SPECTRAL THETA SERIES OF OPERATORS WITH PERIODIC BICHARACTERISTIC FLOW

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Dedicated to Professor Yves Colin de Verdière on the occasion of his 60th birthday.

Abstract. The theta series $\vartheta(z) = \sum \exp(2\pi\mathrm{i} n^2 z)$ is a classical example of a modular form. In this paper we argue that the trace $\vartheta_P(z) = \mathrm{Tr}\, \exp(2\pi\mathrm{i} P^2 z)$, where P is a self-adjoint elliptic pseudo-differential operator of order 1 with periodic bicharacteristic flow, may be viewed as a natural generalization. In particular, we establish approximate functional relations under the action of the modular group. This allows a detailed analysis of the asymptotics of $\vartheta_P(z)$ near the real axis, and the proof of logarithm laws and limit theorems for its value distribution. These asymptotics are in fact distinctly different from those for the 'wave trace' $\mathrm{Tr}\, \exp(-\mathrm{i}Pt)$ whose singularities are well known to be located at the lengths of the periodic orbits of the bicharacteristic flow.

1. Introduction

Let X be a compact C^∞-manifold of dimension d with volume element dx, and P a positive self-adjoint elliptic pseudo-differential operator of order 1, acting on functions of X. The principal symbol of P is $p \in \mathrm{C}^\infty(\mathrm{T}^* X\backslash\{0\}, \mathbb{R}_{>0})$. We assume that the subprincipal symbol of P vanishes. An example for such an operator is $\sqrt{-\Delta+V}$ where Δ is the Laplacian on X and $V \in \mathrm{C}^\infty(X, \mathbb{R}_{>0})$ some smooth potential. P has discrete spectrum,

$$\rho_1 \leq \rho_2 \leq \rho_3 \leq \ldots \to \infty, \tag{1.1}$$

and we will denote by $(\varphi_n)_{n\in\mathbb{N}}$ an orthonormal basis of eigenfunctions in $\mathrm{L}^2(X, dx)$ so that

$$P\varphi_n = \rho_n \varphi_n. \tag{1.2}$$

The trace formulas developed by Chazarain [2], Colin de Verdière [3, 4] and Duistermaat-Guillemin [6] provide an asymptotic expansion of the trace

$$\mathrm{Tr}\, \mathrm{e}^{-\mathrm{i}Pt} = \sum_n \mathrm{e}^{-\mathrm{i}\rho_n t} \tag{1.3}$$

in terms distributions whose singular support is at the lengths of the periodic bicharacteristics of P. Special versions of such trace formulas had appeared earlier, e.g., in Selberg's theory for the spectrum of the Laplacian on a weakly symmetric space [24], and in the the work of Gutzwiller and Balian-Bloch in the context of quantum chaos [12].

In the present paper we replace the first order operator P by P^2 and study the asymptotic behaviour of the spectral theta series

$$\vartheta_P(z) = \mathrm{Tr}\, e(P^2 z) = \sum_n e(\rho_n^2 z), \qquad e(z) = \mathrm{e}^{2\pi\mathrm{i}z}, \tag{1.4}$$

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for z in the complex upper half plane $\mathbb{H}$, when $\operatorname{Im} z \to 0$. The relationship between $\vartheta_P(z)$ and (1.3) is probably best exhibited in the formal relation ($t = 2\pi s$)

$$\vartheta_P(z) = \frac{1}{\sqrt{-2\mathrm{i}z}} \int_{-\infty}^{\infty} \operatorname{Tr} e(-\mathrm{i}Ps)\, e(-s^2/4z)\, ds. \tag{1.5}$$

Colin de Verdiére [3, 4] in fact states the trace formula for (1.3) in terms of an asymptotic relation for $\vartheta_P(z)$, which is uniform in the halfplanes $\operatorname{Im}(-1/z) \geq c > 0$, and it is an open problem to extend the asymptotic analysis outside this domain.

Take for example $P = \sqrt{-\Delta}$, where Δ is the Laplacian on the unit circle. We have (with respect to even test functions)

$$\operatorname{Tr} e(-\mathrm{i}Ps) = \sum_{n\in\mathbb{Z}} e(ns) \tag{1.6}$$

and hence, by Poisson summation,

$$\operatorname{Tr} e(-\mathrm{i}Ps) = \sum_{k\in\mathbb{Z}} \delta(s-k). \tag{1.7}$$

In this case $\vartheta_P(z)$ is of course the classical theta series

$$\vartheta(z) = \sum_{n\in\mathbb{Z}} e(n^2 z), \tag{1.8}$$

and the trace formula (1.7) is encoded, via (1.5), in the functional relation

$$\vartheta(z) = \frac{1}{\sqrt{-2\mathrm{i}z}}\, \vartheta(-1/4z). \tag{1.9}$$

Recall that

$$\vartheta(z+1) = \vartheta(z) \tag{1.10}$$

is the second fundamental functional relation for the classical theta function. The self-reciprocity (1.9) and periodicity (1.10) reflect the strong correlations between the energy and the length spectrum of the Laplacian on the circle, and the fact that the squares of the lengths of periodic orbits are integers. The main objective of this paper is to investigate the functional relations of theta series $\vartheta_P(z)$ for similar operators P whose bicharacteristic flow is still periodic, and to employ the approximate modularity in the asymptotic analysis of $\vartheta_P(z)$. Examples of such P include $\sqrt{-\Delta + V}$ on spheres [11] (or, more generally, Zoll manifolds [1] and rank-one symmetric spaces [10]) and isotropic harmonic oscillators.

One motivation for studying spectral theta functions is their connection with the autocorrelation function

$$C(t) = \int_X u(x,t)\, \overline{u(x,0)}\, dx \tag{1.11}$$

of solutions $u(x,t)$ to the Schrödinger equation $-(2\pi\mathrm{i})^{-1}\partial_t u = P^2 u$: For the initial data $u(x,0) = \sum \mathrm{e}^{-\pi\rho_n^2 y}\varphi_n(x)$ we have $C(t) = \vartheta_P(-t + \mathrm{i}y)$.

A further motivation are spectral statistics. In dimension $d = 2$, Weyl's law implies that the mean spacing between ρ_j^2 is constant, and hence the correctly scaled pair correlation density reads

$$R_2(s) = \sum_{i,j} \delta\big(s - (\rho_i^2 - \rho_j^2)\big)\mathrm{e}^{-2\pi(\rho_i^2+\rho_j^2)y} \tag{1.12}$$

which in turn can be expressed as the Fourier transform of $|\vartheta_P(x+\mathrm{i}y)|^2$ (with respect to x). This approach has been used in [18, 21] for the eigenvalues of the Laplacian on a flat torus.

Outline of the paper. In Section 2 we review the spectral properties of operators P with periodic bicharacteristic flow. In particular the extereme clustering of eigenvalues plays an important role in the main results of this work—the approximate functional relations for $\vartheta_P(z)$ and its value distribution properties. These results are presented in Section 3 as Theorems 3.1–3.5.

A convenient framework for studying the theta functions that appear in the approximate functional relations for $\vartheta_P(z)$ (Theorem 3.1) is the Shale-Weil representation of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, which is introduced in Section 4. Section 5 is devoted to the relevant theta series, their functional relations and asymptotic properties. Most of this material is closely related the results in [19, 20]; we also refer the reader to the monograph [17] for further background on the Shale-Weil representation and theta series. As we shall see in Section 6, Theorem 3.1 implies that the spectral theta series ϑ_P is approximated to leading order (as $y \to 0$) by theta series corresponding to a particular class of cut-off functions that are related to parabolic cylinder functions (Section 7).

The functional relations from Section 5 allow us to view theta series as functions on a homogeneous space of $\Gamma\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of finite volume, where Γ is a certain discrete subgroup of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Following [19, 20], the key idea is then to use ergodic-theoretic methods for the proof of the main theorems stated in Section 3: in Sections 8–10 we state and prove the necessary logarithm laws and equidistribution results, and finally exploit these to prove Theorems 3.2–3.5 in Section 11.

2. Periodic flows and spectral clusters

We will assume throughout the remainder of this paper that the bicharacteristic flow of P is periodic, and we suppose without loss of generality that the smallest period is 2π. This means that every bicharacteristic is closed and has a primitive period of the form $2\pi/m$ for some $m \in \mathbb{N}$. The set of possible primitive periods is finite and will be denoted by $\{2\pi/m_j : j = 0, \ldots, N\}$ with $m_0 = 1$. The periodic orbits corresponding to a given primitive period $2\pi/m_j \neq 2\pi$ form a submanifold of dimension $d_j < d$.

It is well known that, under the above assumptions, the spectrum of P clusters around arithmetic progressions [29, 5]: There is a constant $M > 0$ such that the spectrum of P^2 is contained in the union of intervals

$$I_k = \left[\left(k + \frac{\alpha}{4}\right)^2 - M, \left(k + \frac{\alpha}{4}\right)^2 + M\right], \qquad k = 0, 1, 2, \ldots \tag{2.1}$$

where $\alpha \in \mathbb{Z}$ is the common Maslov index of the 2π-periodic bicharacteristics. It is convenient to relabel the eigenvalues as

$$\rho_n^2 = \left(k + \frac{\alpha}{4}\right)^2 + \mu_{kl} \tag{2.2}$$

where $1 \leq l \leq \delta_k$ and

$$-M \leq \mu_{k1} \leq \ldots \leq \mu_{k\delta_k} \leq M; \tag{2.3}$$

here δ_k defined to be the number of eigenvalues in the kth cluster.

Colin de Verdière's Theorem 1.4 in [5] proves the existence of polynomials $R(t) = b_1 t^{d-1} + \ldots$ and $R_{jl}(t)$ of degrees $d-1$ and $d_j - 1$, respectively such that for k sufficiently large

$$\delta_k = R\left(k + \frac{\alpha}{4}\right) + \sum_{j=1}^{N} \sum_{l=1}^{m_j - 1} R_{jl}(k)\, e(kl/m_j). \tag{2.4}$$

Let us denote the spectral density of the kth cluster by

$$\mu_k(\lambda) = \sum_{l=1}^{\delta_k} \delta(\lambda - \mu_{kl}). \tag{2.5}$$

Theorem 3.1 in [5] states that there are distributions $\mathcal{R}(t)$ and $\mathcal{R}_{jl}(t)$ with support in $[-M, M]$ such that for every $f \in \mathrm{C}^\infty(\mathbb{R})$

$$\int f\, d\mu_k = \int f\, d\mathcal{R}\left(k + \frac{\alpha}{4}\right) + \sum_{j=1}^{N} \sum_{l=1}^{m_j-1} \int f\, d\mathcal{R}_{jl}(k)\, e(kl/m_j), \tag{2.6}$$

and we have the asymptotic expansion

$$\mathcal{R}(x) \sim \nu_1 x^{d-1} + \nu_3 x^{d-3} + \nu_4 x^{d-4} + \ldots, \tag{2.7}$$

$$\mathcal{R}_{jl}(x) \sim \nu_{jl1} x^{d_j-1} + \nu_{jl2} x^{d_j-2} + \nu_{jl3} x^{d_j-3} + \ldots. \tag{2.8}$$

For $f = 1$ this implies in particular that $k^{-(d-1)}\delta_k \to \int d\nu_1$, and thus, in view of (2.4), we have $\int d\nu_1 = b_1$, the leading coefficient of the polynomial $R(t)$.

We refer to [28, 30] for more information on the fine-scale distribution of the eigenvalues in each cluster.

3. The main results

3.1. **Approximate functional relations.**

As we have seen in the last paragraph, ν_1 represents, up to normalization by b_1, a probability measure on $[-M, M]$. Let us denote by $W(t)$ the characteristic function of ν_1,

$$W(t) = \int_{-M}^{M} e(t\lambda) d\nu_1(\lambda). \tag{3.1}$$

Notice that $W(t)$ is analytic.

We denote by $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ the universal cover of $\mathrm{SL}(2,\mathbb{R})$. As a manifold, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ can be identified with $\mathbb{H} \times \mathbb{R}$. (See Sect. 4.1 for details.) A lattice $\Gamma \subset \widetilde{\mathrm{SL}}(2,\mathbb{R})$ is a discrete subgroup such that the homogeneous space $\Gamma\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R})$ has finite volume with respect to Haar measure on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

Theorem 3.1. *There is a continuous function $\Theta : \widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \mathbb{C}$ with the properties that*

(i) *there is a lattice $\Gamma \subset \widetilde{\mathrm{SL}}(2,\mathbb{R})$ such that $\Theta(\tilde\gamma\tilde g) = \Theta(\tilde g)$ for all $\tilde\gamma \in \Gamma$, $\tilde g \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$,*

(ii) *there is a constant $C \geq 0$ such that for all $z = x + \mathrm{i}y \in \mathbb{H}$*

$$\left| y^{(2d-1)/4} \vartheta_P(z) - W(x)\Theta(z, 0) \right| \leq C y^{1/4}.$$

The lattice Γ in fact coincides for even α with the invariance group $\Delta_1(4)$ of the classical theta series (cf. Sect. 4.1), whereas for odd α it is a congruence subgroup thereof.

3.2. **Logarithm laws.**

The following estimates on the growth of spectral theta series generalize the classical studies that go back to Hardy and Littlewood. Our main reference is the paper [8].

Theorem 3.2. *Let $\psi : (0, 1] \to \mathbb{R}_+$ be a non-increasing function such that the integral*

$$\int_0^1 \frac{dy}{y\psi(y)^4} \tag{3.2}$$

diverges (resp. converges). Then for almost every (resp. almost no) $x \in \mathbb{R}$ there is an infinite sequence of $y_1 > y_2 > \ldots \to 0$ such that

$$|\vartheta_P(x+\mathrm{i}y_j)| \geq y_j^{-(2d-1)/4}\psi(y_j). \tag{3.3}$$

In other words for $y < 1$ and almost all x,

$$\vartheta_P(x+\mathrm{i}y) \neq O_x\big(y^{-(2d-1)/4}\psi(y)\big), \tag{3.4}$$

and

$$\vartheta_P(x+\mathrm{i}y) = O_x\big(y^{-(2d-1)/4}\psi(y)\big) \tag{3.5}$$

respectively, depending on whether (3.2) diverges or converges. Compare this behaviour with the heat kernel asymptotics ($x = 0$)

$$\vartheta_P(\mathrm{i}y) \sim c y^{-d/2} \tag{3.6}$$

for some constant $c > 0$. Using Theorem 3.1 and the asymptotic behaviour of theta series in cusps (see in particular Corollary 3.3 in [20]), one may derive similar asymptotics, in fact with the same leading order divergence (up to multiplicative constants), for rational $x \in \mathbb{Q}$. It is also possible to obtain upper bounds under diophantine conditions on x. If x is for instance of bounded type (such as $x = \sqrt{2}$), we have $\vartheta_P(x+\mathrm{i}y) = O_x\big(y^{-(2d-1)/4}\big)$.

By choosing $\psi(y) = (\log y)^{(1\pm\epsilon)/4}$ in Theorem 3.2 with $\epsilon > 0$ arbitrarily small we obtain the following.

Corollary 3.3. *For almost all x*

$$\limsup_{y\to 0} \frac{\log(y^{(2d-1)/4}|\vartheta_P(x+\mathrm{i}y)|)}{\log\log y^{-1}} = \frac{1}{4}. \tag{3.7}$$

3.3. **Limit theorems.** The following limit theorems may be viewed as a generalization of the theorems for the classical theta series [13, 14, 15, 19]. We are interested in the value distribution of $\vartheta_P(x+\mathrm{i}y)$ where x is uniformly distributed in the interval $[a,b]$ in the limit $y \to 0$.

It is not hard to show that the variance has the asymptotics

$$\int_a^b |\vartheta_P(x+\mathrm{i}y)|^2 dx \sim \frac{\Gamma(d-1/2)}{2^{2d}\pi^{d-1/2}}\, y^{-(d-1/2)} \int_a^b |W(t)|^2 dt. \tag{3.8}$$

We therefore normalize $\vartheta_P(z)$ by a factor $a_d y^{(2d-1)/4}$ with

$$a_d = \frac{2^d \pi^{(2d-1)/4}}{\Gamma(d-1/2)^{1/2}}. \tag{3.9}$$

Theorem 3.4. *Let $[a,b] \subset \mathbb{R}$ and g a bounded continuous function $\mathbb{C} \to \mathbb{R}$. Then*

$$\lim_{y\to 0} \int_a^b g\big(a_d y^{(2d-1)/4}\vartheta_P(x+\mathrm{i}y)\big)dx = \int_{\mathbb{C}}\int_a^b g(Z\,W(t))\, d\rho_{d,\alpha}(Z)\, dt \tag{3.10}$$

where $\rho_{d,\alpha}$ is a probability measure on $\mathbb{C}$ with the tail distribution

$$\int_{|Z|>R} d\rho_{d,\alpha}(Z) \sim c_{d,\alpha} R^{-4} \qquad (R \to \infty), \tag{3.11}$$

$$c_{d,\alpha} = \begin{cases} \frac{2^{4(d-1)}\Gamma(d/2)^4}{\pi^3\Gamma(2d-1)} & (\alpha \equiv 0 \bmod 2) \\ \frac{2^{4(d-1)}\Gamma(d/2)^4}{4\pi^3\Gamma(2d-1)} & (\alpha \equiv 1 \bmod 2). \end{cases} \tag{3.12}$$

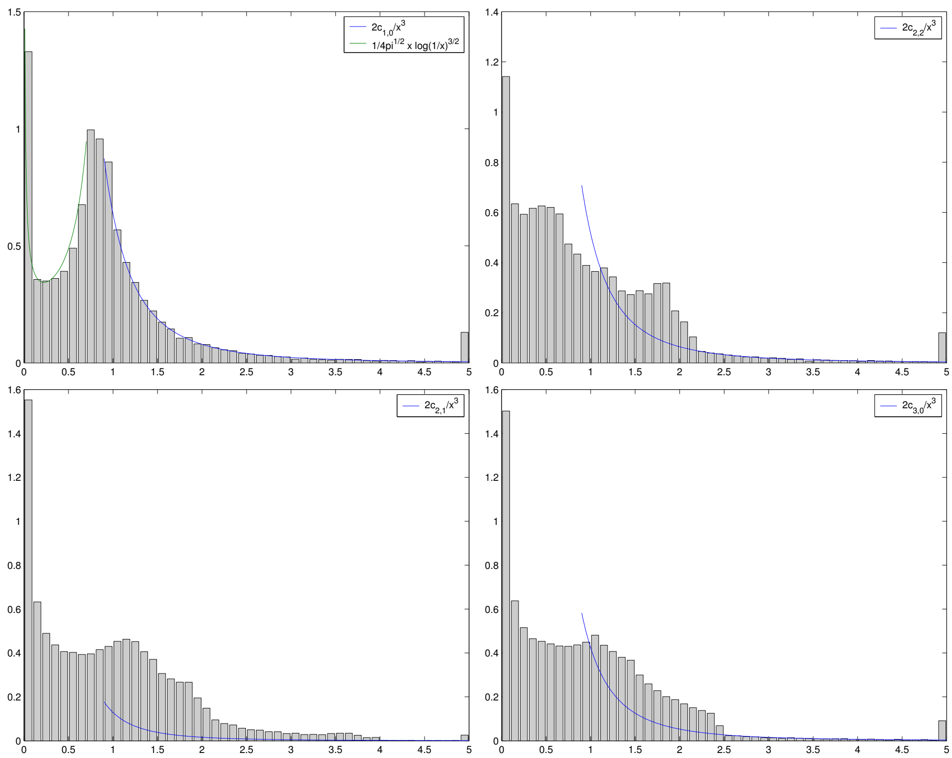

FIGURE 1. Distribution of the absolute value-squared of the spectral theta series for the circle ($d = 1$, $\alpha = 0$), the 2-sphere ($d = 2$, $\alpha = 2$), the 2-sphere with opposite points identified ($d = 2$, $\alpha = 1$) and the 3-sphere ($d = 3$, $\alpha = 0$). In all cases $y = 1/200^2$ and x is sampled over 200^2 random points in $[0, 1)$ with respect to the uniform distribution.

That is, the random variable $a_d y^{(2d-1)/4} \vartheta_P(x + \mathrm{i}y)$ where x is uniformly distributed in $[a, b]$ has a limiting distribution that is given by the product of two independent random variables, $W(t)$ and Z. As we shall see the probability distribution $\rho_{d,\alpha}$ is universal in that it only depends on the dimension d and Maslov index α.

We provide below, Eq. (9.12), and implicit formula for the limiting distribution, which shows for instance that $\rho_{d,\alpha}(Z)$ is invariant under the transformations $Z \mapsto \overline{Z}$ and $Z \mapsto \mathrm{i}Z$. In the case of the classical theta series $\rho_{1,0}(Z)$ is in fact invariant under all rotations about the origin, and furthermore has a singularity at 0:

$$\int_{|Z|<R} d\rho_{d,\alpha}(Z) \sim \big(8\pi \log(1/R)\big)^{-1/2} \qquad (R \to 0). \tag{3.13}$$

Relation (3.11) is a consequence of the divergence of the theta series in several cusps of the homogeneous space that will be introduced later, and similarly (3.13) follows from the superexponential decay in a cusp.

The above asymptotics prove that despite observing square-root cancellation as in the central limit theorem, the limiting distribution of $\theta_P(z)$ is not Gaussian. It is interesting that the numerical experiments displayed in Fig. 2 suggest that a central limit theorem holds when P^2 is replaced by higher powers, i.e., for the

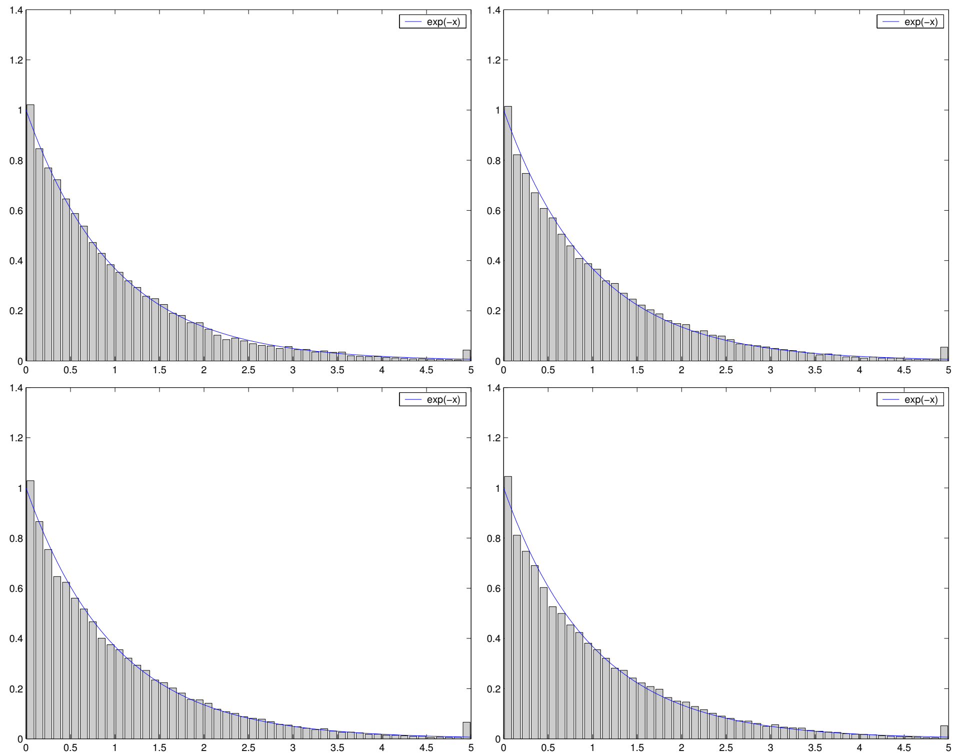

FIGURE 2. The same numerical experiment as in Fig. 1, but with P^2 replaced by P^3. The data seems to follow the distribution $\exp(-x)$ and is therefore consistent with a central limit theorem.

series

$$a_{d,k}\, y^{(2d-1)/2k} \operatorname{Tr} e(P^k z), \qquad a_{d,k} = \frac{2^{(2d-1)/k+1/2} \pi^{(2d-1)/2k}}{\Gamma((2d-1)/k)^{1/2}}. \tag{3.14}$$

3.4. **Correlations.** Our last result states a limit theorem for the joint distribution of $\vartheta_P(x + \mathrm{i}y)$ at n points $\omega_1 x, \ldots, \omega_n x$.

Theorem 3.5. *Suppose $\omega_1, \ldots, \omega_n \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$, and let $[a, b] \subset \mathbb{R}$ and g a bounded continuous function $\mathbb{C}^n \to \mathbb{R}$. Then*

$$\lim_{y\to 0} \int_a^b g\big(a_d y^{(2d-1)/4} \vartheta_P(\omega_1 x + \mathrm{i}y), \ldots, a_d y^{(2d-1)/4} \vartheta_P(\omega_n x + \mathrm{i}y)\big) dx \\ = \int_{\mathbb{C}^n} \int_a^b g(Z_1\, W(t), \ldots, Z_n\, W(t)) \prod_{j=1}^n d\rho_{d,\alpha}(Z_j)\, dt. \tag{3.15}$$

The proof of this theorem uses a sophisticated equidistribution theorem by Shah [25] which exploits Ratner's classification of measures invariant under unipotent flows. I recommend [22] for an excellent introduction to Ratner's theory.

4. Metaplectic representation

This section provides background on the Shale-Weil representation of the universal covering group of $\mathrm{SL}(2, \mathbb{R})$, which provides a natural framework for the theory of theta functions. More details can be found in [17, 19, 20].

4.1. $\mathrm{SL}(2,\mathbb{R})$ **and its universal cover .** The action of $\mathrm{SL}(2,\mathbb{R})$ on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \,:\, \operatorname{Im} z > 0\}$ is defined by fractional linear transformations, i.e.,

$$g : z \mapsto gz = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}). \tag{4.1}$$

The function $\epsilon_g(z) = (cz+d)/|cz+d|$ satisfies the relation $\epsilon_{gh}(z) = \epsilon_g(hz)\,\epsilon_h(z)$. The universal covering group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$ is defined as the set of pairs $[g,\beta_g]$ where $g \in \mathrm{SL}(2,\mathbb{R})$ and β_g is a continuous function of $\mathbb{H}$ such that $\mathrm{e}^{\mathrm{i}\beta_g(z)} = \epsilon_g(z)$. The multiplication law for $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is

$$[g,\beta_g^1][h,\beta_h^2] = [gh,\beta_{gh}^3], \quad \beta_{gh}^3(z) = \beta_g^1(hz) + \beta_h^2(z). \tag{4.2}$$

We may identify $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with $\mathbb{H} \times \mathbb{R}$ via $[g,\beta_g] \mapsto (z,\phi) = (g\mathrm{i}, \beta_g(\mathrm{i}))$. The action of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ on $\mathbb{H} \times \mathbb{R}$ is then canonically defined by

$$[g,\beta_g](z,\phi) = (gz, \phi + \beta_g(z)). \tag{4.3}$$

The Haar measure of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ reads in these coordinates

$$d\mu = y^{-2} dx\, dy\, d\phi. \tag{4.4}$$

The congruence subgroup

$$\Gamma_1(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,:\, a \equiv d \equiv 1,\, c \equiv 0 \bmod 4 \right\}, \tag{4.5}$$

lifts to the following discrete subgroup of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$,

$$\Delta_1(4) = \left\{ [\gamma,\beta_\gamma] \,:\, \gamma \in \Gamma_1(4),\ \mathrm{e}^{\mathrm{i}\beta_\gamma(z)/2} = j_\gamma(z) \right\}, \tag{4.6}$$

where

$$j_\gamma(z) = \left(\frac{c}{d}\right) \left(\frac{cz+d}{|cz+d|} \right)^{1/2}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(4). \tag{4.7}$$

Here $z^{1/2}$ denotes the principal branch of the square-root of z, i.e., the one for which $-\pi/2 < \arg z^{1/2} \leq \pi/2$; $\left(\frac{c}{d}\right)$ denotes the generalized quadratic residue symbol.

4.2. **Shale-Weil representation.** For every $g \in \mathrm{SL}(2,\mathbb{R})$ we have the unique Iwasawa decomposition

$$g = n_x\, a_y\, k_\phi = (z,\phi), \tag{4.8}$$

where $z = x + \mathrm{i}y \in \mathbb{H}$, $\phi \in [0,2\pi)$, and

$$n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a_y = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}, \quad k_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

This can be extended to an Iwasawa decomposition of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, which of course corresponds to the parametrization introduced after (4.2). We have for any element $\tilde{g} = [g,\beta_g] \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$

$$\tilde{g} = [g,\beta_g] = \tilde{n}_x\, \tilde{a}_y\, \tilde{k}_\phi = [n_x, 0]\ [a_y, 0]\ [k_\phi, \beta_{k_\phi}] \tag{4.9}$$

The Shale-Weil representation is usually defined as a projective representation of $\mathrm{SL}(2,\mathbb{R})$, which becomes a true representation of the metaplectic (i.e., double) cover of $\mathrm{SL}(2,\mathbb{R})$. It is therefore also a proper representation of the universal cover $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. In view of the decomposition (4.9) it is sufficient to define the representation on the three factors. For any $f \in \mathrm{L}^2(\mathbb{R})$ we set (cf. [17])

$$[R(\tilde{n}_x)f](t) = e(t^2 x) f(t), \tag{4.10}$$

$$[R(\tilde{a}_y)f](t) = y^{1/4} f(y^{1/2} t), \tag{4.11}$$

and
(4.12)

$$[R(\tilde{k}_\phi)f](t) = \begin{cases} e(-\sigma_\phi/8)\, f(t) & (\phi = 0 \bmod 2\pi) \\ e(-\sigma_\phi/8)\, f(-t) & (\phi = \pi \bmod 2\pi) \\ \dfrac{e(-\sigma_\phi/8)\, 2^{1/2}}{|\sin\phi|^{1/2}} \displaystyle\int_{\mathbb{R}} e\left[\frac{(t^2+t'^2)\cos\phi - 2tt'}{\sin\phi}\right] & f(t')\, dt' \\ & (\phi \neq 0 \bmod \pi) \end{cases}$$

where

$$\sigma_\phi = \begin{cases} 2\nu & \text{if } \phi = \nu\pi, \\ 2\nu+1 & \text{if } \nu\pi < \phi < (\nu+1)\pi. \end{cases}$$

Let us denote by $\mathcal{S}_\eta(\mathbb{R})$ the space of functions with the property that

$$\sup_{t,\phi}(1+|t|)^\eta |[R(K_\phi)f](t)| < \infty. \tag{4.13}$$

Schwartz functions satisfy this condition for η arbitrarily large. Other examples will be discussed in Section 7.

5. THETA SERIES

We now introduce a general class of theta series. Their functional relations will allow us to view them as functions on a homogeneous space of the form $\Gamma\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R})$ where Γ is a lattice in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. The theta functions that appear in Theorem 3.1 correspond to special choices of cut-off function f, see Sections 6 and 7 for details.

5.1. **Even theta series.** For $f \in \mathcal{S}_\eta(\mathbb{R})$, $\eta > 1$, and $(z,\phi) \in \mathbb{H}\times\mathbb{R} \simeq \widetilde{\mathrm{SL}}(2,\mathbb{R})$ we define the theta series

$$\Theta_f^+(z,\phi) := \Theta_f^+(\tilde{g}) := \sum_{n\in\mathbb{Z}} [R(\tilde{g})f](n), \tag{5.1}$$

with $\tilde{g} = \tilde{n}_x \tilde{a}_y \tilde{k}_\phi$. More explicitly,

$$\Theta_f^+(z,\phi) = y^{1/4} \sum_{n\in\mathbb{Z}} f_\phi(ny^{1/2})\, e(n^2 x), \tag{5.2}$$

where $f_\phi = R(\tilde{k}_\phi)f$. In view of (4.13), the series in (5.1), (5.2) converges therefore absolutely and uniformly for (z,ϕ) with z in any compact set in $\mathbb{H}$.

We call Θ_f^+ an even theta series as it is invariant under $f(t) \mapsto f(-t)$. That is, $\Theta_f^+ = 0$ for f odd. Note that the choice $f(t) = \exp(-2\pi t^2)$ leads to the classical theta series $\Theta_f^+(z,0) = y^{1/4}\sum_{n\in\mathbb{Z}} e(n^2 z)$.

It is well known that Θ_f^+ is invariant under the discrete subgroup $\Delta_1(4)$ (see e.g. [19], Proposition 3.1), i.e.,

$$\Theta_f^+(\tilde{\gamma}\tilde{g}) = \Theta_f^+(\tilde{g}) \tag{5.3}$$

for all $\tilde{\gamma} \in \Delta_1(4)$. We may therefore view Θ_f^+ as a continuous function on the manifold $\mathcal{Y}$. $\mathcal{Y}$ has three cusps located at $z = 0$, $1/2$, and ∞. The theta series is unbounded in two of them (cf. [19], Proposition 3.2):

$$\Theta_f^+(z,\phi) = \begin{cases} e^{\mathrm{i}\pi/4} f_{\phi_0}(0)\, y_0^{1/4} + O(y_0^{-(2\eta-1)/4}) & (y_0 \geq 1) \\ O(y_{1/2}^{-(2\eta-1)/4}) & (y_{1/2} \geq 1) \\ f_{\phi_\infty}(0)\, y_\infty^{1/4} + O(y_\infty^{-(2\eta-1)/4}) & (y_\infty \geq 1), \end{cases} \tag{5.4}$$

with the cuspidal coordinates

$$\begin{cases} (z_0,\phi_0) = (-(4z)^{-1}, \phi + \arg z) \\ (z_{1/2},\phi_{1/2}) = (-(4z-2)^{-1}, \phi + \arg(z-1/2)) \\ (z_\infty,\phi_\infty) = (z,\phi). \end{cases} \tag{5.5}$$

5.2. **Odd theta series.** The transformation formulas (5.3) and asymptotic relations (5.4) are a consequence of two fundamental functional relations, which we state for the slightly generalized theta series

$$\Theta_f(z,\phi;\xi) = y^{1/4} \mathrm{e}^{-\mathrm{i}\pi\xi_1\xi_2} \sum_{n\in\mathbb{Z}} f_\phi\big((n-\xi_2)y^{1/2}\big)\, e\big((n-\xi_2)^2 x + n\xi_1\big), \tag{5.6}$$

where $\xi = {}^{\mathrm{t}}(\xi_1,\xi_2) \in \mathbb{R}^2$. We then have

$$\Theta_f\left(z+\frac{1}{2},\phi;\begin{pmatrix}\xi_1+\xi_2+\frac{1}{2}\\ \xi_2\end{pmatrix}\right) = \mathrm{e}^{-\mathrm{i}\pi\xi_2/2}\Theta_f(z,\phi;\xi) \tag{5.7}$$

and

$$\Theta_f\left(-\frac{1}{4z},\phi+\arg z;\begin{pmatrix}-\xi_2\\ \xi_1\end{pmatrix}\right) = \mathrm{e}^{-\mathrm{i}\pi/4}\Theta_f(z,\phi;\xi). \tag{5.8}$$

The first formula follows from elementary substitution, while the second is a consequence of Poisson summation. So Θ_f^+ is in fact quasi-invariant under the group Γ generated by the elements

$$\tilde\gamma_1 = \left[\begin{pmatrix}1&1\\0&1\end{pmatrix},0\right], \qquad \tilde\gamma_2 = \left[\begin{pmatrix}0&-1/2\\2&0\end{pmatrix},\arg\right], \tag{5.9}$$

i.e.,

$$\Theta_f^+(\tilde\gamma\tilde g) = \chi(\tilde\gamma)\Theta_f^+(\tilde g) \tag{5.10}$$

for all $\tilde\gamma\in\Gamma$ where $\chi(\tilde\gamma)$ is the character of Γ defined by $\chi(\tilde\gamma_1)=1$ and $\chi(\tilde\gamma_2) = \mathrm{e}^{-\mathrm{i}\pi/4}$.

We define the odd partner of Θ_f^+ by

$$\begin{aligned}\Theta_f^-(z,\phi) &= \mathrm{e}^{\mathrm{i}\pi/4}\Theta_f\left(z,\phi;\begin{pmatrix}\frac12\\ \frac12\end{pmatrix}\right)\\ &= y^{1/4}\sum_{n\in\mathbb{Z}}(-1)^n f_\phi\left(\left(n-\frac12\right)y^{1/2}\right) e\left(\left(n-\frac12\right)^2 x\right).\end{aligned} \tag{5.11}$$

Note that Θ_f^- vanishes for even f. Eqs. (5.7) and (5.8) imply

$$\Theta_f^-\left(z+\frac12,\phi\right) = \mathrm{e}^{\mathrm{i}\pi/4}\Theta_f^-(z,\phi) \tag{5.12}$$

$$\Theta_f^-\left(-\frac{1}{4z},\phi+\arg z\right) = \mathrm{e}^{\mathrm{i}\pi/4}\Theta_f^-(z,\phi), \tag{5.13}$$

respectively. These relations prove that Θ_f^- is variant under the discrete subgroup

$$\Delta(4) = \left\{[\gamma,\beta_\gamma] \;:\; \gamma\in\Gamma(4),\ \mathrm{e}^{\mathrm{i}\beta_\gamma(z)/2} = j_\gamma(z)\right\}, \tag{5.14}$$

with the principal congruence subgroup

$$\Gamma(4) = \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z}) \;:\; a\equiv d\equiv 1,\, b\equiv c\equiv 0 \bmod 4\right\}. \tag{5.15}$$

Hence Θ_f^- is a continuous function on the homogeneous space $\mathcal{Y}_- = \Delta(4)\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R})$. We furthermore note that the function $|\Theta_f^-(z/2)|$ is invariant under the elements

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{5.16}$$

and hence may be viewed as a continuous function on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$. This space has one cusp at $z\to\infty$, where $\Theta_f^-(z,\phi)\to 0$. This shows that, unlike the even theta series, Θ_f^- is a bounded function.

5.3. **Some useful formulas.** Subsequent application of (5.8) and (5.7) yields

$$\Theta_f\left(z,\phi;\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}\right) = \mathrm{e}^{\mathrm{i}\pi/4}\Theta_f^+\left(-\frac{1}{4z}-\frac{1}{2},\phi+\arg z\right). \tag{5.17}$$

Setting $f_\pm(w)=\frac{1}{2}(f(w)\pm f(-w))$, we have

$$\Theta_{f_\pm}\left(z,\phi;\begin{pmatrix} 0 \\ \frac{3}{4} \end{pmatrix}\right) = \pm\Theta_{f_\pm}\left(z,\phi;\begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}\right). \tag{5.18}$$

In the even case we average average over the two and then use (5.17),

$$\begin{aligned}\Theta_{f_+}\left(z,\phi;\begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}\right) &= \frac{1}{\sqrt{2}}\Theta_{f_+}\left(\frac{z}{4},\phi;\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}\right) \\ &= \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2}}\Theta_{f_+}\left(-\frac{1}{z}-\frac{1}{2},\phi+\arg z;0\right) \\ &= \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2}}\Theta_f^+\left(-\frac{1}{z}-\frac{1}{2},\phi+\arg z\right).\end{aligned} \tag{5.19}$$

Likewise, in odd case we have

$$\begin{aligned}\Theta_{f_-}\left(z,\phi;\begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}\right) &= \frac{1}{\sqrt{2}}\Theta_{f_-}\left(\frac{z}{4},\phi;\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}\right) \\ &= \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2}}\Theta_f^-\left(\frac{z}{4},\phi\right).\end{aligned} \tag{5.20}$$

So

$$\Theta_f\left(z,\phi;\begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}\right) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2}}\Theta_f^+\left(-\frac{1}{z}-\frac{1}{2},\phi+\arg z\right)+\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2}}\Theta_f^-\left(\frac{z}{4},\phi\right). \tag{5.21}$$

We summarize the above results in group notation, for $\alpha\in\mathbb{Z}$:

$$\Theta_f\left(\tilde g;\begin{pmatrix} 0 \\ \frac{\alpha}{4} \end{pmatrix}\right) = \begin{cases} \Theta_f^+(\tilde g) & (\alpha\equiv 0 \bmod 4) \\ \mathrm{e}^{\mathrm{i}\pi/4}\Theta_f^+\left(\left[\begin{pmatrix} -1 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix},\arg\right]\tilde g\right) & (\alpha\equiv 2 \bmod 4) \\ \widetilde\Theta_f^\pm\left(\left[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix},0\right]\tilde g\right) & (\alpha\equiv \pm 1 \bmod 4) \end{cases} \tag{5.22}$$

where

$$\widetilde\Theta_f^\pm(\tilde g) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2}}\Theta_f^+\left(\left[\begin{pmatrix} -1 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix},\arg\right]\tilde g\right)\pm\frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{2}}\Theta_f^-(\tilde g). \tag{5.23}$$

Because

$$\left[\begin{pmatrix} -1 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix},\arg\right]\Delta(4)\left[\begin{pmatrix} -1 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix},\arg\right]^{-1}\subset\Delta_1(4) \tag{5.24}$$

we have

$$\widetilde\Theta_f^\pm(\tilde\gamma\tilde g) = \widetilde\Theta_f^\pm(\tilde g) \tag{5.25}$$

for all $\tilde{\gamma} \in \Delta(4)$.

6. Proof of Theorem 3.1

In view of (2.6),

$$\begin{aligned} \vartheta_P(z) &= \sum_{k=0}^{\infty} \sum_{i=l}^{\delta_k} e\left\{ \left[\left(k+\frac{\alpha}{4}\right)^2 + \mu_{kl} \right] z \right\} \\ &= \sum_{k=0}^{\infty} e\left\{ \left(k+\frac{\alpha}{4}\right)^2 z \right\} \left[\left(k+\frac{\alpha}{4}\right)^{d-1} W(z) + O(k^{d-2}) \right]. \end{aligned} \tag{6.1}$$

Now

$$W(z) = W(x) + O(y), \tag{6.2}$$

and furthermore,

$$\left| \sum_{k=0}^{\infty} O(k^{d-2}) e\left\{ \left(k+\frac{\alpha}{4}\right)^2 z \right\} \right| \ll \sum_{k=0}^{\infty} k^{d-2} \mathrm{e}^{-2\pi\left(k+\frac{\alpha}{4}\right)^2 y} \ll y^{-(d-1)/2}. \tag{6.3}$$

Therefore

$$\begin{aligned} & a_d y^{(2d-1)/4} \vartheta_P(z) \\ &= a_d y^{(2d-1)/4} W(x) \sum_{k=0}^{\infty} \left(k+\frac{\alpha}{4}\right)^{d-1} e\left\{ \left(k+\frac{\alpha}{4}\right)^2 z \right\} + O(y^{1/4}) \\ &= W(x) \Theta_f\left(z, 0; \begin{pmatrix} 0 \\ -\frac{\alpha}{4} \end{pmatrix}\right) + O(y^{1/4}) \end{aligned} \tag{6.4}$$

with

$$f(t) = \begin{cases} 0 & (t \le 0) \\ a_d t^{d-1} \mathrm{e}^{-2\pi t^2} & (t > 0). \end{cases} \tag{6.5}$$

So

$$\Theta(z) = \Theta_f\left(z, 0; \begin{pmatrix} 0 \\ -\frac{\alpha}{4} \end{pmatrix}\right), \tag{6.6}$$

and hence Theorem 3.1 follows from the results of Section 5, provided $f \in \mathcal{S}_\eta(\mathbb{R})$ with $\eta > 1$, i.e., satisfies condition (4.13). This will be resolved in the next section, which at the same time provides some explicit formulae for f_ϕ.

7. Some explicit formulae

7.1. **Parabolic cylinder functions.** To work out $f_\phi(t)$ corresponding to f as in (6.5) we may assume without loss of generality $0 < \phi < \pi$ since $f_{\phi+\pi}(t) = \mathrm{e}^{-\mathrm{i}\pi/2} f_\phi(-t)$. We have

$$\begin{aligned} f_\phi(t) &= a_d \mathrm{e}^{-\mathrm{i}\pi/4} 2^{1/2} (\sin\phi)^{-1/2} e\left(t^2 \cot\phi\right) \int_0^\infty t'^{d-1} e\left[\frac{t'^2 \mathrm{e}^{\mathrm{i}\phi} - 2tt'}{\sin\phi} \right] dt' \\ &= \frac{a_d \mathrm{e}^{-\mathrm{i}\phi/2} 2^{1/2} e\left(t^2 \cot\phi\right)}{(1-\mathrm{i}\cot\phi)^{(d-1)/2}} \int_0^\infty t'^{d-1} \mathrm{e}^{-2\pi t'^2 - 4\pi\mathrm{i}tt'(1+\mathrm{i}\cot\phi)^{1/2}} dt' \\ &= \frac{a_d \mathrm{e}^{-\mathrm{i}\phi/2} 2^{1/2} \Gamma(d) \mathrm{e}^{-\pi t^2(1-\mathrm{i}\cot\phi)}}{(4\pi)^{d/2}(1-\mathrm{i}\cot\phi)^{(d-1)/2}} D_{-d}\big(\mathrm{i}t[4\pi(1+\mathrm{i}\cot\phi)]^{1/2}\big) \end{aligned} \tag{7.1}$$

where $D_p(z)$ is a parabolic cylinder function (cf. [9] 3.462).

The asymptotic behaviour of $D_p(z)$ as $z \to \infty$ (cf. [9] 9.246) implies

$$f_\phi(t) \sim \frac{a_d \mathrm{e}^{-\mathrm{i}\phi/2} 2^{1/2}\Gamma(d)}{(4\pi)^d} (\sin\phi)^{d-1}(1+\mathrm{i}\cot\phi)^{-1/2} e(t^2\cot\phi)(\mathrm{i}t)^{-d}, \tag{7.2}$$

for $t \to \infty$, and hence $|f_\phi(t)| \ll |t|^{-d}$ where the implied constant is independent of ϕ. This means $f \in \mathcal{S}_d(\mathbb{R})$ for $d > 1$.

7.2. **Hermite polynomials.** As we have seen in the previous section the theta series Θ_f can be decomposed into Θ_f^+ and Θ_f^- which depend only on the even or odd part of f, respectively. In this case, and d suitably odd or even, we can select instead

$$f(t) = \frac{a_d}{2}\, t^{d-1} \mathrm{e}^{-2\pi t^2}, \tag{7.3}$$

as a test function and expect better analytic properties. Indeed,

(7.4)

$$\begin{aligned} f_\phi(t) &= \frac{a_d \mathrm{e}^{-\mathrm{i}\pi\sigma_\phi/4}}{|2\sin\phi|^{1/2}} e\left(t^2\cot\phi\right) \int_{-\infty}^{\infty} t'^{d-1} e\left(\frac{t'^2\mathrm{e}^{\mathrm{i}\phi} - 2tt'}{\sin\phi}\right) dt' \\ &= \frac{a_d \mathrm{e}^{-\mathrm{i}\pi\sigma_\phi/4}}{|2\sin\phi|^{1/2}} e\left(t^2\cot\phi\right) \left(-\frac{\sin\phi}{4\pi\mathrm{i}}\right)^{d-1} \frac{d^{d-1}}{dt^{d-1}} \int_{-\infty}^{\infty} e\left(\frac{t'^2\mathrm{e}^{\mathrm{i}\phi} - 2tt'}{\sin\phi}\right) dt' \\ &= \frac{a_d \mathrm{e}^{-\mathrm{i}\phi/2}}{2} \left(-\frac{\sin\phi}{4\pi\mathrm{i}}\right)^{d-1} e\left(t^2\cot\phi\right) \frac{d^{d-1}}{dt^{d-1}} e\left(t^2(\mathrm{i}-\cot\phi)\right) \\ &= \frac{a_d \mathrm{e}^{-\mathrm{i}\phi/2}}{2} \left(\frac{\sin\phi}{\mathrm{i}\sqrt{8\pi}}\right)^{d-1} (1+\mathrm{i}\cot\phi)^{(d-1)/2} H_{d-1}\big(2\pi t(1+\mathrm{i}\cot\phi)^{1/2}\big) \mathrm{e}^{-2\pi t^2}, \end{aligned}$$

where H_ν are the Hermite polynomials. The above calculation is best checked first in the case $0 < \phi < \pi$ and then extended to all ϕ using the relation $f_{\phi+\pi}(t) = \mathrm{e}^{-\mathrm{i}\pi/2} f_\phi(-t)$. Note that the above implies the bound $|f_\phi(t)| \ll |t|^{d-1}\mathrm{e}^{-2\pi t^2}$ where the implied constant is independent of ϕ.

We will now change gear and collect the ergodic-theoretic results required for the proof of Theorems 3.2, 3.4 and 3.5.

8. Logarithm laws

Let Γ be a non-uniform lattice in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, i.e., the homogeneous space $\mathcal{Y} = \Gamma\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is non-compact and has finite volume. The right actions

$$\mathcal{Y} \to \mathcal{Y}, \qquad \Gamma\tilde{g}_0 \mapsto \Gamma\tilde{g}_0\tilde{a}_{\exp t} \tag{8.1}$$

and

$$\mathcal{Y} \to \mathcal{Y}, \qquad \Gamma\tilde{g}_0 \mapsto \Gamma\tilde{g}_0\tilde{n}_x \tag{8.2}$$

define flows on $\mathcal{Y}$. Under the natural projection $\pi : \mathcal{Y} \to \mathcal{Y}/\{\tilde{k}_\phi : \phi \in \mathbb{R}\}$, the above flows become the classical geodesic and horocycle flows on the hyperbolic surface $\Gamma\backslash\mathbb{H} = \mathcal{Y}/\{\tilde{k}_\phi : \phi \in \mathbb{R}\}$. We denote by $\mathrm{dist}(z, z')$ the geodesic distance on $\Gamma\backslash\mathbb{H} = \mathcal{Y}/\{\tilde{k}_\phi : \phi \in \mathbb{R}\}$. We recall that the horocycle $\{\tilde{n}_x : x \in \mathbb{R}\}$ parametrizes the unstable direction of the geodesic flow for $t \to -\infty$.

The following theorem is of a similar type as the dynamical logarithm laws first discussed by Sullivan [27], and may be viewed as a special case of Kleinbock and Margulis [16].

Theorem 8.1. *Fix $\tilde{g}_0 \in \mathcal{Y}$. Let $r : [1, \infty) \to \mathbb{R}_+$ be a non-decreasing function such that the integral*

$$\int_1^\infty \mathrm{e}^{-r(t)} dt \tag{8.3}$$

diverges (resp. converges). Then, for almost every (resp. almost no) $x \in \mathbb{R}$ there is an infinite sequence of $t_1 < t_2 < \ldots \to \infty$ such that

$$\operatorname{dist}(\pi(\tilde{g}_0), \pi(\tilde{g}_0 \tilde{n}_x \tilde{a}_{\exp -t_j})) \geq r(t_j). \tag{8.4}$$

By taking $r(t) = (1 \pm \epsilon) \log t$ with $\epsilon > 0$ small, the theorem implies that for almost all x we have

$$\limsup_{y \to 0} \frac{\operatorname{dist}(\pi(\tilde{g}_0), \pi(\tilde{g}_0 \tilde{n}_x \tilde{a}_y))}{\log \log y^{-1}} = 1. \tag{8.5}$$

Theorem 8.1 can be generalized and strengthened in several ways [16]. On may for example assume in addition that the sequence $\{t_j\}$ is such that the points $\Gamma \tilde{g}_0 \tilde{n}_x \tilde{a}_{\exp -t_j}$ are contained in only one of the several possible cusps. In the case of $\Delta_1(4)$ we take this cusp to be the one at ∞, where the asymptotic behaviour (5.4) says that

$$\log |\Theta_f^+(\tilde{g}_0 \tilde{n}_x \tilde{a}_{\exp -t_j})|^4 \sim \operatorname{dist}(\pi(\tilde{g}_0), \pi(\tilde{g}_0 \tilde{n}_x \tilde{a}_{\exp -t_j})) + \log |f_{\phi_j}(0)|^4. \tag{8.6}$$

We may assume in addition that the sequence t_j is chosen optimally, in the sense that $\tilde{g}_0 \tilde{n}_x \tilde{a}_{\exp -t_j}$ corresponds to the highest point in the cusp. At this point $\phi_j = \pi/4$, which explains the significance of the term $f_{\pi/4}(0)$ in the following corollary.

Corollary 8.2. *Suppose $f \in \mathcal{S}_\eta(\mathbb{R})$ for some $\eta > 1$ and $f_{\pi/4}(0) \neq 0$. Let $\psi : (0, 1] \to \mathbb{R}_+$ be a non-increasing function such that the integral*

$$\int_0^1 \frac{dy}{y \psi(y)^4} \tag{8.7}$$

diverges (resp. converges). Then, for almost every (resp. almost no) $x \in \mathbb{R}$ there is an infinite sequence of $y_1 > y_2 > \ldots \to 0$ such that

$$|\Theta_f^+(\tilde{g}_0 \tilde{n}_x \tilde{a}_{y_j})| \geq \psi(y_j). \tag{8.8}$$

This means in particular that for almost all x

$$\limsup_{y \to 0} \frac{\log |\Theta_f^+(\tilde{g}_0 \tilde{n}_x \tilde{a}_y)|}{\log \log y^{-1}} = \frac{1}{4}. \tag{8.9}$$

Since $\Theta_f^-(\tilde{g})$ is bounded, the above statements also hold for $\widetilde{\Theta}_f^\pm(\tilde{g})$, and thus for

$$\Theta_f\left(\tilde{g}, \begin{pmatrix} 0 \\ \frac{\alpha}{4} \end{pmatrix}\right).$$

9. Equidistribution

We will now consider the equidistribution of $\tilde{a}_y$-translates of arcs of horocycles, and discuss the implication on the value distribution properties of theta series. We follow the approach developed in [19].

Theorem 9.1. *Fix an interval $[a, b]$, $\tilde{g}_0 \in \widetilde{\mathrm{SL}}(2, \mathbb{R})$ and a bounded continuous function $F : \mathcal{Y} \to \mathbb{R}$. Then*

$$\lim_{y \to 0} \int_a^b F(\tilde{g}_0 \tilde{n}_x \tilde{a}_y) dx = \frac{b - a}{\mu(\mathcal{Y})} \int_{\mathcal{Y}} F(\tilde{g}) \, d\mu(\tilde{g}). \tag{9.1}$$

This theorem may be proved by exploiting the mixing property of the diagonal flow generated by $\tilde{a}_y$ (using the same ideas as in [7]), or by using the classification of measures invariant under the horocycle flow (see Theorem 10.1 below). Special cases, such as translates of arcs of closed horocycles, can also be handled analytically and permit the calculation of explicit convergence rates, cf. [23, 20, 26] and references therein.

The following theorem is a slight generalization that allows for x-dependent test functions.

Theorem 9.2. *Fix an interval* $[a, b]$*,* $\tilde{g}_0 \in \widetilde{\mathrm{SL}}(2, \mathbb{R})$ *and a bounded continuous function* $F : [a, b] \times \mathcal{Y} \to \mathbb{R}$*. Then*

$$\lim_{y\to 0} \int_a^b F(x, \tilde{g}_0 \tilde{n}_x \tilde{a}_y) dx = \frac{1}{\mu(\mathcal{Y})} \int_{\mathcal{Y}} \int_a^b F(t, \tilde{g})\, dt\, d\mu(\tilde{g}). \tag{9.2}$$

Proof. We assume in this proof that the Haar measure is normalized, i.e., $\mu(\mathcal{Y}) = 1$. As a first step let us suppose that F has compact support, and hence is uniformly continuous. Therefore, given $\delta > 0$ there exists $\epsilon > 0$ such that

$$F(x_0, \tilde{g}) - \delta \le F(x, \tilde{g}) \le F(x_0, \tilde{g}) + \delta \tag{9.3}$$

for all $x \in [x_0, x_0 + \epsilon]$, $x_0 \in [a, b - \epsilon]$, $\tilde{g} \in \widetilde{\mathrm{SL}}(2, \mathbb{R})$. Now, for $n = [2(b - a)/\epsilon]$

$$\int_a^b F(x, \tilde{g}_0 \tilde{n}_x \tilde{a}_y) dx = \sum_{j=0}^{n-1} \int_{a+(b-a)j/n}^{a+(b-a)(j+1)/n} F(x, \tilde{g}_0 \tilde{n}_x \tilde{a}_y) dx \tag{9.4}$$

$$\le \sum_{j=0}^{n-1} \int_{a+(b-a)j/n}^{a+(b-a)(j+1)/n} F(a + (b - a)j/n, \tilde{g}_0 \tilde{n}_x \tilde{a}_y) dx + (b - a)\delta.$$

Theorem 9.1 implies

$$\lim_{y\to 0} \int_{a+(b-a)j/n}^{a+(b-a)(j+1)/n} F(a + (b - a)j/n, \tilde{g}_0 \tilde{n}_x \tilde{a}_y) dx = \frac{b - a}{n} \int_{\mathcal{Y}} F(a + (b - a)j/n, \tilde{g}) d\mu(\tilde{g}) \tag{9.5}$$

and hence

$$\limsup_{y\to 0} \int_a^b F(x, \tilde{g}_0 \tilde{n}_x \tilde{a}_y) dx \le \frac{b - a}{n} \sum_{j=0}^{n-1} \int_{\mathcal{Y}} F(a + (b - a)j/n, \tilde{g}) d\mu(\tilde{g}) + (b - a)\delta. \tag{9.6}$$

The sum over j is a Riemann sum with discrepancy δ, so

$$\limsup_{y\to 0} \int_a^b F(x, \tilde{g}_0 \tilde{n}_x \tilde{a}_y) dx \le \int_a^b \int_{\mathcal{Y}} F(t, \tilde{g}) d\mu(\tilde{g}) dt + (b - a)2\delta. \tag{9.7}$$

By the same argument,

$$\liminf_{y\to 0} \int_a^b F(x, \tilde{g}_0 \tilde{n}_x \tilde{a}_y) dx \ge \int_a^b \int_{\mathcal{Y}} F(t, \tilde{g}) d\mu(\tilde{g})\, dt + (b - a)2\delta, \tag{9.8}$$

and thus, since δ can be arbitrarily small

$$\lim_{y\to 0} \int_a^b F(x, \tilde{g}_0 \tilde{n}_x \tilde{a}_y) dx = \int_a^b \int_{\mathcal{Y}} F(t, \tilde{g}) d\mu(\tilde{g})\, dt. \tag{9.9}$$

This proves the statement of the theorem for compactly supported F. We will now extend this result to bounded continuous F.

Given $\delta > 0$, write $F = F_1 + F_2$ where F_1 is continuous and has compact support, and F_2 is bounded continuous with the properties that $|F_2| \leq K$ (for some $K > 0$) and the set of $\tilde{g} \in \mathcal{M}$ for which $F(t, \tilde{g}) \neq 0$ has measure $< \delta/K$. This can be achieved by choosing the support of F_2 sufficiently deep in the cusps. The function

$$\overline{F}_2(\tilde{g}) = \sup_{t\in[a,b]} |F_2(t,\tilde{g})| \tag{9.10}$$

is bounded continuous on $\mathcal{Y}$. The proof of the theorem now follows from the observation that

$$\begin{aligned} \limsup_{y\to 0} \int_a^b |F(x, \tilde{g}_0 \tilde{n}_x \tilde{a}_y)| dx &\leq \limsup_{y\to 0} \int_a^b \overline{F}(x, \tilde{g}_0 \tilde{n}_x \tilde{a}_y) dx \\ &= (b-a) \int_{\mathcal{Y}} \overline{F}(\tilde{g}) d\mu(\tilde{g}) \\ &< (b-a)\delta, \end{aligned} \tag{9.11}$$

where the equality is again a consequence of Theorem 9.1. □

Corollary 9.3. *Fix an interval* $[a, b]$ *and let* $h : [a, b] \to \mathbb{C}$ *be continuous,* $g : \mathbb{C} \to \mathbb{R}$ *bounded continuous. Then*

$$\begin{aligned} \lim_{y\to 0} \int_a^b g\left[h(x) \Theta_f \left(x + \mathrm{i}y, 0; \begin{pmatrix} 0 \\ \frac{\alpha}{4} \end{pmatrix} \right) \right] dx \\ = \frac{1}{\mu(\mathcal{Y})} \int_a^b \int_{\mathcal{Y}} g\big(h(t) X_f(\tilde{g})\big) d\mu(\tilde{g})\, dt \end{aligned} \tag{9.12}$$

where

$$X_f(\tilde{g}) = \begin{cases} \Theta_f^+(\tilde{g}) & (\alpha \equiv 0 \bmod 4) \\ \mathrm{e}^{\mathrm{i}\pi/4} \Theta_f^+(\tilde{g}) & (\alpha \equiv 2 \bmod 4) \\ \widetilde{\Theta}_f^{\pm}(\tilde{g}) & (\alpha \equiv \pm 1 \bmod 4). \end{cases} \tag{9.13}$$

Proof. Set $F(t, \tilde{g}) = g(h(t) X(\tilde{g}))$ in Theorem 9.2, with the appropriate choices of $\tilde{g}_0$ (these are determined by relation (5.22)). □

That is, the limiting distribution is given by the product of two independent random variables, $h(t)$ and $X(\tilde{g})$ with t and $\tilde{g}$ uniformly distributed on $[a, b]$ and $\mathcal{Y}$ respectively.

10. Equidistribution in products

Let G be a Lie group and Γ a lattice in G. Shah's theorem 1.4 in [25] implies the following.

Theorem 10.1. *Suppose* G *contains a Lie subgroup* H *isomorphic to* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ *(we denote the corresponding embedding by* $\varphi : \widetilde{\mathrm{SL}}(2,\mathbb{R}) \to G$*), such that the set* $\Gamma\backslash\Gamma H$ *is dense in* $\Gamma\backslash G$*. Let* $F : \Gamma\backslash G \to \mathbb{R}$ *be bounded continuous. Then*

$$\lim_{y\to 0} \int_a^b F\big(\varphi(\tilde{n}_x \tilde{a}_y)\big) dx = \frac{b-a}{\mu_G(\Gamma\backslash G)} \int_{\Gamma\backslash G} f d\mu_G \tag{10.1}$$

where μ_G *is the Haar measure of* G*.*

This theorem in turn can be used to prove the following statement. As before $\mathcal{Y} = \Gamma\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R})$ where Γ is a non-uniform lattice.

Theorem 10.2. *Fix an interval* $[a,b]$, $\tilde{g}_0 \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$ *and a bounded continuous function* $F : \mathcal{Y}^n \to \mathbb{R}$. *Suppose* $\tilde{n}_r \in \Gamma$ *for some* $r \in \mathbb{Q}$, *and that* $\omega_1, \ldots, \omega_n \in \mathbb{R}$ *are linearly independent over* $\mathbb{Q}$. *Then*

$$\lim_{y\to 0} \int_a^b F(\tilde{g}_0 \tilde{n}_{\omega_1 x} \tilde{a}_y, \ldots, \tilde{g}_0 \tilde{n}_{\omega_n x} \tilde{a}_y) dx = \frac{b-a}{\mu(\mathcal{Y})^n} \int_{\mathcal{Y}^n} F \, d^n\mu. \tag{10.2}$$

Proof. We may assume without loss of generality that $\omega_j > 0$. We use the embedding

$$\varphi : \tilde{g} \to \big(\tilde{a}_{\sqrt{\omega_1}} \tilde{g} \tilde{a}_{\sqrt{\omega_1}}^{-1}, \ldots, \tilde{a}_{\sqrt{\omega_n}} \tilde{g} \tilde{a}_{\sqrt{\omega_n}}^{-1}\big) \tag{10.3}$$

and note that in particular

$$\varphi(\tilde{n}_x \tilde{a}_y) = (\tilde{n}_{\omega_1 x} \tilde{a}_y, \ldots, \tilde{n}_{\omega_n x} \tilde{a}_y). \tag{10.4}$$

To establish the density required in Shah's theorem, note that by Weyl's equidistribution theorem for the Kronecker flow the set

$$\begin{aligned}&\{(\tilde{n}_{m_1 r}, \ldots, \tilde{n}_{m_n r})\varphi(\tilde{n}_x \tilde{a}_y) : (m_1, \ldots, m_n) \in \mathbb{Z}^n,\ x \in \mathbb{R}\} \\ &\qquad = \{(\tilde{n}_{\omega_1 x + m_1 r} \tilde{a}_y, \ldots, \tilde{n}_{\omega_n x + m_n r} \tilde{a}_y) : (m_1, \ldots, m_n) \in \mathbb{Z}^n,\ x \in \mathbb{R}\}\end{aligned} \tag{10.5}$$

is dense in

$$\{(\tilde{n}_{x_1} \tilde{a}_y, \ldots, \tilde{n}_{x_n} \tilde{a}_y) : (x_1, \ldots, x_n) \in \mathbb{R}^n\}, \tag{10.6}$$

and, since by Theorem 9.1 the set $\Gamma\backslash\Gamma\{\tilde{n}_x \tilde{a}_y : x \in \mathbb{R}\}$ is dense in $\mathcal{Y}$, the set (10.6) is dense in $\mathcal{Y}^n$. □

The next statements follows from the analogous arguments as for Theorem 9.2 and Corollary 9.3, respectively.

Theorem 10.3. *Fix a bounded continuous function* $F : [a,b] \times \mathcal{Y}^n \to \mathbb{R}$. *Then, under the assumptions of Theorem 10.2,*

$$\lim_{y\to 0} \int_a^b F(x, \tilde{g}_0 \tilde{n}_{\omega_1 x} \tilde{a}_y, \ldots, \tilde{g}_0 \tilde{n}_{\omega_n x} \tilde{a}_y) dx = \frac{1}{\mu(\mathcal{Y})^n} \int_{\mathcal{Y}^n} \int_a^b F(t)\, dt\, d^n\mu. \tag{10.7}$$

Corollary 10.4. *Fix an interval* $[a,b]$ *and let* $h_1, \ldots, h_n : [a,b] \to \mathbb{C}$ *be continuous,* $g : \mathbb{C}^n \to \mathbb{R}$ *bounded continuous. Suppose* $\omega_1, \ldots, \omega_n \in \mathbb{R}$ *are linearly independent over* $\mathbb{Q}$. *Then*

$$\begin{aligned}\lim_{y\to 0} \int_a^b g\Bigg[\ldots, h_j(x)\Theta_{f_j}\Bigg(\omega_j x + \mathrm{i}y, 0; \begin{pmatrix} 0 \\ \frac{\alpha}{4} \end{pmatrix}\Bigg), \ldots\Bigg] dx \\ = \frac{1}{\mu(\mathcal{Y})^n} \int_{\mathcal{Y}} \int_a^b g\big(\ldots, h_j(t) X_{f_j}(\tilde{g}_j), \ldots\big)\, dt\end{aligned} \tag{10.8}$$

with X_f *as in* (9.13).

11. Proof of Theorems 3.2, 3.4, 3.5

11.1. **Proof of Theorem 3.2.** Theorem 3.2 follows from Theorem 3.1, Corollary 8.2 and the analyticity of $W(t)$. The latter implies that $W(t)$ has at most finitely many zeros in any given compact interval, and hence $|\log W(t)| < \infty$ for almost all t.

11.2. **Proof of Theorem 3.4.** This theorem follows directly from Theorem 3.1 and Corollary 9.3. The asymptotics of the tail distribution follows from [19], p. 144: For general $f \in \mathcal{S}_\eta(\mathbb{R})$ $(\eta > 1)$

$$\frac{\mu\{\tilde{g} \in \Delta_1(4)\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R}) : |\Theta_f^+(\tilde{g})| > R\}}{\mu(\Delta_1(4)\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R}))} \sim \frac{1}{\pi^2 R^4} \int_0^\pi |f_\phi(0)|^4 d\phi \tag{11.1}$$

as $R \to \infty$. Recall that Θ_f^- is bounded so does not contribute to the tails.

For the choice (6.5), and α even,

$$\begin{aligned} c_{d,\alpha} &= \frac{1}{\pi^2} \int_0^\pi |f_\phi(0)|^4 d\phi \\ &= \frac{4a_d^4}{\pi^2} \left| \int_0^\infty t^{d-1} \mathrm{e}^{-2\pi t^2} dt \right|^4 \int_0^\pi (\sin\phi)^{2(d-1)} d\phi \\ &= a_d^4 \frac{\Gamma(d/2)^4}{(2\pi)^{2(d+1)}} B(d-1/2, 1/2) \\ &= a_d^4 \frac{\Gamma(d/2)^4}{(2\pi)^{2(d+1)}} \frac{\Gamma(d-1/2)\Gamma(1/2)}{\Gamma(d)} \\ &= \frac{2^{2(d-1)}}{\pi^{5/2}} \frac{\Gamma(d/2)^4}{\Gamma(d-1/2)\Gamma(d)} \\ &= \frac{2^{4(d-1)}\Gamma(d/2)^4}{\pi^3\Gamma(2d-1)}. \end{aligned} \tag{11.2}$$

For α odd, we have (recall (5.21) which suggests to substitute $R \mapsto \sqrt{2}\,R$)

$$c_{d,\alpha} = \frac{2^{4(d-1)}\Gamma(d/2)^4}{4\pi^3\Gamma(2d-1)}. \tag{11.3}$$

11.3. **Proof of Theorem 3.5.** Analogous to the proof of Theorem 3.4, we apply Theorem 3.1 and Corollary 10.4.

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References

[1] A.L. Besse, Manifolds all of whose geodesics are closed. Springer-Verlag, Berlin-New York, 1978.
[2] J. Chazarain, Formule de Poisson pour les variétés riemanniennes. Invent. Math. 24 (1974), 65–82.
[3] Y. Colin de Verdière, Spectre du laplacien et longueurs des géodésiques périodiques. I. Compositio Math. 27 (1973), 83–106.
[4] Y. Colin de Verdière, Spectre du laplacien et longueurs des géodésiques périodiques. II. Compositio Math. 27 (1973), 159–184.
[5] Y. Colin de Verdière, Sur le spectre des opérateurs elliptiques à bicaractéristiques toutes périodiques. Comment. Math. Helv. 54 (1979), no. 3, 508–522.
[6] J.J. Duistermaat and V.W. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics. Invent. Math. 29 (1975), no. 1, 39–79.
[7] A. Eskin and C. McMullen, Mixing, counting, and equidistribution in Lie groups. Duke Math. J. 71 (1993), no. 1, 181–209.
[8] H. Fiedler, W. Jurkat and O. Körner, Asymptotic expansions of finite theta series, Acta Arithm. 32 (1977), 129–146.
[9] I.S. Gradshteyn and I.M. Ryzhik, Table of integrals, series, and products. Fourth edition. Academic Press, New York-London 1965.
[10] V. Guillemin, Some spectral results on rank one symmetric spaces. Advances in Math. 28 (1978), no. 2, 129–137.

[11] V. Guillemin, Some spectral results for the Laplace operator with potential on the n-sphere. Advances in Math. 27 (1978), no. 3, 273–286.
[12] M.C. Gutzwiller. Chaos in classical and quantum mechanics. Interdisciplinary Applied Mathematics, 1. Springer-Verlag, New York, 1990.
[13] W.B. Jurkat and J.W. van Horne, The proof of the central limit theorem for theta sums, Duke Math. J. (1981) 873–885.
[14] W.B. Jurkat and J.W. van Horne, On the central limit theorem for theta series, Michigan Math. J. 29 (1982) 65–77.
[15] W.B. Jurkat and J.W. van Horne, The uniform central limit theorem for theta sums, Duke Math. J. 50 (1983) 649–666.
[16] D.Y. Kleinbock and G.A. Margulis, Logarithm laws for flows on homogeneous spaces. Invent. Math. 138 (1999), no. 3, 451–494.
[17] G. Lion and M. Vergne, The Weil representation, Maslov index and theta series. Progress in Mathematics, 6. Birkhuser, Boston, Mass., 1980.
[18] J. Marklof, Spectral form factors of rectangle billiards. Comm. Math. Phys. 199 (1998), no. 1, 169–202.
[19] J. Marklof, Limit theorems for theta sums. Duke Math. J. 97 (1999), no. 1, 127–153.
[20] J. Marklof, Theta sums, Eisenstein series, and the semiclassical dynamics of a precessing spin. Emerging applications of number theory (Minneapolis, MN, 1996), 405–450, IMA Vol. Math. Appl., 109, Springer, New York, 1999.
[21] J. Marklof, Pair correlation densities of inhomogeneous quadratic forms. Ann. of Math. (2) 158 (2003), no. 2, 419–471.
[22] D.W. Morris, Ratner's theorems on unipotent flows. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2005.
[23] P. Sarnak, Asymptotic behavior of periodic orbits of the horocycle flow and Eisenstein series. Comm. Pure Appl. Math. 34 (1981), no. 6, 719–739.
[24] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. J. Indian Math. Soc. (N.S.) 20 (1956), 47–87.
[25] N.A. Shah, Limit distributions of expanding translates of certain orbits on homogeneous spaces. Proc. Indian Acad. Sci. Math. Sci. 106 (1996), no. 2, 105–125.
[26] A. Strömbergsson, On the uniform equidistribution of long closed horocycles. Duke Math. J. 123 (2004), no. 3, 507–547.
[27] D. Sullivan, Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics. Acta Math. 149 (1982), no. 1, 215–237.
[28] A. Uribe and S. Zelditch, Spectral statistics on Zoll surfaces. Comm. Math. Phys. 154 (1993), no. 2, 313–346.
[29] A. Weinstein, Asymptotics of eigenvalue clusters for the Laplacian plus a potential. Duke Math. J. 44 (1977), no. 4, 883–892.
[30] S. Zelditch, Fine structure of Zoll spectra. J. Funct. Anal. 143 (1997), no. 2, 415–460.

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