NL2592 **Cat map**

The cat map is perhaps the simplest area-preserving transformation which exhibits a high degree of chaos. In the development of the theory of dynamical systems it served as a guiding example to illustrate new concepts such as entropy (Sinai 1959) and Markov partitions (Adler & Weiss 1967). The cat map owes its name to an illustration by V.I. Arnold showing the image of a cat before and after the application of the map. In the mathematical literature it is also referred to as *hyperbolic toral automorphism*.

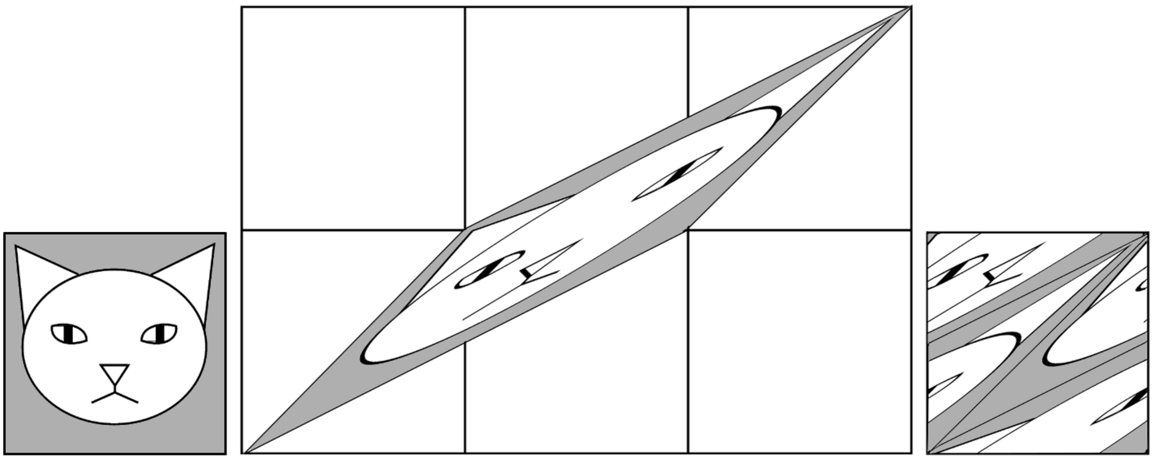

Figure 1. The cat map: the image of a cat in the unit square (left) is stretched by the matrix A (middle) and then re-assembled by cutting and translating (without rotation) the different parts of the cat's face back into the unit square (right).

The torus M, which topologically has the shape of a doughnut, may be described by the points in the unit square (see figure) where opposite sides are identified. Alternatively, one may think of a point on M as a point in the plane $\mathbb{R}^2$ modulo integer translations in $\mathbb{Z}^2$. This yields a representaion of M as the coset space $\mathbb{R}^2/\mathbb{Z}^2 := \{x + \mathbb{Z}^2 : x \in \mathbb{R}^2\}$. The cat map is now a mapping $\phi : M \to M$ defined by $x \mapsto Ax \pmod{\mathbb{Z}^2}$, where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{1}$$

provided $a, b, c, d \in \mathbb{Z}$ are chosen such that $|\det A| = 1$ and A has eigenvalues $\lambda_\pm$ with modulus not equal to one. (This implies that both eigenvalues are real and distinct.) The matrix A used in our illustration is

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{2}$$

Let us now explore some of the dynamical properties which show that the cat map is indeed completely "chaotic" ($\rightarrow$ Chaotic Dynamics). *Sensitivity on initial conditions* is measured by the rate of divergence of two nearby points, x and $x + \delta x$, under iterations of ϕ. For any smooth map ϕ the Taylor expansion for small δx yields

$\phi(x+\delta x) = \phi(x) + D\phi_x \delta x + O(\delta x^2)$ where $D\phi_x$ is the differential of ϕ at x; it may be viewed as a linear map from TM_x to $TM_{\phi(x)}$, the tangent spaces at x and $\phi(x)$, respectively. Since in the case of the cat map ϕ is linear the above Taylor expansion is in fact exact, i.e., $\phi(x+\delta x) = \phi(x) + D\phi_x\,\delta x$ with $D\phi_x = A$. If $v_\pm$ are the eigenvectors of A corresponding to the eigenvalues $\lambda_\pm$, let us denote by E_x^+ and E_x^- the subspaces of TM_x spanned by v_+ and v_-, respectively. Since $\lambda_+ \neq \lambda_-$, we have

$$TM_x = E_x^+ \oplus E_x^- . \tag{3}$$

Furthermore, since $|\lambda_+| = 1/|\lambda_-| > 1$, we find

$$\|D\phi_x(\xi)\| \geq \mathrm{e}^{\lambda}\|\xi\| \quad \text{if } \xi \in E_x^+, \tag{4}$$

$$\|D\phi_x(\xi)\| \leq \mathrm{e}^{-\lambda}\|\xi\| \quad \text{if } \xi \in E_x^-, \tag{5}$$

with $\lambda = \ln|\lambda_+| > 0$ and $\|\cdot\|$ is the euclidean norm. Hence ϕ is *expanding* in the direction of v_+, and *contracting* in the direction of v_-, which will be therefore referred to as the *unstable* and *stable* directions, respectively. [The inequalities (4) and (5) are here in fact equalities; inequalities become necessary in the case of more general Anosov maps, if one seeks uniform bounds with λ independent of x.] For the nth forward or backward iterates of the map (n a positive integer), we have, by the above arguments with ϕ replaced by $\phi^{\pm n}$,

$$\|D\phi_x^n(\xi)\| \geq \mathrm{e}^{n\lambda}\|\xi\|, \quad \|D\phi_x^{-n}(\xi)\| \leq \mathrm{e}^{-n\lambda}\|\xi\|, \quad \text{if } \xi \in E_x^+, \tag{6}$$

$$\|D\phi_x^n(\xi)\| \leq \mathrm{e}^{-n\lambda}\|\xi\|, \quad \|D\phi_x^{-n}(\xi)\| \geq \mathrm{e}^{n\lambda}\|\xi\|, \quad \text{if } \xi \in E_x^-. \tag{7}$$

The expansion/contraction is thus exponentially fast in time. Relations (3), (6) and (7) are equivalent to the statement that the cat map is an *Anosov system*. Special features of Anosov systems are ergodicity, mixing, structural stability, exponential proliferation of periodic orbits and positive entropy h.

There is a particularly simple formula for the *entropy* h due to Sinai (1959) which states that $h = \lambda = \ln|\lambda_+|$. The number of fixed points of the nth iterate ϕ^n is equal to $|\det(1-A^n)| = |(1-\lambda_+^n)(1-\lambda_-^n)|$ and is therefore asymptotically given by $\exp(nh)$, for n large. The notion of *ergodicity* implies that for any $f \in \mathrm{L}^1(M, dx)$ we have for almost every $x \in M$ (that is, for all x up to a set of Lebesgue measure zero)

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(\phi^n x) = \langle f\rangle, \qquad \langle f\rangle := \int_M f(x)\,dx. \tag{8}$$

Mixing means that, if $f, g \in \mathrm{L}^2(M, dx)$, then

$$\lim_{n\to\pm\infty} \int_M f(\phi^n x)\, g(x)\, dx = \langle f\rangle\langle g\rangle. \tag{9}$$

Notice that Lebesgue measure is the natural invariant probability measure for ϕ, since $|\det A| = 1$. Although the mixing property follows from general arguments for Anosov systems, it can be proved for ϕ directly by means of elementary Fourier analysis. What is more, the rate of convergence in (9) is in fact exponentially fast in n for suitably smooth test functions f, g ("exponential decay of correlations"), and super-exponentially fast for analytic ones.

A powerful tool in the analysis of dynamical systems are *Markov partitions.* In the case of the cat map the torus is divided into a finite collection of non-overlapping parallelograms $P_1, \ldots, P_N$ whose sides point in the directions of the eigenvectors v_+ and v_-, such that if $\phi(P_i)$ (or $\phi^{-1}(P_i)$) intersects with P_j, then it extends all the way across P_j. Let us construct an N by N matrix B whose coefficients are $B_{ij} = 1$ if the intersection of $\phi(P_i)$ with P_j is non-empty, and $B_{ij} = 0$ otherwise. A symbolic description of the dynamics of the cat map can now be obtained as follows. Consider doubly infinite sequences of the form $\cdots b_{-2}b_{-1}\underline{b_0}b_1b_2\cdots$, where b_n is an integer $1, \ldots, N$ with the condition that the number b_n can be followed by b_{n+1} only if $B_{b_n\, b_{n+1}} = 1$. To each such sequence we can associate a point x on M by requiring that for every $n \in \mathbb{Z}$ the parallelogram R_{b_n} contains the iterate $\phi^n(x)$. The *symbolic dynamics* of ϕ is now given by shifting the mark by one step to the right: the new word $\cdots b_{-1}b_0\underline{b_1}b_2b_3\cdots$ indeed represents the point $\phi(x)$. The dynamical properties of ϕ are thus encoded in the matrix B. In particular, the higher the rate of mixing of ϕ, the smaller the number of coefficients B_{ij} which are zero. This in turn means we have less restrictions on $b_n \mapsto b_{n+1}$ and a typical orbit will be represented by a more "random" sequence of symbols.

Cat maps, as well as higher-dimensional toral automorphisms, are featured in most text books on dynamical systems and ergodic theory. An introduction to the basic concepts may be found in Arnold & Avez (1968), and more advanced topics, such as entropy, Markov partitions and structural stability, are discussed for example in Adler & Weiss (1970), Katok & Hasselblatt (1995), Pollicott & Yuri (1998) and Shub (1986).

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See also Chaotic Dynamics

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