find arbitrarily large λ such that*

$$R_2[-a,a](\lambda) \geq \frac{\log\lambda}{\log\log\log\lambda}.$$

(ii) *For $\alpha \in C$, there exists an infinite sequence $L_1 < L_2 < \cdots \to \infty$ such that*

$$\lim_{j\to\infty} R_2[-a,a](L_j) = \pi a.$$

Thus the diophantine conditions in Theorem 1 are indeed necessary. Part (i) and (ii) of Theorem 2 follow from the logarithmic divergence at rational α and from Theorem 1, respectively, by a typical Baire-category argument, see Section 8 in [**10**]. The key to Theorem 1 is the value distribution of Jacobi theta sums, see next section.

Theorems 1 and 2 illustrate the subtle dependence of spectral correlations on the choice of parameter: While almost all values (in measure) lead to the expected answer, topologically generic choices do not. This remarkable fact had been pointed out first by Sarnak [**11**] in the case of flat tori, where he established convergence to Poisson for almost all flat tori. His result has recently been improved by Eskin, Margulis and Mozes [**6**], who characterized all "good" tori by diophantine conditions.

[1] A set of first Baire category is a countable union of nowhere dense sets. Sets of second category are all those sets, which are not of first category.

Triple and higher correlations are presently much less well understood. A number of results on higher-dimensional flat tori are due to VanderKam [**15**], where the increased dimension of the moduli space facilitates the averaging, very much in the spirit of ideas of Sinai [**14**] and Major [**7**].

3. Spectral form factors and Jacobi theta sums

The spectral form factor

$$K_2(t,\lambda)=\frac{1}{D\lambda}\Big|\sum_{j:\lambda_j\le\lambda}\mathrm{e}^{2\pi\mathrm{i}\lambda_j t}\Big|^2$$

is essentially the Fourier transform of the pair correlation density. We have

$$R_2[a,b](\lambda)=\int_{-\infty}^{\infty}K_2(t,\lambda)\hat\chi_{[a,b]}(t)\,dt-\chi_{[a,b]}(0)+o(1)$$

where $\hat\chi_{[a,b]}$ is the Fourier transform of the characteristic function $\chi_{[a,b]}$ of the interval $[a,b]$. Convergence problems may be avoided by smoothing $\chi_{[a,b]}$ slightly.

In the case when the eigenvalues λ_j are given by values at integers of quadratic forms, the exponential sum defining $K_2(t,\lambda)$ is a theta sum. In the case discussed here, it is a variant of Jacobi's theta sum, namely

$$\Theta_f(\tau,\phi|\boldsymbol{\xi})=v^{1/2}\sum_{(m,n)\in\mathbb{Z}^2}f_\phi((m-y)v^{1/2},nv^{1/2})e(\tfrac12(m-y)^2u+\tfrac12 n^2u+mx),$$

with $\tau=u+\mathrm{i}v$, $\boldsymbol{\xi}=\begin{pmatrix}x\\y\end{pmatrix}$ and $f_\phi=U^\phi f$, where U^ϕ is a certain one-parameter-group of unitary operators ($U^0=\mathrm{id}$) acting on smooth functions $f\in\mathrm{L}^2(\mathbb{R}^2)$, see [**10**], Section 3 for details. One finds that

$$K_2(t,\lambda)=\frac{1}{4D}|\Theta_f(\tau,\phi|\boldsymbol{\xi})|^2+O(\lambda^{-1})$$

for $u=2t$, $v=\lambda^{-1}$, $\phi=0$ and $(x,y)=(0,\alpha)$. The function f is set to be a (smoothed) characteristic function, defining the energy window for the λ_j. f may for instance be taken as $f(\omega,w)=\chi_{[0,1]}(\omega^2+w^2)$, so that $0\le\lambda_j/\lambda\le1$.

The crucial idea is now that the function $|\Theta_f(\tau,\phi|\boldsymbol{\xi})|^2$ can be identified with a function on a quotient manifold $M=\Gamma\backslash(\mathrm{SL}(2,\mathbb{R})\ltimes\mathbb{R}^2)$, with Γ a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})\ltimes\mathbb{R}^2$. The manifold M is non-compact but has finite volume with respect to Haar measure. Furthermore the average

$$\int|\Theta_f(2t+\mathrm{i}v,\phi|\boldsymbol{\xi})|^2\hat\chi(t)\,dt=\frac12\int|\Theta_f(u+\mathrm{i}v,\phi|\boldsymbol{\xi})|^2\hat\chi_{[a,b]}(\frac{u}{2})\,du$$

is an average along a unipotent orbit, which is expanding as $v=\lambda^{-1}\to0$. Following Ratner's classification of measures invariant under unipotent flows, it can be shown that the orbit becomes equidistributed on M with respect to Haar measure, as long as α is irrational [**13, 10**].

The equidistribution theorem must not, however, be applied directly in our situation since $|\Theta_f(u+\mathrm{i}v,\phi|\boldsymbol{\xi})|^2$ is unbounded, diverging in the cusp at infinity as

$$|\Theta_f(u+\mathrm{i}v,\phi|\boldsymbol{\xi})|^2\sim v|f_\phi(-yv^{1/2},0)|^2\qquad(v\to\infty)$$

uniformly for $y\in[-\frac12,\frac12]$. Compare [**10**], Proposition 3.13.

In fact, a small arc of the orbit in the neighbourhood of $u = 0$, which runs into the cusp, gives a non-vanishing contribution; one can show ([**10**], Lemma 7.3) that for any fixed $\epsilon > 0$,

$$\int_{|u|<v^{1-\epsilon}} |\Theta_f(u+\mathrm{i}v,\phi|\boldsymbol{\xi})|^2 \hat{\chi}(\frac{u}{2})\,du \to 2\pi^2 \hat{\chi}_{[a,b]}(0) = 2\pi^2(b-a),$$

as $v \to 0$. The remaining part of the orbit $|u| > v^{1-\epsilon}$ becomes equidistributed, under the condition that α is diophantine ([**10**], Theorem 6.3).[2] We find, as $v \to 0$,

$$\int_{|u|>v^{1-\epsilon}} |\Theta_f(u+\mathrm{i}v,\phi|\boldsymbol{\xi})|^2 \hat{\chi}(\frac{u}{2})\,du \to \frac{1}{\mathrm{vol}(M)} \int_M |\Theta_f|^2 d\mu \int \hat{\chi}_{[a,b]}(\frac{u}{2})\,du.$$

Analogous to the proof of Lemma A.8 in [**10**], one can work out

$$\frac{1}{\mathrm{vol}(M)} \int_M |\Theta_f|^2 d\mu = 2\pi$$

and obviously

$$\int \hat{\chi}_{[a,b]}(\frac{u}{2})\,du = 2\chi_{[a,b]}(0).$$

Collecting all contributions, we therefore have (recall $D = \pi/2$)

$$\int K_2(t,\lambda)\hat{\chi}_{[a,b]}(t)\,dt \to \frac{1}{8D}\big(2\pi^2(b-a) + 4\pi\chi_{[a,b]}(0)\big) = \frac{\pi}{2}(b-a) + \chi_{[a,b]}(0)$$

as $\lambda \to \infty$. Hence

$$\lim_{\lambda\to\infty} R_2[a,b](\lambda) = \frac{\pi}{2}(b-a)$$

as claimed in Theorem 1.

References

[1] M.V. Berry and M. Tabor, Level clustering in the regular spectrum, *Proc. Roy. Soc.* A **356** (1977) 375-394.

[2] P.M. Bleher, Trace formula for quantum integrable systems, lattice point problem, and small divisors, in: D. Hejhal et al. (eds.), *Emerging Applications of Number Theory*, IMA Volumes in Mathematics and its Applications, Vol. 109 (Springer, New York, 1999) pp. 1-38.

[3] O. Bohigas, M.-J. Giannoni and C. Schmit, Characterization of chaotic quantum spectra and universality of level fluctuation laws, *Phys. Rev. Lett.* **52** (1984) 1-4.

[4] Z. Cheng and J.L. Lebowitz, Statistics of energy levels in integrable quantum systems, *Phys. Rev.* A **44** (1991) 3399-3402.

[5] Z. Cheng, J.L. Lebowitz and P. Major, On the number of lattice points between two enlarged and randomly shifted copies of an oval, *Probab. Theory Related Fields* **100** (1994) 253-268.

[6] A. Eskin, G. Margulis and S. Mozes, Quadratic forms of signature (2,2) and eigenvalue spacings on rectangular 2-tori, preprint.

[7] P. Major, Poisson law for the number of lattice points in a random strip with finite area, *Prob. Theo. Rel. Fields* **92** (1992) 423-464.

[8] J. Marklof, Spectral form factors of rectangle billiards, *Comm. Math. Phys.* **199** (1998) 169-202.

[9] J. Marklof, The Berry-Tabor conjecture, *Proceedings of the 3rd European Congress of Mathematics*, Barcelona 2000 (Birkhäuser, to appear).

[10] J. Marklof, Pair correlation densities of inhomogeneous quadratic forms, preprint 2000.

[11] P. Sarnak, Values at integers of binary quadratic forms, *Harmonic Analysis and Number Theory* (Montreal, PQ, 1996), 181-203, CMS Conf. Proc. **21**, Amer. Math. Soc., Providence, RI, 1997.

[2] If α is not diophantine, there will be subsequences of v along which small arcs of the orbit gain too much weight in the cusp when integrated over the unbounded theta sum; this results in the divergence observed in Theorem 2.

[12] P. Sarnak, Quantum chaos, symmetry and zeta functions. Lecture I: Quantum chaos, *Curr. Dev. Math.* (1997) 84-101.

[13] N.A. Shah, Limit distributions of expanding translates of certain orbits on homogeneous spaces, *Proc. Indian Acad. Sci., Math. Sci.* **106** (1996) 105-125.

[14] Ya.G. Sinai, Poisson distribution in a geometrical problem, *Adv. Sov. Math., AMS Publ.* **3** (1991) 199-215.

[15] J.M. VanderKam, Correlations of eigenvalues on multi-dimensional flat tori, *Comm. Math. Phys.* **210** (2000) 203-223.

School of Mathematics, University of Bristol, Bristol BS8 1TW, U.K.
E-mail address: j.marklof@bristol.ac.uk