Level spacing statistics and integrable dynamics

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Abstract. Level spacing statistics of quantum systems, which have a completely integrable classical limit, are expected to follow locally the statistics of a Poisson process, according to a conjecture of Berry and Tabor. I will report on a recent proof of this fact in the case of two-point statistics of a ring threaded by Aharonov-Bohm flux lines.

1. Introduction

The theory of quantum chaos is concerned with quantum systems which possess a classical limit. When the classical dynamics is chaotic, one finds the level spacing statistics typically follow those of suitable random matrix ensembles [3]. If, in contrast, the classical dynamics is completely integrable the statistics can in general be modeled by a Poisson process [1]. Although these observations are supported by overwhelming numerical evidence, only a few rigorous results are available, mostly in the integrable case. For recent reviews on the state-of-the-art, the reader is referred to [12, 2, 8, 9]. Here I will discuss a family of integrable systems previously studied in [4, 5] and [10], for which the connection to the Poisson model can be understood rigorously in the case of two-point statistics.

The eigenvalues studied in [4, 5, 10] are of the form \( \lambda_j = (m-\alpha)^2 + (n-\beta)^2 \), where \( \alpha, \beta \) are constants and \( m, n \) run over the integers. Let us here focus on the special case when \( \beta = 0 \). The \( \lambda_j \) can then be interpreted as the energy eigenvalues (in suitable units) of a quantum particle constrained to a cylindrical ring with length \( \pi \) and radius one, which is threaded by an Aharonov-Bohm flux \( \alpha \). More precisely,

\[
\lambda_j = (m-\alpha)^2 + n^2
\]

where \( m \in \mathbb{Z} \) and \( n = 1, 2, 3, \ldots \), if we assume Dirichlet boundary conditions on the cylinder’s rim.

2. Pair correlation

The mean density \( D \) of the sequence of \( \lambda_j \) is clearly

\[
D := \lim_{\lambda \to \infty} \frac{1}{\lambda} \# \{ j : \lambda_j \leq \lambda \} = \frac{\pi}{2}.
\]

(Recall: the number of lattice points in a semicircle of radius \( \sqrt{\lambda} \) is asymptotically \( \pi \lambda/2 \).)
The pair correlation function of the eigenvalue sequence $\lambda_j$ is now defined as

$$R_2[a, b](\lambda) = \frac{1}{D\lambda} \# \{j \neq k : \lambda_j \leq \lambda, \lambda_k \leq \lambda, a \leq \lambda_k - \lambda_j \leq b\}.$$  

It is well known that if the $\lambda_j$ come from a Poisson process with mean density $D$, one has

$$\lim_{\lambda \to \infty} R_2[a, b](\lambda) = D(b - a)$$

almost surely. In the case, when the $\lambda_j$ are the energy levels defined above, we have the following results, cf. [10].

We shall call $\alpha$ diophantine if there exist constants $\kappa, C > 0$ such that

$$|\alpha - \frac{p}{q}| > \frac{C}{q^\kappa}$$

for all $p, q \in \mathbb{Z}$. The smallest possible value of $\kappa$ is $\kappa = 2$. We will say $\alpha$ is of type $\kappa$.

**Theorem 1 ([10], Theorem A.10).** Assume $\alpha$ is diophantine. Then

$$\lim_{\lambda \to \infty} R_2[a, b](\lambda) = \frac{\pi}{2} (b - a).$$

This is clearly in accordance with the Poisson model. In the case of rational values of $\alpha$ the spectrum is highly degenerate. One has

$$R_2[-a, a](\lambda) \sim c_\alpha \log \lambda \quad (\lambda \to \infty)$$

for any $a > 0$, and some constant $c_\alpha$ depending only on $\alpha$. This in turn can be used to show that the previous theorem is in fact wrong for topologically generic $\alpha$:

**Theorem 2.** For any $a > 0$, there exists a set $C \subset [0, 1]$ of second Baire category, for which the following holds.\(^1\)

(i) For $\alpha \in C$, we find arbitrarily large $\lambda$ such that

$$R_2[-a, a](\lambda) \geq \frac{\log \lambda}{\log \log \log \lambda}.$$  

(ii) For $\alpha \in C$, there exists an infinite sequence $L_1 < L_2 < \cdots \to \infty$ such that

$$\lim_{j \to \infty} R_2[-a, a](L_j) = \pi a.$$  

Thus the diophantine conditions in Theorem 1 are indeed necessary. Part (i) and (ii) of Theorem 2 follow from the logarithmic divergence at rational $\alpha$ and from Theorem 1, respectively, by a typical Baire-category argument, see Section 8 in [10]. The key to Theorem 1 is the value distribution of Jacobi theta sums, see next section.

Theorems 1 and 2 illustrate the subtle dependence of spectral correlations on the choice of parameter: While almost all values (in measure) lead to the expected answer, topologically generic choices do not. This remarkable fact had been pointed out first by Sarnak [11] in the case of flat tori, where he established convergence to Poisson for almost all flat tori. His result has recently been improved by Eskin, Margulis and Mozes [6], who characterized all “good” tori by diophantine conditions.

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\(^1\)A set of first Baire category is a countable union of nowhere dense sets. Sets of second category are all those sets, which are not of first category.
Triple and higher correlations are presently much less well understood. A number of results on higher-dimensional flat tori are due to VanderKam [15], where the increased dimension of the moduli space facilitates the averaging, very much in the spirit of ideas of Sinai [14] and Major [7].

3. Spectral form factors and Jacobi theta sums

The spectral form factor

\[ K_2(t, \lambda) = \frac{1}{D\lambda} \left| \sum_{j: \lambda_j \leq \lambda} e^{2\pi i \lambda_j t} \right|^2 \]

is essentially the Fourier transform of the pair correlation density. We have

\[ R_2[a, b](\lambda) = \int_{-\infty}^{\infty} K_2(t, \lambda) \hat{\chi}_{[a, b]}(t) \, dt - \chi_{[a, b]}(0) + o(1) \]

where \( \hat{\chi}_{[a, b]} \) is the Fourier transform of the characteristic function \( \chi_{[a, b]} \) of the interval \([a, b]\). Convergence problems may be avoided by smoothing \( \chi_{[a, b]} \) slightly.

In the case when the eigenvalues \( \lambda_j \) are given by values at integers of quadratic forms, the exponential sum defining \( K_2(t, \lambda) \) is a theta sum. In the case discussed here, it is a variant of Jacobi’s theta sum, namely

\[ \Theta_f(\tau, \phi|\xi) = v^{1/2} \sum_{(m, n)\in\mathbb{Z}^2} f_{\phi}(m - y)v^{1/2}, nu^{1/2})e(\frac{1}{2}(m - y)^2u + \frac{1}{2}m^2u + mx), \]

with \( \tau = u + iv \), \( \xi = \begin{pmatrix} x \\ y \end{pmatrix} \) and \( f_{\phi} = U^\phi f \), where \( U^\phi \) is a certain one-parameter-group of unitary operators \((U^0 = \text{id})\) acting on smooth functions \( f \in L^2(\mathbb{R}^2) \), see [10], Section 3 for details. One finds that

\[ K_2(t, \lambda) = \frac{1}{4D} \left| \Theta_f(\tau, \phi|\xi) \right|^2 + O(\lambda^{-1}) \]

for \( u = 2t, v = \lambda^{-1}, \phi = 0 \) and \( (x, y) = (0, \alpha) \). The function \( f \) is set to be a (smoothed) characteristic function, defining the energy window for the \( \lambda_j \). \( f \) may for instance be taken as \( f(\omega, w) = \chi_{[0, 1]}(\omega^2 + w^2) \), so that \( 0 \leq \lambda_j/\lambda \leq 1 \).

The crucial idea is now that the function \( \left| \Theta_f(\tau, \phi|\xi) \right|^2 \) can be identified with a function on a quotient manifold \( M = \Gamma \backslash (\text{SL}(2, \mathbb{R}) \times \mathbb{R}^2) \), with \( \Gamma \) a discrete subgroup of \( \text{SL}(2, \mathbb{R}) \times \mathbb{R}^2 \). The manifold \( M \) is non-compact but has finite volume with respect to Haar measure. Furthermore the average

\[ \int |\Theta_f(2t + iv, \phi|\xi)|^2 \hat{\chi}(t) \, dt = \frac{1}{2} \int |\Theta_f(u + iv, \phi|\xi)|^2 \hat{\chi}_{[a, b]}(\frac{u}{2}) \, du \]

is an average along a unipotent orbit, which is expanding as \( v = \lambda^{-1} \to 0 \). Following Ratner’s classification of measures invariant under unipotent flows, it can be shown that the orbit becomes equidistributed on \( M \) with respect to Haar measure, as long as \( \alpha \) is irrational [13, 10].

The equidistribution theorem must not, however, be applied directly in our situation since \( |\Theta_f(u + iv, \phi|\xi)|^2 \) is unbounded, diverging in the cusp at infinity as

\[ |\Theta_f(u + iv, \phi|\xi)|^2 \sim v|f_{\phi}(-yv^{1/2}, 0)|^2 \quad (v \to \infty) \]

uniformly for \( y \in [-\frac{1}{2}, \frac{1}{2}] \). Compare [10], Proposition 3.13.
In fact, a small arc of the orbit in the neighbourhood of \( u = 0 \), which runs into the cusp, gives a non-vanishing contribution; one can show ([10], Lemma 7.3) that for any fixed \( \epsilon > 0 \),

\[
\int_{|u| < v^{1-\epsilon}} |\Theta_f(u+iv, \phi(\xi)|^2 \hat{\chi}\left(\frac{u}{2}\right) du \to 2\pi^2 \hat{\chi}_{[a,b]}(0) = 2\pi^2 (b-a),
\]
as \( v \to 0 \). The remaining part of the orbit \( |u| > v^{1-\epsilon} \) becomes equidistributed, under the condition that \( \alpha \) is diophantine ([10], Theorem 6.3).\(^2\) We find, as \( v \to 0 \),

\[
\int_{|u| > v^{1-\epsilon}} |\Theta_f(u+iv, \phi(\xi)|^2 \hat{\chi}\left(\frac{u}{2}\right) du \to \frac{1}{\text{vol}(M)} \int_M |\Theta_f|^2 d\mu \int \hat{\chi}_{[a,b]}\left(\frac{u}{2}\right) du.
\]

Analogous to the proof of Lemma A.8 in [10], one can work out

\[
\frac{1}{\text{vol}(M)} \int_M |\Theta_f|^2 d\mu = 2\pi
\]

and obviously

\[
\int \hat{\chi}_{[a,b]}\left(\frac{u}{2}\right) du = 2\chi_{[a,b]}(0).
\]

Collecting all contributions, we therefore have (recall \( D = \pi/2 \))

\[
\int K_2(t, \lambda) \hat{\chi}_{[a,b]}(t) dt \to \frac{1}{8D} \left(2\pi^2 (b-a) + 4\pi \chi_{[a,b]}(0)\right) = \frac{\pi}{2} (b-a) + \chi_{[a,b]}(0)
\]
as \( \lambda \to \infty \). Hence

\[
\lim_{\lambda \to \infty} R_2[a,b](\lambda) = \frac{\pi}{2} (b-a)
\]
as claimed in Theorem 1.

References


\(^2\)If \( \alpha \) is not diophantine, there will be subsequences of \( v \) along which small arcs of the orbit gain too much weight in the cusp when integrated over the unbounded theta sum; this results in the divergence observed in Theorem 2.


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