Random Lattices in the Wild: from Pólya's Orchard to Quantum Oscillators

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Point processes are statistical models that describe the distribution of discrete events in space and time. Applications are everywhere, from galaxies to elementary particles. My aim here is to convince you that there is an exotic but interesting class of point processes — random lattices — that have fascinating connections with various branches of mathematics and some basic models in physics.

So what is a random lattice? First of all, a *lattice* in dimension one is any non-trivial discrete subgroup of the additive group of real numbers $\mathbb{R}$. (*Non-trivial* means anything but the group of one element.) The additive group of integers $\mathbb{Z}$ is an example and, up to rescaling by a constant factor, it is in fact the only example. Now in order to turn $\mathbb{Z}$ into a random object, let us translate $\mathbb{Z}$ by a real number α to obtain the set $\mathcal{S}(\alpha) = \mathbb{Z}+\alpha$, and then view α as a random variable uniformly distributed in the unit interval $[0,1]$. The choice of the unit interval is natural since α and $\alpha+1$ will lead to the same shifted lattice $\mathcal{S}(\alpha)$. With this, $\mathcal{S}(\alpha)$ becomes a random set, which we take (for the purposes of this discussion) to be synonymous with *random point process*. One can check that $\mathcal{S}(\alpha)$ is a translation-stationary random point process, i.e., $\mathcal{S}(\alpha)+t$ has the same distribution as $\mathcal{S}(\alpha)$ for every choice of $t \in \mathbb{R}$ — a simple consequence of the fact that α is assumed to be uniformly distributed in $[0,1]$. A random point process describes the probability of finding k points in a given set B. In the present setting, for B a bounded interval of length $|B|$ and integer $k \geq 0$, we have that

$$\mathbb{P}\Big(|\mathcal{S}(\alpha) \cap B| = k\Big) = \max\Big(1 - \big|k - |B|\big|\,,\, 0\Big).$$

It is not difficult to see that the expected number of points in B is $|B|$, which means that the process has *intensity one* — compare this with the corresponding probabilities for a Poisson process!

The above construction has produced a simple instance of a point process in $\mathbb{R}$. Independent superpositions of one-dimensional randomly shifted lattices explain for example the limiting gap distribution of the fractional parts of the sequence $\log n$, with $n = 1,2,3\ldots$ [14]. But the fun really starts in dimension two!

Poisson process

A homogeneous Poisson process with intensity one in $\mathbb{R}$ can be realised as a sequence of random points where the distances between consecutive points are independent random variables with an exponential distribution. That is, the probability that a gap is larger than s is e^{-s}. It follows that the probability of having k points in the interval B is given by the Poisson distribution

$$\frac{|B|^k}{k!}\,\mathrm{e}^{-|B|}.$$

Two-dimensional random lattices

To construct a two-dimensional random lattice, we begin with the integer lattice $\mathbb{Z}^2$. We could proceed as before and define a random point process in $\mathbb{R}^2$ by shifting $\mathbb{Z}^2$ randomly by a vector $\boldsymbol{\alpha}$, say, uniformly distributed in $[0,1]^2$. This is fine, but there is a more interesting avenue. Unlike in dimension one, we have a non-trivial group of linear volume-preserving transformations acting on $\mathbb{R}^2$. We can use this action, rather than the group of translations as above, to randomise $\mathbb{Z}^2$ and thus produce a two-dimensional random lattice with a fundamental cell of volume one. Here is how it works. We represent elements in $\mathbb{R}^2$ as row vectors $\boldsymbol{x} = (x_1, x_2)$. A linear transformation is then represented by real matrix multiplication from the right,

$$\boldsymbol{x} \mapsto \boldsymbol{x}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax_1 + cx_2, bx_1 + dx_2).$$

Volume is preserved if and only if the determinant has modulus one, that is $|ad - bc| = 1$. We will only need to consider the case where also orientation is preserved, which means $ad - bc = 1$. Such matrices form a group, which we will label as $\mathrm{SL}(2,\mathbb{R})$. L stands for *linear* and S for *special* (referring to the unit determinant). To produce our first example of a random lattice in $\mathbb{R}^2$, consider the sheared lattice

$$\mathscr{P}_1(u) = \mathbb{Z}^2 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

Note that, $\mathscr{P}(u+1) = \mathscr{P}(u)$, and it is therefore natural to consider u as a random variable uniformly distributed on $[0,1]$. This turns $\mathscr{P}(u)$ into a random set, a random point process. A similar construction is possible for the randomly rotated lattice

$$\mathscr{R}_1(\phi) = \mathbb{Z}^2 \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

where ϕ is uniformly distributed in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. It is a fact that any matrix in $M \in \mathrm{SL}(2,\mathbb{R})$ can be uniquely written as a product of a shear, stretch and rotation matrix

$$M = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

where u is real, v is real and positive, and $-\pi < \phi \leq \pi$. This is known as the Iwasawa decomposition of $\mathrm{SL}(2,\mathbb{R})$, and provides a parametrisation of $\mathrm{SL}(2,\mathbb{R})$ in terms of (u,v,ϕ). It follows that any choice of random elements (u,v,ϕ) yields a random lattice $\mathbb{Z}^2 M$. The above examples of randomly sheared or rotated lattices are simply special cases! But is there a particular natural choice of probability measure for (u,v,ϕ) that plays the role of a uniform measure? One could start with u uniformly distributed in $[0,1]$ and ϕ uniformly distributed in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, as above — but what is a natural *uniform* probabilty measure on the positive axis for v? The answer is highly non-trivial, but has a beautiful geometric interpretation. The key to the solution is the modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, where now all matrix coefficients are restricted to integers. It is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ and in fact precisely the subgroup of all $\gamma \in \mathrm{SL}(2,\mathbb{R})$ such that $\mathbb{Z}^2\gamma = \mathbb{Z}^2$. This means that M and γM lead to the same lattice $\mathbb{Z}^2 M$, and we can therefore restrict our attention to only one representative of the coset $\Gamma M = \{\gamma M \mid \gamma \in \Gamma\}$. A convenient set of such representatives is for example given by

$$\mathscr{F} = \Big\{(u,v,\theta) \in \mathbb{R}^3 \mid -\tfrac{1}{2} < u < \tfrac{1}{2}, \quad u^2 + v^2 > 1,\ v > 0,\ -\tfrac{\pi}{2} < \phi < \tfrac{\pi}{2}\Big\}$$

(we should also include about half of the boundary). This set is called a *fundamental domain* of the Γ-action, just as the unit interval is a fundamental domain of the $\mathbb{Z}$-action on $\mathbb{R}$. The most natural *uniform* measure on $\mathscr{F}$ is obtained from the Haar measure of $\mathrm{SL}(2,\mathbb{R})$, restricted to $\mathscr{F}$ and normalised as a probability measure. Explicitly, this Haar probability measure is

$$\mu_{\mathscr{F}} = \frac{3}{\pi^2} \frac{du\, dv\, d\phi}{v^2}.$$

Geometers will have spotted the intriguing similarity with formulas from hyperbolic geometry: The group $\mathrm{SL}(2,\mathbb{R})$ acts on the upper complex halfplane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ by Möbius (fractional linear) transformations

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The Möbius transformation for M as in the Iwasawa decomposition maps i to $u + \mathrm{i}v$, and thus the Möbius action really comes from group multiplication in $\mathrm{SL}(2,\mathbb{R})$. In fact, we can identify $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$ with the unit tangent bundle of the modular surface $\Gamma\backslash\mathbb{H}$, where the angle $\theta = -2\phi$ parametrises the direction of the tangent vector at the point $\tau = u + \mathrm{i}v$. With this identification, the Haar probability measure $\mu_{\mathscr{F}}$ becomes the natural invariant measure for the geodesic and horocycle flows for the modular surface.

Haar probability measure

If (x_1, x_2, x_3) is a uniformly distributed random vector in the unit cube $(-\frac{1}{2}, \frac{1}{2})^3$, then

$$(u,v,\phi) = \left(\sin(\tfrac{\pi}{3}x_1), \frac{\cos(\frac{\pi}{3}x_1)}{\frac{1}{2} - x_2}, \pi x_3\right).$$

is a random element in $\mathscr{F}$ distributed according to the Haar probability measure $\mu_{\mathscr{F}}$.

A key property of Haar measure on $\mathrm{SL}(2,\mathbb{R})$ is that it is invariant under left and right multiplication by its group elements. This implies that (using the invariance under right multiplication) for M distributed according to $\mu_{\mathscr{F}}$, the random lattices $\mathbb{Z}^2 M$ and $\mathbb{Z}^2 M g$ have the same distribution for every element $g \in \mathrm{SL}(2,\mathbb{R})$. In other words, the random point process $\mathbb{Z}^2 M$ is $\mathrm{SL}(2,\mathbb{R})$-stationary! The process is, however, not translation-stationary since the origin is always realised. But even with the origin removed, the random process $\mathbb{Z}^2 M \setminus \{\mathbf{0}\}$ is not translation-stationary (as the formulas below will show). Nevertheless, Siegel's famous mean value formula (published in 1945) shows that its intensity measure is the standard Lebesgue measure dy.

Siegel's mean value formula

Motivated by questions in the geometry of numbers, Siegel proved that for any measurable function $f : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$,

$$\int_{\mathcal{F}} \Bigg(\sum_{\boldsymbol{x} \in \mathbb{Z}^2 M \setminus \{\mathbf{0}\}} f(\boldsymbol{x}) \Bigg) \, d\mu_{\mathcal{F}} = \int_{\mathbb{R}^2} f(\boldsymbol{y}) \, d\boldsymbol{y}.$$

Siegel's formula in fact works for lattices in arbitrary dimension d. In 1998 it was generalised by Veech to general $\mathrm{SL}(d,\mathbb{R})$-stationary point processes in $\mathbb{R}^d$. (Veech in fact proved it for a more general class of random locally finite Borel measures in $\mathbb{R}^d$).

One challenge is now to work out the probability

$$\mathbb{P}\Big(|\mathbb{Z}^2 M \cap B| = k \Big)$$

for a given Borel set B. This turns out to be more difficult than one would think, despite the explicit and simple form of the Haar probability measure. The problem is the domain of integration! Let us specialise to the case of lattice points in a strip.

Lattice points in a strip

Consider the lattice $\mathbb{Z}^2 M$ restricted to the vertical strip

$$\mathcal{Z}_{w,R} = \big(w - R, w + R\big) \times (0, \infty),$$

the green strip in Figure 1. For simplicity (and because it's all that is needed for our applications below) we assume that $-R < w < R$, so that the vertical axis interesects $\mathcal{Z}_{w,R}$. We can now look for the lattice point in the strip with the lowest height, i.e., with the smallest positive x_2-coordinate. For typical lattices this point will be unique, and we will denote it by $\boldsymbol{q}$.

It is remarkable that, for any given lattice $\mathbb{Z}^2 M$, there are at most three possible choices for $\boldsymbol{q}$: the two basis vectors $\boldsymbol{r}, \boldsymbol{s}$ of $\mathbb{Z}^2 M$ with minimal height in the larger vertical strip between $-2R$ and $2R$ (see Figure 1), and their sum $\boldsymbol{r} + \boldsymbol{s}$. This fact, and its link to the famous three gap theorem for circle rotations, is explained in [15]. This pretty observation enables us to calculate the distribution of the minimal height vector $\boldsymbol{q}$ [12].

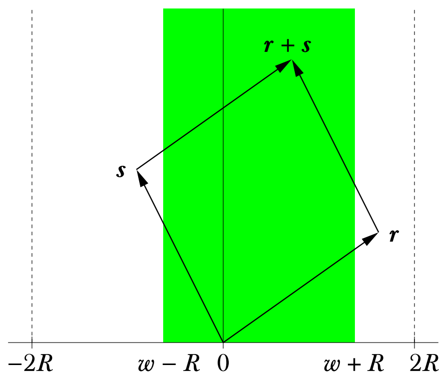

Figure 1. The two linearly independent lattice vectors with lowest and second-lowest heights in the vertical strip between $-2R$ and $2R$ form a basis. One can show that at any vertical strip of width one (in green) contains at least one of the three points, and hence the minimal height vector $\boldsymbol{q}$ is either $\boldsymbol{r}$, $\boldsymbol{s}$ or $\boldsymbol{r} + \boldsymbol{s}$.

Distribution of the lattice point with minimal height

If $\mathbb{Z}^2 M$ is a Haar random lattice, then the minimal height vector $\boldsymbol{q} = (q_1, q_2)$ in $\mathcal{Z}_{w,R}$ is distributed according to the probability measure $K_{w,R}(\boldsymbol{q}) d\boldsymbol{q}$ with density $K_{w,R}(q_1, q_2)$ given by

$$\frac{6}{\pi^2} \, H\Bigg(1 + \frac{q_2^{-1} - \max\left(|w|, |q_1 - w|\right) - R}{|q_1|} \Bigg)$$

$$\text{where } H(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \leq x. \end{cases}$$

The density $K_{w,R}(\boldsymbol{q})$ evidently depends on the choice of w, which proves that the random process $\mathbb{Z}^2 M \setminus \{\mathbf{0}\}$ is not translation-stationary. The $\mathrm{SL}(2,\mathbb{R})$-stationarity of our random lattice implies on the other hand that all distribution functions must be invariant under a simultaneously scaling of the horizontal and vertical directions by factors of $\lambda > 0$

and λ^{-1}, respectively. And indeed, the invariance

$$K_{\lambda w, \lambda R}(\lambda q_1, \lambda^{-1} q_2) = K_{w,R}(q_1, q_2)$$

is consistent with the explicit formula above.

If one is only interested in the height q_2 of $\boldsymbol{q}$ but not its direction, simply integrate over $q_1 \in [w-R, w+R]$. The result of this integration can be found in [12, Eq. (26)]. There is nothing to prevent us to further average over w, thus providing the distribution of the minimal height for a randomly shifted strip. The result of this second integration is as follows.

Distribution of minimal height on average

For a Haar random lattice $\mathbb{Z}^2 M$ the minimal height of a lattice point in the strip $\mathfrak{T}_{w,R}$, on average over w, is distributed according to the probability measure $P_R(q_2)\,dq_2 = 2R\,P(2Rq_2)\,dq_2$, with $P(s)$ given by (see also Figure 2)

$$\frac{6}{\pi^2} \times \begin{cases} 1 & (s \le 1) \\ \frac{1}{s} + 2\left(1 - \frac{1}{s}\right)^2 \log\left(1 - \frac{1}{s}\right) & \\ \quad -\frac{1}{2}\left(1 - \frac{2}{s}\right)^2 \log\left|1 - \frac{2}{s}\right| & (s > 1). \end{cases}$$

The first moment is $\int_0^1 sP(s)ds = 1$. There is, however, a *heavy tail*: for s large, we have

$$P(s) \sim \frac{4}{\pi^2} s^{-3}.$$

So already the second moment diverges! Compare this with the exponential distribution in Figure 2, which we would have obtained for minimum height points from a Poisson point process with unit intensity, in a strip of unit width.

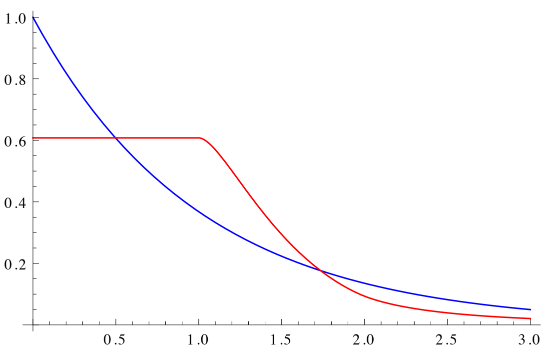

Figure 2. The exponential density e^{-s} (blue) vs. $P(s)$ (red).

Let us now discuss two natural examples where these distributions can be found in the 'wild'. The first describes visibility in Pólya's orchard or — equivalently — the free path length in the periodic Lorentz gas, and the second the energy level statistics for quantum harmonic oscillators.

Figure 3. The author in a perfectly periodic orchard: A poplar plantation near Pordenone, Italy.

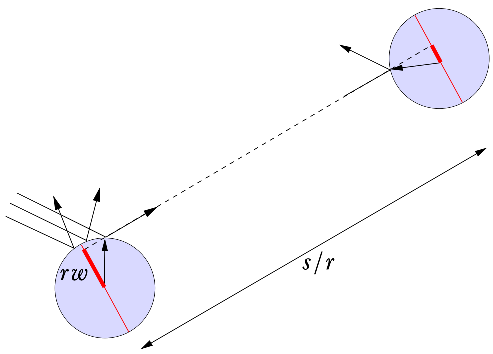

Figure 4. Intercollision flight of a particle in the Lorentz gas with scatterers of radius r. The free path length s is measured in units of $1/r$ and the the exit parameter w in units of r.

Pólya's orchard and the Lorentz gas

Pólya asked how far one could see in a forest, if all tree trunks had the same radius r and were either (a) randomly located or (b) planted on a perfect periodic grid. The same question arises in the study of the free path length for the two-dimensional Lorentz gas, where in the simplest setting a particle moves along straight lines in an array of spherical scatterers, see Figure 4. Let us here focus on the periodic setting, where the trees/scatterers are centered at points of $\mathbb{Z}^2$. What is the visibility, or free path length, with the observer at a given tree looking in direction $(-\sin\phi, \cos\phi)$? Is there a limit distribution when r is small and θ random?

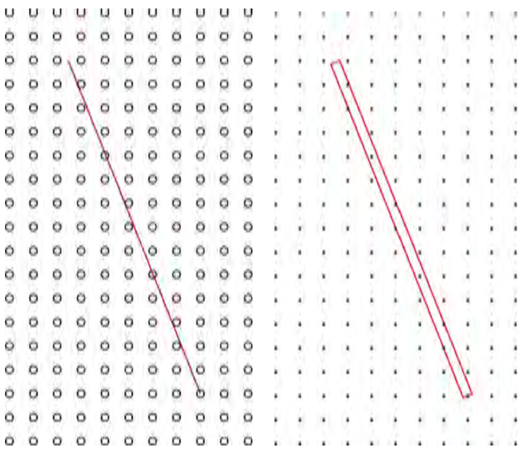

Figure 5. Left: A ray of length s/r in direction $(-\sin\phi, \cos\phi)$ intersecting k tree trunks of (small) radius r. Right: A rectangle containing k lattice points pointing the same direction, same length and width $2r$.

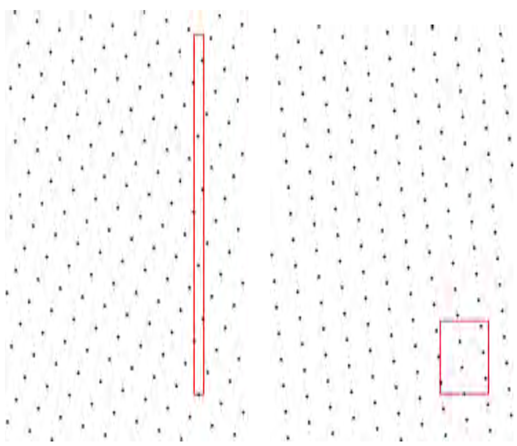

Figure 6. Left: The configuration in Figure 5 (right) rotated clockwise by ϕ. Right: The configuration on the left rescaled in the horizontal and vertical directions by factors of r^{-1} and r, respectively. The rectangle has now width 2 and height s.

The number of tree trunks of radius r intersecting a ray of length s/r is the same as the number of lattice points in a rectangle of width $2r$ and length s/r, see Figure 5. Now let's rotate the whole picture clockwise as in Figure 6 (left). The rectangle is now vertical, and instead of the lattice $\mathbb{Z}^2$ we have the rotated lattice

$$\mathcal{R}_1(\phi) = \mathbb{Z}^2 \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

which we have met before. Finally, we stretch the picture as described in Figure 6 (right), and obtain the rectangle of height s and width 2 — the r-dependence is gone! On the flipside, the underlying lattice has now transformed to the r-dependent lattice

$$\mathcal{R}_r(\phi) = \mathbb{Z}^2 \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} r^{-1} & 0 \\ 0 & r \end{pmatrix}.$$

The visibility, or free path length, can now be expressed as the minimal height of lattice points in the strip $\mathcal{Z}_{w,1}$, where w describes the offset of the ray relative to the center of the initial tree trunk; see Figure 6. (For example $w = 0$ means the ray emerges from its centre as in Figure 5.) The condition $|w| < 1$ ensures we are sitting somewhere on the tree trunk. In the context of the Lorentz gas, the fact that the minimal height can only take three values as w varies is known as Thom's problem, in turn a close variant of Slater's problem. The key fact we will now use is the following:

Randomly rotated lattices

If ϕ is a uniformly distributed random variable in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then the random lattice $\mathcal{R}_r(\phi)$ converges in distribution to the Haar random lattice $\mathbb{Z}^2 M$ as $r \to 0$.

This statement is a consequence of the equidistribution of large circles in the homogeneous space $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$. The convergence implies that the limit distribution for the minimal height vector $\boldsymbol{q}$ in the lattice $\mathcal{R}_r(\phi)$ restricted to the strip $\mathcal{Z}_{w,1}$ is given by the density $K_{w,1}(\boldsymbol{q})$, and the corresponding distribution of the free path length is $P_1(s) = 2P(2s)$, see Figure 7. Note that if we had measured visibility in units of the diameter $2s$ rather than radius r, the limit distribution would be $P(s)$.

In the case of the Lorentz gas, $P_1(s)$ was in fact first found by the physicist Dahlqvist [3] in 1997, and only in 2007 established rigorously by number theorists Boca and Zaharescu [1], who employed analytic methods based on continued fractions and Farey sequences. The density $K_{w,1}(\boldsymbol{q})$ plays an important role in describing particles in transport in the periodic Lorentz gas, and in 2008 was calculated independently by Caglioti and Golse [2] by continued fraction techniques, and by Strömbergsson and the author [12] via random lattices. The principal advantage of the latter method is that it works in any dimension [13] and even extends to aperiodic, quasicrystalline point configurations! Now, on to the second 'real-world' appearance of random lattices.

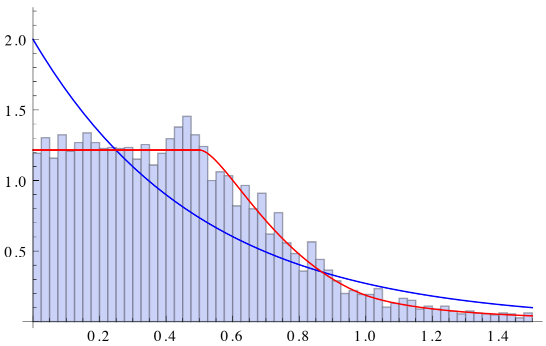

Figure 7. The distribution of free path for the periodic Lorentz gas with scatterers of radius $r = 10^{-8}$, sampled over 6000 initial conditions. Theoretical curves are the exponential density $2e^{-2s}$ (blue) vs. $P_1(s) = 2P(2s)$ (red). The data was computed using the algorithm in [9].

Quantum oscillators

In quantum mechanics, the energy levels of bound states can only take specific discrete ('quantized') values. One of the simplest and most fundamental quantum systems with a purely discrete spectrum is the harmonic oscillator. In two space dimensions, its energy levels are given by

$$E_{m,n} = (m + \tfrac{1}{2})\hbar\omega_1 + (n + \tfrac{1}{2})\hbar\omega_2$$

where $m, n = 0, 1, 2, \dots$ run through the non-negative integers. The quantities ω_1, ω_2 are positive reals, the *oscillation frequencies* and $\hbar$ denotes Planck's constant. If we measure energy in units of $\hbar\omega_2$, we have the simpler expression

$$\epsilon_{m,n} = (m + \tfrac{1}{2})u + (n + \tfrac{1}{2}), \qquad u = \frac{\omega_1}{\omega_2}.$$

Of particular significance are the spacings between energy levels, as they determine the emission spectrum of the system. After a little thought, you can convince yourself that the spacings between consecutive levels $\epsilon_{m,n}$ in the interval $[E, E+1)$ are the same as the gaps between the fractional parts ξ_m of the sequence mu, where $m = 0, \dots, N-1$ and N is number of $\epsilon_{m,n}$ in $[E, E+1)$. The three gap theorem mentioned earlier thus implies that we have the same phenomenon for the energy levels for a harmonic oscillator, at least for intervals of length one. A numerical illustration of this fact is given in Figure 8.

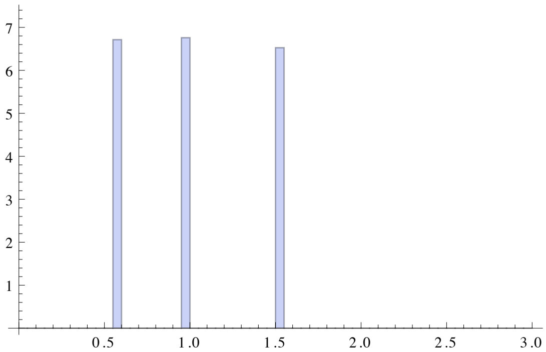

Figure 8. The gap distribution for the fractional parts of nu, with $n = 1, \dots, 50000$ and $u = \pi$.

One can show, however, that the distribution in Figure 8 will not converge as N becomes large. The only hope to see a limit is to introduce a further average over u. Using the approach in [15], we can express the gap between ξ_m and its nearest neighbour to the right as the minimal height of all lattice points in the strip $\mathcal{X}_{w,1/2}$ (of width one), with $w = \frac{m}{N} - \frac{1}{2}$ and the lattice

$$\mathcal{P}_N(u) = \mathbb{Z}^2 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N^{-1} & 0 \\ 0 & N \end{pmatrix}.$$

As in the case of randomly rotated lattices, also here we have a limit theorem.

Randomly sheared lattices

If u is a uniformly distributed random variable in $[0, 1]$, then the random lattice $\mathcal{P}_N(u)$ converges in distribution to the Haar random lattice $\mathbb{Z}^2 M$ as $N \to \infty$.

This fact is based on the equidistribution of long closed horocycles on $\Gamma\backslash \mathrm{SL}(2, \mathbb{R})$, which was proved by Zagier in 1979 in the case of the modular surface, and for more general discrete subgroups Γ by Sarnak in 1981. The most powerful extension of results of this type (as well as the rotational averages used for Pólya's orchard) is due to Ratner in the early 1990s [16]. It describes equidistribution of unipotent orbits on quotients $\Gamma\backslash G$ where G is now a general Lie group. (Horocycles are special examples of unipotent orbits.) Recent breakthroughs that build on Ratner's work include the deep measure classification and equidistribution theorems for moduli spaces by Eskin, Mirzakhani and Mohammadi. For an introduction to dynamics on homogeneous spaces and their relevance in number theory I recommend the excellent textbook by Einsiedler and Ward [4].

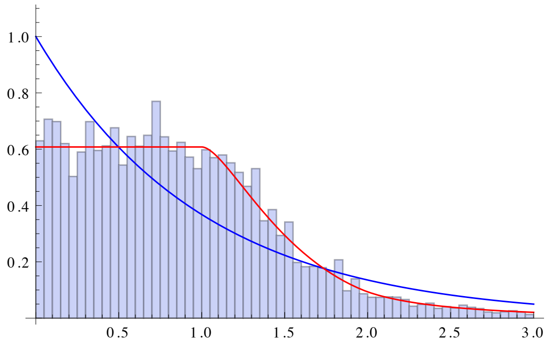

Figure 9. The gap distribution for the fractional parts of nu, with $n = 1, \ldots, 2000$ and u sampled over 2000 randomly chosen points in $[0,1]$. Theoretical curves are the exponential density e^{-s} (blue) vs. $P(s)$ (red).

By the same reasoning we used earlier for Pólya's orchard, the convergence of randomly sheared lattices to Haar distributed random lattices establishes the convergence of the gap distribution for the fractional parts of mu. The one difference is we now sum over $w = \frac{m}{N} - \frac{1}{2}$ $(m = 0, \ldots, N-1)$ rather than integrate — but this discrete average can be treated as a Riemann sum which approximates the Riemann integral for N large. We can conclude that the gaps between fractional parts on mu, and thus the energy level spacings for quantum oscillators, have the same limit distribution as the free path length in the periodic Lorentz gas! Figure 9 compares numerical data with the theoretical prediction.

The explicit form of the level spacing distribution for quantum oscillators (in Figure 9) was first established by Greenman [7] in 1996, following previous work on the problem by Berry and Tabor (1977), Bohigas, Giannoni and Pandey (1989), Bleher (1990-91), Pandey and Ramaswamy (1992), Mazel and Sinai (1992); see [10] for details and references. Greenman's paper predates Dahlqvist's and Boca and Zaharescu's work on the Lorentz gas; and perhaps more remarkably, the likeness of the two distributions seems to have been overlooked even in the recent literature [17]! That the two are the same is evident of course by simply staring at the explicit formulas, and perhaps no surprise given the similarity of their arithmetic setting. The beauty of using lattices is that we have a conceptual understanding of why the limit distributions must coincide: random rotations and random shears both converge to the same Haar probability measure — a non-trivial fact!

Other applications

We can construct random lattices that are not only $\mathrm{SL}(2,\mathbb{R})$-stationary but also translation-stationary as follows. Take the randomly shifted lattice $\mathbb{Z}^2 + \boldsymbol{\alpha}$ with $\boldsymbol{\alpha}$ uniformly distributed in the unit square $[0,1]^2$ (recall our construction in dimension one), then apply a linear transformation to obtain the random affine lattice $(\mathbb{Z}^2 + \boldsymbol{\alpha})M$ with M distributed in $\mathscr{F}$ with respect to Haar measure. This point process is now translation-stationary and it has intensity one. In fact, also its second moment coincides with that of a Poisson point process; again a consequence of Siegel's mean value formula [5, App. B]. In 2004, Elkies and McMullen [6] proved that the limiting gap distribution for the fractional parts of $\sqrt{n}$, $n = 1, 2, 3, \ldots$ can be derived via a random affine lattice. The proof uses equidistribution of certain nonlinear horocycles, which is a consequence of Ratner's measure classification theorem. The distribution found by Elkies and McMullen also describes the limiting distribution for directions in a fixed affine lattice [13].

Random lattices appeared in the probability literature in Kallenberg's disproof of the Davidson conjecture [8] on the classification of line processes which have (almost surely) no parallel lines. The counterexamples to the conjecture were constructed using two-dimensional random affine lattices restricted to a vertical strip, where each lattice point represents a line via the standard linear parametrisation. This is particularly impressive as Kallenberg was unaware of Siegel's classical construction in the geometry of numbers, as clarified by Kingman; see the quote at the end of Kallenberg's paper.

Other examples where random lattices play an important role are the value distribution of quadratic forms, such as in Margulis' proof of the Oppenheim conjecture, the Hall distribution describing the gaps between Farey fractions, random Diophantine approximation, diameters of random Caley graphs of abelian groups, the Frobenius problem, hitting times for integrable dynamical systems, deviations of ergodic averages of toral translations, etc. And how about random lattices in non-Euclidean settings?

But these are stories for another day!

Take home message

- Random lattices are important point processes with connections to ergodic theory, geometry, number theory, combinatorics, probability and physics.
- The level spacing distribution of a quantum oscillator equals the free path distribution of the periodic Lorentz gas.

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FURTHER READING

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