# THE ASYMPTOTIC DISTRIBUTION OF FROBENIUS NUMBERS

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ABSTRACT. The Frobenius number $F(\boldsymbol{a})$ of an integer vector $\boldsymbol{a}$ with positive coprime coefficients is defined as the largest number that does not have a representation as a positive integer linear combination of the coefficients of $\boldsymbol{a}$. We show that if $\boldsymbol{a}$ is taken to be random in an expanding $d$-dimensional domain, then $F(\boldsymbol{a})$ has a limit distribution, which is given by the probability distribution for the covering radius of a certain simplex with respect to a $(d-1)$-dimensional random lattice. This result extends recent studies for $d = 3$ by Arnold, Bourgain-Sinai and Shur-Sinai-Ustinov. The key features of our approach are (a) a novel interpretation of the Frobenius number in terms of the dynamics of a certain group action on the space of $d$-dimensional lattices, and (b) an equidistribution theorem for a multidimensional Farey sequence on closed horospheres.

## 1. INTRODUCTION

Let us denote by $\widehat{\mathbb{Z}}^d = \{\boldsymbol{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d : \gcd(a_1, \ldots, a_d) = 1\}$ the set of primitive lattice points, and by $\widehat{\mathbb{Z}}^d_{\geq 2}$ the subset with coefficients $a_j \geq 2$. Given $\boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2}$, it is well known that any sufficiently large integer $N > 0$ can be represented in the form

$$N = \boldsymbol{m} \cdot \boldsymbol{a} \tag{1.1}$$

with $\boldsymbol{m} \in \mathbb{Z}^d_{\geq 0}$. Frobenius was interested in the largest integer $F(\boldsymbol{a})$ that fails to have a representation of this type. That is,

$$F(\boldsymbol{a}) = \max \mathbb{Z} \setminus \{\boldsymbol{m} \cdot \boldsymbol{a} > 0 : \boldsymbol{m} \in \mathbb{Z}^d_{\geq 0}\}. \tag{1.2}$$

We will refer to $F(\boldsymbol{a})$ as the *Frobenius number* of $\boldsymbol{a}$. In the case of two variables ($d = 2$) Sylvester showed that

$$F(\boldsymbol{a}) = a_1 a_2 - a_1 - a_2. \tag{1.3}$$

No such explicit formulas are known in higher dimensions, cf. [13], [14], [19]. The present paper will discuss a new interpretation of the Frobenius number in terms of the dynamics of a certain flow $\Phi^t$ on the space of lattices $\Gamma\backslash G$, with $G := \mathrm{SL}(d, \mathbb{R})$, $\Gamma := \mathrm{SL}(d, \mathbb{Z})$. This dynamical interpretation is a key step in the proof of the following limit theorem on the asymptotic distribution of the Frobenius number $F(\boldsymbol{a})$, where $\boldsymbol{a}$ is randomly selected from the set $T\mathcal{D} = \{T\boldsymbol{x} : \boldsymbol{x} \in \mathcal{D}\}$, with $T$ large and $\mathcal{D}$ a fixed bounded subset of $\mathbb{R}^d_{\geq 0}$.

**Theorem 1.** *Let $d \geq 3$. There exists a continuous non-increasing function $\Psi_d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\Psi_d(0) = 1$, such that for any bounded set $\mathcal{D} \subset \mathbb{R}^d_{\geq 0}$ with boundary of Lebesgue measure zero, and any $R \geq 0$,*

$$\lim_{T\to\infty} \frac{1}{T^d} \# \left\{ \boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2} \cap T\mathcal{D} : \frac{F(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} > R \right\} = \frac{\mathrm{vol}(\mathcal{D})}{\zeta(d)} \Psi_d(R). \tag{1.4}$$

Variants of Theorem 1 were previously known only in dimension $d = 3$, cf. [7], [21]; see also [3], [4] for related studies and open conjectures, and [2], [7] for results in higher dimensions. The scaling of $F(\boldsymbol{a})$ used in Theorem 1 is consistent with numerical experiments [5, Section 5].

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We will furthermore establish that the limit distribution $\Psi_d(R)$ is given by the distribution of the covering radius of the simplex

$$\Delta = \{ \boldsymbol{x} \in \mathbb{R}_{\geq 0}^{d-1} : \boldsymbol{x} \cdot \boldsymbol{e} \leq 1 \}, \qquad \boldsymbol{e} := (1, 1, \ldots, 1), \tag{1.5}$$

with respect to a random lattice in $\mathbb{R}^{d-1}$. Here, the *covering radius* (sometimes also called *inhomogeneous minimum*) of a set $K \subset \mathbb{R}^{d-1}$ with respect to a lattice $\mathcal{L} \subset \mathbb{R}^{d-1}$ is defined as the infimum of all $\rho > 0$ with the property that $\mathcal{L} + \rho K = \mathbb{R}^{d-1}$.

To state this result precisely, let $\mathbb{Z}^{d-1}A$ be a lattice in $\mathbb{R}^{d-1}$ with $A \in G_0 := \mathrm{SL}(d-1, \mathbb{R})$. The *space of lattices* (of unit covolume) is $\Gamma_0 \backslash G_0$ with $\Gamma_0 := \mathrm{SL}(d-1, \mathbb{Z})$. We denote by $\mu_0$ the unique $G_0$-right invariant probability measure on $\Gamma_0 \backslash G_0$; an explicit formula for $\mu_0$ is given in Section 3.

**Theorem 2.** *Let $\rho(A)$ be the covering radius of the simplex $\Delta$ with respect to the lattice $\mathbb{Z}^{d-1}A$. Then*

$$\Psi_d(R) = \mu_0\big(\{ A \in \Gamma_0 \backslash G_0 : \rho(A) > R \}\big). \tag{1.6}$$

The connection between Frobenius numbers and lattice free simplices is well understood [9], [16]. In particular, Theorem 2 connects nicely to the sharp lower bound of [1] (see also [15]):

$$\frac{F(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} \geq \rho_*, \qquad \text{with } \rho_* := \inf_{A \in \Gamma_0 \backslash G_0} \rho(A). \tag{1.7}$$

It is proved in [1] that $\rho_* > ((d-1)!)^{1/(d-1)} > 0$, and so in particular

$$\Psi_d(R) = 1 \qquad \text{for} \quad 0 \leq R < \rho_*. \tag{1.8}$$

An explicit formula for $\Psi_d(R)$ has recently been derived in dimension $d = 3$ by different techniques, cf. [21]. In this case $\rho_* = \sqrt{3}$.

It is amusing to note that all of the above statements also hold in the trivial case $d = 2$, except for the continuity of the limit distribution: By Sylvester's formula (1.3)

$$\Psi_2(R) = \begin{cases} 1 & (R < 1) \\ 0 & (R \geq 1). \end{cases} \tag{1.9}$$

The covering radius of the simplex $\Delta = [0, 1]$ with respect to the lattice $\mathbb{Z}$ is $\rho(1) = 1$. $\mathbb{Z}$ is of course the unique element in the space of one-dimensional lattices of unit covolume, and hence (1.9) follows also formally from (1.6).

We now give a brief outline of the paper. Section 2 explains the aforementioned dynamical interpretation of the Frobenius number in terms of the right action of a one-parameter subgroup $\Phi^t$ on the space of lattices $\Gamma \backslash G$: We show that there is a function $W_\delta$ of $\Gamma \backslash G$ that produces, when evaluated along a certain orbit of $\Phi^t$, the Frobenius number $F(\boldsymbol{a})$. This observation is the crucial step in the application of an equidistribution theorem for multidimensional Farey sequences on closed horospheres in $\Gamma \backslash G$, which is proved in Section 3. A useful variant of this theorem is discussed in Section 4. Section 5 exploits the equidistribution theorem to give upper and lower bounds for the lim sup and lim inf of (1.4), respectively, and the purpose of the remaining Sections 6 and 7 is to show that the lim sup and lim inf coincide. This is achieved by relating the limit distribution $\Psi_d(R)$ to the covering radius of a simplex with respect to a random lattice (Section 6), and proving that $\Psi_d(R)$ is continuous (Section 7).

The results of Sections 3 and 4 provide a new approach to Schmidt's work [17] on the distribution of (primitive) sublattices of $\mathbb{Z}^d$. Appendix A illuminates this connection by deriving a generalization of Schmidt's Theorem 3 in the case of primitive sublattices of rank $d-1$.

## 2. Dynamical interpretation

Let $G := \mathrm{SL}(d,\mathbb{R})$ and $\Gamma := \mathrm{SL}(d,\mathbb{Z})$, and define

$$n_+(\boldsymbol{x}) = \begin{pmatrix} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{x} & 1 \end{pmatrix}, \qquad n_-(\boldsymbol{x}) = \begin{pmatrix} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{x} \\ \boldsymbol{0} & 1 \end{pmatrix}, \qquad \Phi^t = \begin{pmatrix} \mathrm{e}^{-t} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & \mathrm{e}^{(d-1)t} \end{pmatrix}. \tag{2.1}$$

The right action

$$\Gamma\backslash G \to \Gamma\backslash G, \qquad \Gamma M \mapsto \Gamma M \Phi^t \tag{2.2}$$

defines a flow on the space of lattices $\Gamma\backslash G$. The horospherical subgroups generated by $n_+(\boldsymbol{x})$ and $n_-(\boldsymbol{x})$ parametrize the stable and unstable directions of the flow $\Phi^t$ as $t\to\infty$. This can be seen as follows. Let $d: G\times G \to \mathbb{R}_{\geq 0}$ be a left $G$-invariant Riemannian metric on $G$, i.e., $d(hM,hM') = d(M,M')$ for all $h,M,M'\in G$. We may choose $d$ in such a way that

$$d\big(n_\pm(\boldsymbol{x}), n_\pm(\boldsymbol{x}')\big) \leq \|\boldsymbol{x}-\boldsymbol{x}'\|, \tag{2.3}$$

where $\|\cdot\|$ the standard euclidean norm. Note that $n_-(\boldsymbol{x})\Phi^t = \Phi^t n_-(\mathrm{e}^{dt}\boldsymbol{x})$. Hence, for any $M\in G$,

$$d\big(M n_-(\boldsymbol{x})\Phi^t, M\Phi^t\big) = d\big(M\Phi^t n_-(\mathrm{e}^{dt}\boldsymbol{x}), M\Phi^t\big) = d\big(n_-(\mathrm{e}^{dt}\boldsymbol{x}), 1_d\big) \leq \mathrm{e}^{dt}\|\boldsymbol{x}\|, \tag{2.4}$$

which explains the interpretation of $n_-(\boldsymbol{x})$ as an element in the *unstable* horospherical subgroup. The argument for $n_+(\boldsymbol{x})$ as the stable analogue is identical.

In the following we will represent functions on $\Gamma\backslash G$ as left $\Gamma$-invariant functions on $G$, i.e., functions $f: G\to\mathbb{R}$ that satisfy $f(\gamma M) = f(M)$ for all $\gamma\in\Gamma$. The left $G$-invariant metric $d(\,\cdot\,,\,\cdot\,)$ yields thus a Riemannian metric $d_\Gamma(\,\cdot\,,\,\cdot\,)$ on $\Gamma\backslash G$ by setting

$$d_\Gamma(M,M') := \min_{\gamma\in\Gamma} d(M,\gamma M'). \tag{2.5}$$

Indeed, the left $G$-invariance of $d$ implies $d_\Gamma(\gamma M, M') = d_\Gamma(M,M') = d_\Gamma(M,\gamma M')$ for any $\gamma\in\Gamma$.

The aim of the present section is to identify a function $W_\delta$ on $\Gamma\backslash G$ that, when evaluated along a specific orbit of the flow $\Phi^t$, produces the Frobenius number. (As we shall see below, the situation is slightly more complicated in that $W_\delta$ also depends on additional variables in $\mathbb{R}^{d-1}$.)

We will assume throughout that $\boldsymbol{a}\in\widehat{\mathbb{Z}}^d_{\geq 2}$. Following [8], [18] we reduce the Frobenius problem modulo $a_d$. For $r\in\mathbb{Z}/a_d\mathbb{Z}$ set

$$F_r(\boldsymbol{a}) = \max (r + a_d\mathbb{Z}) \setminus \{\boldsymbol{m}\cdot\boldsymbol{a} > 0 : \boldsymbol{m}\in\mathbb{Z}^d_{\geq 0},\ \boldsymbol{m}\cdot\boldsymbol{a}\equiv r \bmod a_d\} \tag{2.6}$$

Then

$$F(\boldsymbol{a}) = \max_{r \bmod a_d} F_r(\boldsymbol{a}). \tag{2.7}$$

Consider the smallest positive integer that has a representation in $r \bmod a_d$,

$$N_r(\boldsymbol{a}) = \min\{\boldsymbol{m}\cdot\boldsymbol{a} > 0 : \boldsymbol{m}\in\mathbb{Z}^d_{\geq 0},\ \boldsymbol{m}\cdot\boldsymbol{a}\equiv r \bmod a_d\}. \tag{2.8}$$

Then $F_r(\boldsymbol{a}) = N_r(\boldsymbol{a}) - a_d$. We have in fact

$$N_r(\boldsymbol{a}) = \begin{cases} a_d & (r\equiv 0 \bmod a_d) \\ \min\{\boldsymbol{m}'\cdot\boldsymbol{a}' : \boldsymbol{m}'\in\mathbb{Z}^{d-1}_{\geq 0},\ \boldsymbol{m}'\cdot\boldsymbol{a}'\equiv r \bmod a_d\} & (r\not\equiv 0 \bmod a_d) \end{cases} \tag{2.9}$$

with $\boldsymbol{a}' = (a_1,\ldots,a_{d-1})$. In view of (2.7) we conclude

$$F(\boldsymbol{a}) = \max_{r\not\equiv 0 \bmod a_d} N_r(\boldsymbol{a}) - a_d. \tag{2.10}$$

We assume in the following $a_1,\ldots,a_{d-1}\leq a_d\leq T$, and $0<\delta\leq\frac{1}{2}$. For $r\not\equiv 0 \bmod a_d$ we then have

$$N_r(\boldsymbol{a}) = \min\left\{\boldsymbol{m}'\cdot\boldsymbol{a}' : \boldsymbol{m}\in\mathbb{Z}^{d-1}_{\geq 0}\times\mathbb{Z},\ \big|\boldsymbol{m}\cdot\boldsymbol{a} - r\big| < \frac{\delta a_d}{T}\right\}. \tag{2.11}$$

For $\boldsymbol{\xi} = (\boldsymbol{\xi}', \xi_d) \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, set

$$N(\boldsymbol{a}, \boldsymbol{\xi}, T) := \min_+ \left\{ (\boldsymbol{m}' + \boldsymbol{\xi}') \cdot \boldsymbol{a}' : \boldsymbol{m} + \boldsymbol{\xi} \in (\mathbb{Z}^d + \boldsymbol{\xi}) \cap \mathbb{R}_{\geq 0}^{d-1} \times \mathbb{R},\ \big| (\boldsymbol{m} + \boldsymbol{\xi}) \cdot \boldsymbol{a} \big| < \frac{\delta a_d}{T} \right\}, \tag{2.12}$$

where $\min_+$ is defined by

$$\min_+ \mathcal{A} = \begin{cases} \min \mathcal{A} \cap \mathbb{R}_{\geq 0} & (\mathcal{A} \cap \mathbb{R}_{\geq 0} \neq \emptyset) \\ 0 & (\mathcal{A} \cap \mathbb{R}_{\geq 0} = \emptyset). \end{cases} \tag{2.13}$$

It is evident that $N(\boldsymbol{a}, \boldsymbol{\xi}, T)$ is indeed well defined as a function of $\boldsymbol{\xi} \in \mathbb{T}^d$, and furthermore $N_r(\boldsymbol{a}) = N(\boldsymbol{a}, (\boldsymbol{0}, -\frac{r}{a_d}), T)$.

**Lemma 1.** *Let $\boldsymbol{a} = (a_1, \ldots, a_d) \in \widehat{\mathbb{Z}}_{\geq 2}^d$ with $a_1, \ldots, a_{d-1} \leq a_d \leq T$, $0 < \delta \leq \frac{1}{2}$. Then*

$$F(\boldsymbol{a}) = \sup_{\boldsymbol{\xi} \in \mathbb{R}^d/\mathbb{Z}^d} N(\boldsymbol{a}, \boldsymbol{\xi}, T) - \boldsymbol{e} \cdot \boldsymbol{a}, \tag{2.14}$$

*where $\boldsymbol{e} = (1, 1, \ldots, 1)$.*

*Proof.* Substituting $\xi_d$ by $\xi_d - \boldsymbol{\xi}' \cdot \frac{\boldsymbol{a}'}{a_d}$, we have

(2.15)

$$\begin{aligned} &\sup_{\boldsymbol{\xi} \in \mathbb{R}^d/\mathbb{Z}^d} N(\boldsymbol{a}, \boldsymbol{\xi}, T) \\ &= \sup_{\substack{\boldsymbol{\xi}' \in [0,1)^{d-1} \\ \xi_d \in \mathbb{T}^1}} \min_+ \left\{ (\boldsymbol{m}' + \boldsymbol{\xi}') \cdot \boldsymbol{a}' : \boldsymbol{m} + \boldsymbol{\xi} \in (\mathbb{Z}^d + \boldsymbol{\xi}) \cap \mathbb{R}_{\geq 0}^{d-1} \times \mathbb{R},\ \big| \boldsymbol{m} \cdot \boldsymbol{a} + \xi_d a_d \big| < \frac{\delta a_d}{T} \right\} \\ &= \sup_{\substack{\boldsymbol{\xi}' \in [0,1)^{d-1} \\ \xi_d \in \mathbb{T}^1}} \min_+ \left\{ (\boldsymbol{m}' + \boldsymbol{\xi}') \cdot \boldsymbol{a}' : \boldsymbol{m} \in \mathbb{Z}_{\geq 0}^{d-1} \times \mathbb{Z},\ \big| \boldsymbol{m} \cdot \boldsymbol{a} + \xi_d a_d \big| < \frac{\delta a_d}{T} \right\} \\ &= \sup_{\xi_d \in \mathbb{T}^1} \min_+ \left\{ \boldsymbol{m}' \cdot \boldsymbol{a}' : \boldsymbol{m} \in \mathbb{Z}_{\geq 0}^{d-1} \times \mathbb{Z},\ \big| \boldsymbol{m} \cdot \boldsymbol{a} + \xi_d a_d \big| < \frac{\delta a_d}{T} \right\} + \boldsymbol{e} \cdot \boldsymbol{a}', \end{aligned}$$

where $\boldsymbol{e} = (1, 1, \ldots, 1)$. The second equality follows from the fact that for $1 \leq j < d$, $m_j + \xi_j \geq 0$ implies $m_j \geq 0$ since $m_j \in \mathbb{Z}$ and $\xi_j \in [0, 1)$. We observe that, since $\frac{\delta a_d}{T} \leq \frac{1}{2}$ and $\boldsymbol{m} \cdot \boldsymbol{a} \in \mathbb{Z}$, we can replace in the inequality $|\boldsymbol{m} \cdot \boldsymbol{a} + \xi_d a_d| < \frac{\delta a_d}{T}$ the quantity $\xi_d a_d$ by its nearest integer, say $s$. That is, (2.15) equals

$$\sup_{s \bmod a_d} \min_+ \left\{ \boldsymbol{m}' \cdot \boldsymbol{a}' : \boldsymbol{m} \in \mathbb{Z}_{\geq 0}^{d-1} \times \mathbb{Z},\ \big| \boldsymbol{m} \cdot \boldsymbol{a} + s \big| < \frac{\delta a_d}{T} \right\} + \boldsymbol{e} \cdot \boldsymbol{a}'. \tag{2.16}$$

The case $s \equiv 0 \bmod a_d$ does not contribute (because then $\boldsymbol{m} = \boldsymbol{0}$ achieves 0 as minimum). Since $0 \leq a_j \leq a_d$ we thus obtain

$$\max_{r \not\equiv 0 \bmod a_d} N_r(\boldsymbol{a}) = \sup_{\boldsymbol{\xi} \in \mathbb{R}^d/\mathbb{Z}^d} N(\boldsymbol{a}, \boldsymbol{\xi}, T) - \boldsymbol{e} \cdot \boldsymbol{a}', \tag{2.17}$$

and the lemma follows from (2.10). □

Let $W_\delta$ denote the function $\mathbb{R}_{\geq 0}^{d-1} \times G \to \mathbb{R}$, $(\boldsymbol{\alpha}, M) \mapsto W_\delta(\boldsymbol{\alpha}, M)$, given by

$$W_\delta(\boldsymbol{\alpha}, M) = \sup_{\boldsymbol{\xi} \in \mathbb{T}^d} \min_+ \big\{ (\boldsymbol{m} + \boldsymbol{\xi}) M \cdot (\boldsymbol{\alpha}, 0) : \boldsymbol{m} \in \mathbb{Z}^d,\ (\boldsymbol{m} + \boldsymbol{\xi}) M \in \mathcal{R}_\delta \big\} \tag{2.18}$$

where $\mathcal{R}_\delta = \mathbb{R}_{\geq 0}^{d-1} \times (-\delta, \delta)$. Note that for every $\gamma \in \Gamma$

$$\begin{aligned} W_\delta(\boldsymbol{\alpha}, \gamma M) &= \sup_{\boldsymbol{\xi} \in \mathbb{T}^d} \min_+ \big\{ (\boldsymbol{m} + \boldsymbol{\xi}) \gamma M \cdot (\boldsymbol{\alpha}, 0) : \boldsymbol{m} \in \mathbb{Z}^d,\ (\boldsymbol{m} + \boldsymbol{\xi}) \gamma M \in \mathcal{R}_\delta \big\} \\ &= \sup_{\boldsymbol{\xi} \in \mathbb{T}^d \gamma} \min_+ \big\{ (\boldsymbol{m} + \boldsymbol{\xi}) M \cdot (\boldsymbol{\alpha}, 0) : \boldsymbol{m} \in \mathbb{Z}^d \gamma,\ (\boldsymbol{m} + \boldsymbol{\xi}) M \in \mathcal{R}_\delta \big\}. \end{aligned} \tag{2.19}$$

Both $\mathbb{Z}^d$ and $\mathbb{T}^d$ are $\Gamma$-invariant; thus

$$W_\delta(\boldsymbol{\alpha},\gamma M)=W_\delta(\boldsymbol{\alpha},M) \tag{2.20}$$

for all $\boldsymbol{\alpha}\in\mathbb{R}_{\geq 0}^{d-1}$, $M\in G$ and $\gamma\in\Gamma$.

Combining Definition (2.18) with Lemma 1 (set $t=\frac{\log T}{d-1}$) we obtain:

**Theorem 3.** *Let $\boldsymbol{a}=(a_1,\ldots,a_d)\in\widehat{\mathbb{Z}}_{\geq 2}^d$ with $a_1,\ldots,a_{d-1}\leq a_d\leq \mathrm{e}^{(d-1)t}$, and $0<\delta\leq\frac{1}{2}$. Then*

$$F(\boldsymbol{a})=\mathrm{e}^t W_\delta\big(\boldsymbol{a}',n_-(\widehat{\boldsymbol{a}})\Phi^t\big)-\boldsymbol{e}\cdot\boldsymbol{a}, \tag{2.21}$$

*where*

$$\widehat{\boldsymbol{a}}=\frac{\boldsymbol{a}'}{a_d}=\left(\frac{a_1}{a_d},\ldots,\frac{a_{d-1}}{a_d}\right). \tag{2.22}$$

## 3. Farey sequences on horospheres

Denote by $\mu=\mu_G$ the Haar measure on $G=\mathrm{SL}(d,\mathbb{R})$, normalized so that it represents the unique right $G$-invariant probability measure on the homogeneous space $\Gamma\backslash G$, where $\Gamma=\mathrm{SL}(d,\mathbb{Z})$. By Siegel's volume formula

$$d\mu(M)\frac{dt}{t}=\big(\zeta(2)\zeta(3)\cdots\zeta(d)\big)^{-1}\ \det(X)^{-d}\prod_{i,j=1}^{d}dX_{ij}, \tag{3.1}$$

where $X=(X_{ij})=t^{1/d}M\in\mathrm{GL}^+(d,\mathbb{R})$ with $M\in G$, $t>0$, cf. [10], [22]. We will also use the notation $\mu_0$ for the right $G_0$-invariant probability measure on $\Gamma_0\backslash G_0$, with $G_0=\mathrm{SL}(d-1,\mathbb{R})$ and $\Gamma_0=\mathrm{SL}(d-1,\mathbb{Z})$.

Consider the subgroups

$$H=\Big\{M\in G:(\boldsymbol{0},1)M=(\boldsymbol{0},1)\Big\}=\left\{\begin{pmatrix}A & {}^{\mathrm{t}}\boldsymbol{b}\\ \boldsymbol{0} & 1\end{pmatrix}:A\in G_0,\ \boldsymbol{b}\in\mathbb{R}^{d-1}\right\} \tag{3.2}$$

and

$$\Gamma_H=\Gamma\cap H=\left\{\begin{pmatrix}\gamma & {}^{\mathrm{t}}\boldsymbol{m}\\ \boldsymbol{0} & 1\end{pmatrix}:\gamma\in\Gamma_0,\ \boldsymbol{m}\in\mathbb{Z}^{d-1}\right\}. \tag{3.3}$$

Note that $H$ and $\Gamma_H$ are isomorphic to $\mathrm{ASL}(d-1,\mathbb{R})$ and $\mathrm{ASL}(d-1,\mathbb{Z})$, respectively. We normalize the Haar measure $\mu_H$ of $H$ so that it becomes a probability measure on $\Gamma_H\backslash H$; explicitly:

$$d\mu_H(M)=d\mu_0(A)\,d\boldsymbol{b},\qquad M=\begin{pmatrix}A & {}^{\mathrm{t}}\boldsymbol{b}\\ \boldsymbol{0} & 1\end{pmatrix}. \tag{3.4}$$

The following states the classical equidistribution theorem for $\Phi^t$-translates of the closed horospheres $\Gamma\backslash\Gamma\{n_-(\boldsymbol{x}):\boldsymbol{x}\in\mathbb{T}^{d-1}\}$ on $\Gamma\backslash G$; cf. [11, Section 5].

**Theorem 4.** *Let $\lambda$ be a Borel probability measure on $\mathbb{T}^{d-1}$, absolutely continuous with respect to Lebesgue measure, and let $f:\mathbb{T}^{d-1}\times\Gamma\backslash G\to\mathbb{R}$ be bounded continuous. Then*

$$\lim_{t\to\infty}\int_{\mathbb{T}^{d-1}}f\big(\boldsymbol{x},n_-(\boldsymbol{x})\Phi^t\big)\,d\lambda(\boldsymbol{x})=\int_{\mathbb{T}^{d-1}\times\Gamma\backslash G}f(\boldsymbol{x},M)\,d\lambda(\boldsymbol{x})\,d\mu(M). \tag{3.5}$$

A standard probabilistic argument [20, Chapter III] allows to reformulate the above statement in terms characteristic functions of subsets of $\mathbb{T}^{d-1}\times\Gamma\backslash G$.

**Theorem 5.** *Take $\lambda$ as in Theorem 4, and let $\mathcal{A}\subset\mathbb{T}^{d-1}\times\Gamma\backslash G$. Then*

$$\liminf_{t\to\infty}\lambda\big(\big\{\boldsymbol{x}\in\mathbb{T}^{d-1}:\big(\boldsymbol{x},n_-(\boldsymbol{x})\Phi^t\big)\in\mathcal{A}\big\}\big)\geq(\lambda\times\mu)(\mathcal{A}^\circ) \tag{3.6}$$

*and*

$$\limsup_{t\to\infty}\lambda\big(\big\{\boldsymbol{x}\in\mathbb{T}^{d-1}:\big(\boldsymbol{x},n_-(\boldsymbol{x})\Phi^t\big)\in\mathcal{A}\big\}\big)\leq(\lambda\times\mu)(\overline{\mathcal{A}}). \tag{3.7}$$

*Remark* 3.1. This shows that Theorem 4 can be extended to test functions $f$ that are characteristic functions of subsets of $\mathbb{T}^{d-1} \times \Gamma\backslash G$ with boundary of $(\lambda \times \mu)$-measure zero [11, Sect. 5.3], and thus also to functions that are the product of such a characteristic function and a bounded continuous function.

We will now replace the absolutely continuous measure $\lambda$ by equally weighted point masses at the elements of the Farey sequence

$$\mathcal{F}_Q = \left\{ \frac{\boldsymbol{p}}{q} \in [0,1)^{d-1} : (\boldsymbol{p}, q) \in \widehat{\mathbb{Z}}^d,\ 0 < q \leq Q \right\}, \tag{3.8}$$

for $Q \in \mathbb{N}$. Note that

$$|\mathcal{F}_Q| \sim \frac{Q^d}{d\,\zeta(d)} \qquad (Q \to \infty). \tag{3.9}$$

It will be notationally convenient to also allow general $Q \in \mathbb{R}_{\geq 1}$ in the definition (3.8) of $\mathcal{F}_Q$; note that $\mathcal{F}_Q = \mathcal{F}_{[Q]}$ where $[Q]$ is the integer part of $Q$.

**Theorem 6.** *Fix $\sigma \in \mathbb{R}$. Let $f : \mathbb{T}^{d-1} \times \Gamma\backslash G \to \mathbb{R}$ be bounded continuous. Then, for $Q = \mathrm{e}^{(d-1)(t-\sigma)}$,*

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{|\mathcal{F}_Q|} \sum_{\boldsymbol{r}\in\mathcal{F}_Q} f\big(\boldsymbol{r}, n_-(\boldsymbol{r})\Phi^t\big) \\ = d(d-1)\mathrm{e}^{d(d-1)\sigma} \int_\sigma^\infty \int_{\mathbb{T}^{d-1}\times\Gamma_H\backslash H} \widetilde{f}(\boldsymbol{x}, M\Phi^{-s})\, d\boldsymbol{x}\, d\mu_H(M)\, \mathrm{e}^{-d(d-1)s} ds \end{aligned} \tag{3.10}$$

*with $\widetilde{f}(\boldsymbol{x}, M) := f(\boldsymbol{x}, {}^{\mathrm{t}}M^{-1})$.*

*Remark* 3.2. The identical argument as in Remark 3.1 permits the extension of Theorem 6 to any test function $f$ which is the product of a bounded continuous function and the characteristic function of a subset $\mathcal{A} \subset \mathbb{T}^{d-1} \times \Gamma\backslash G$, where $\widetilde{\mathcal{A}} = \{(\boldsymbol{x}, M) : (\boldsymbol{x}, {}^{\mathrm{t}}M^{-1}) \in \mathcal{A}\}$ has boundary of measure zero with respect to $d\boldsymbol{x}\, d\mu_H\, ds$.

*Proof of Theorem 6.* **Step 0: Uniform continuity.** By choosing the test function $f(\boldsymbol{x}, M) = f_0(\boldsymbol{x}, M\Phi^{-\sigma})$ with $f_0 : \mathbb{T}^{d-1} \times \Gamma\backslash G \to \mathbb{R}$ bounded continuous, it is evident that we only need to consider the case $\sigma = 0$. We may also assume without loss of generality that $f$, and thus $\widetilde{f}$, have compact support. That is, there is $\mathcal{C} \subset G$ compact such that $\operatorname{supp} f, \operatorname{supp} \widetilde{f} \subset \mathbb{T}^{d-1} \times \Gamma\backslash\Gamma\mathcal{C}$. The generalization to bounded continuous functions follows from a standard approximation argument.

Since $f$ is continuous and has compact support, it is uniformly continuous. That is, given any $\delta > 0$ there exists $\epsilon > 0$ such that for all $(\boldsymbol{x}, M), (\boldsymbol{x}', M') \in \mathbb{R}^{d-1} \times G$,

$$\|\boldsymbol{x} - \boldsymbol{x}'\| < \epsilon, \qquad d(M, M') < \epsilon \tag{3.11}$$

implies

$$\big|f(\boldsymbol{x}, M) - f(\boldsymbol{x}', M')\big| < \delta. \tag{3.12}$$

The plan is now to first establish (3.10) for the set

$$\mathcal{F}_{Q,\theta} = \left\{ \frac{\boldsymbol{p}}{q} \in [0,1)^{d-1} : (\boldsymbol{p}, q) \in \widehat{\mathbb{Z}}^d,\ \theta Q < q \leq Q \right\}, \tag{3.13}$$

for any $\theta \in (0,1)$. The constant $\theta$ will remain fixed until the very last step of this proof.

**Step 1: Thicken the Farey sequence.** The plan is to reduce the statement to Theorem 4. To this end, we thicken the set $\mathcal{F}_{Q,\theta}$ as follows: For $\epsilon > 0$ (we will in fact later use the $\epsilon$ from Step 0), let

$$\mathcal{F}_Q^\epsilon = \bigcup_{\boldsymbol{r}\in\mathcal{F}_{Q,\theta}+\mathbb{Z}^{d-1}} \big\{\boldsymbol{x} \in \mathbb{R}^{d-1} : \|\boldsymbol{x} - \boldsymbol{r}\| < \epsilon\mathrm{e}^{-dt}\big\}. \tag{3.14}$$

Note that $\mathcal{F}_Q^\epsilon$ is symmetric with respect to $\boldsymbol{x} \mapsto -\boldsymbol{x}$. A short calculation yields

$$\mathcal{F}_Q^\epsilon = \bigcup_{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d} \{\boldsymbol{x} \in \mathbb{R}^{d-1} : \boldsymbol{a}\, n_+(\boldsymbol{x})\Phi^{-t} \in \mathfrak{C}_\epsilon\}, \tag{3.15}$$

where

$$\mathfrak{C}_\epsilon = \{(y_1,\ldots,y_d) \in \mathbb{R}^d : \|(y_1,\ldots,y_{d-1})\| < \epsilon y_d,\ \theta < y_d \le 1\}. \tag{3.16}$$

Let

$$\mathcal{H}_\epsilon = \bigcup_{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d} \mathcal{H}_\epsilon(\boldsymbol{a}), \qquad \mathcal{H}_\epsilon(\boldsymbol{a}) = \{M \in G : \boldsymbol{a}M \in \mathfrak{C}_\epsilon\}. \tag{3.17}$$

The bijection (cf. [22])

$$\Gamma_H\backslash\Gamma \to \widehat{\mathbb{Z}}^d, \qquad \Gamma_H\gamma \mapsto (\boldsymbol{0},1)\gamma \tag{3.18}$$

allows us to rewrite

$$\mathcal{H}_\epsilon = \bigcup_{\gamma\in\Gamma_H\backslash\Gamma} \mathcal{H}_\epsilon((\boldsymbol{0},1)\gamma) = \bigcup_{\gamma\in\Gamma/\Gamma_H} \gamma\mathcal{H}_\epsilon^1, \qquad \text{with } \mathcal{H}_\epsilon^1 = \mathcal{H}_\epsilon((\boldsymbol{0},1)). \tag{3.19}$$

Now

$$\begin{aligned}\mathcal{H}_\epsilon^1 =& \{M \in G : (\boldsymbol{0},1)M \in \mathfrak{C}_\epsilon\}\\ =& H\{M_{\boldsymbol{y}} : \boldsymbol{y} \in \mathfrak{C}_\epsilon\}\end{aligned} \tag{3.20}$$

with $H$ as in (3.2), and $M_{\boldsymbol{y}} \in G$ such that $(\boldsymbol{0},1)M_{\boldsymbol{y}} = \boldsymbol{y}$. Since $\boldsymbol{y} \in \mathfrak{C}_\epsilon$ implies $y_d > 0$, we may choose

$$M_{\boldsymbol{y}} = \begin{pmatrix} y_d^{-1/(d-1)} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{y}' & y_d \end{pmatrix}, \qquad \boldsymbol{y}' = (y_1,\ldots,y_{d-1}). \tag{3.21}$$

**Step 2: Prove disjointness.** We will now prove the following claim: *Given a compact subset $\mathcal{C} \subset G$, there exists $\epsilon_0 > 0$ such that*

$$\gamma\mathcal{H}_\epsilon^1 \cap \mathcal{H}_\epsilon^1 \cap \Gamma\mathcal{C} = \emptyset \tag{3.22}$$

*for every $\epsilon \in (0,\epsilon_0]$, $\gamma \in \Gamma \setminus \Gamma_H$.*

To prove this claim, note that (3.22) is equivalent to

$$\mathcal{H}_\epsilon((\boldsymbol{p},q)) \cap \mathcal{H}_\epsilon^1 \cap \Gamma\mathcal{C} = \emptyset \tag{3.23}$$

for every $(\boldsymbol{p},q) \in \widehat{\mathbb{Z}}^d$, $(\boldsymbol{p},q) \neq (\boldsymbol{0},1)$. For

$$M = \begin{pmatrix} A & {}^{\mathrm{t}}\boldsymbol{b} \\ \boldsymbol{0} & 1 \end{pmatrix} M_{\boldsymbol{y}}, \qquad M_{\boldsymbol{y}} = \begin{pmatrix} y_d^{-1/(d-1)} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{y}' & y_d \end{pmatrix}, \tag{3.24}$$

we have

$$(\boldsymbol{p},q)M = (\boldsymbol{p}Ay_d^{-1/(d-1)} + (\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)\boldsymbol{y}', (\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)y_d), \tag{3.25}$$

and thus $M \in \mathcal{H}_\epsilon((\boldsymbol{p},q)) \cap \mathcal{H}_\epsilon^1$ if and only if

$$\|\boldsymbol{p}Ay_d^{-1/(d-1)} + (\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)\boldsymbol{y}'\| < \epsilon(\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)y_d, \tag{3.26}$$

$$\theta < (\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)y_d \le 1, \tag{3.27}$$

and

$$\|\boldsymbol{y}'\| < \epsilon y_d, \qquad \theta < y_d \le 1. \tag{3.28}$$

Relations (3.27) and (3.28) imply $\|(\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)\boldsymbol{y}'\| < \epsilon(\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)y_d \le \epsilon$ and so, by (3.26), $\|\boldsymbol{p}Ay_d^{-1/(d-1)}\| < 2\epsilon(\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)y_d \le 2\epsilon$. That is, $\|\boldsymbol{p}A\| < 2\epsilon y_d^{1/(d-1)}$ and hence

$$\|\boldsymbol{p}A\| < 2\epsilon. \tag{3.29}$$

Let us now suppose $M \in \Gamma\mathcal{C}$ with $\mathcal{C}$ compact. The set

$$\mathcal{C}' = \mathcal{C}\{M_{\boldsymbol{y}}^{-1} : \boldsymbol{y} \in \overline{\mathfrak{C}}_\epsilon\} \tag{3.30}$$

is still compact, by the compactness of $\overline{\mathfrak{C}}_\epsilon$ (the closure of $\mathfrak{C}_\epsilon$) in $\mathbb{R}^d \setminus \{\mathbf{0}\}$. In view of (3.24) we obtain

$$\begin{pmatrix} A & {}^{\mathrm{t}}\boldsymbol{b} \\ \mathbf{0} & 1 \end{pmatrix} \in \Gamma\mathcal{C}', \tag{3.31}$$

and so $A \in \Gamma_0\mathcal{C}_0$ for some compact $\mathcal{C}_0 \subset G_0$.

Mahler's compactness criterion then shows that

$$I := \inf_{A\in\Gamma_0\mathcal{C}_0} \inf_{\boldsymbol{p}\in\mathbb{Z}^{d-1}\setminus\{\mathbf{0}\}} \|\boldsymbol{p}A\| > 0. \tag{3.32}$$

Now choose $\epsilon_0$ such that $0 < 2\epsilon_0 < I$. Then (3.29) implies $\boldsymbol{p} = \mathbf{0}$ and therefore $q = 1$. The claim is proved.

**Step 3: Apply Theorem 4.** Step 2 implies that, for $\mathcal{C} \subset G$ compact, there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0]$

$$\mathcal{H}_\epsilon \cap \Gamma\mathcal{C} = \bigcup_{\gamma\in\Gamma/\Gamma_H} \big(\gamma\mathcal{H}_\epsilon^1 \cap \Gamma\mathcal{C}\big) \tag{3.33}$$

is a disjoint union. Hence, if $\chi_\epsilon$ and $\chi_\epsilon^1$ are the characteristic functions of the sets $\mathcal{H}_\epsilon$ and $\mathcal{H}_\epsilon^1$, respectively, we have

$$\chi_\epsilon(M) = \sum_{\gamma\in\Gamma_H\setminus\Gamma} \chi_\epsilon^1(\gamma M), \tag{3.34}$$

for all $M \in \Gamma\mathcal{C}$. Evidently $\mathcal{H}_\epsilon^1$ and thus $\mathcal{H}_\epsilon$ have boundary of $\mu$-measure zero. We furthermore set $\widetilde{\chi}_\epsilon(M) := \chi_\epsilon({}^{\mathrm{t}}M^{-1})$, and note that $\chi_\epsilon\big(n_+(\boldsymbol{x})\Phi^{-t}\big) = \chi_\epsilon\big(n_+(-\boldsymbol{x})\Phi^{-t}\big)$ is the characteristic function of the set $\mathcal{F}_Q^\epsilon$; recall (3.15) and the remark after (3.14). Therefore

$$\begin{aligned} \int_{\mathcal{F}_Q^\epsilon/\mathbb{Z}^{d-1}} f\big(\boldsymbol{x}, n_-(\boldsymbol{x})\Phi^t\big) d\boldsymbol{x} &= \int_{\mathbb{T}^{d-1}} f\big(\boldsymbol{x}, n_-(\boldsymbol{x})\Phi^t\big) \chi_\epsilon\big(n_+(-\boldsymbol{x})\Phi^{-t}\big) d\boldsymbol{x} \\ &= \int_{\mathbb{T}^{d-1}} f\big(\boldsymbol{x}, n_-(\boldsymbol{x})\Phi^t\big) \widetilde{\chi}_\epsilon\big(n_-(\boldsymbol{x})\Phi^t\big) d\boldsymbol{x}, \end{aligned} \tag{3.35}$$

and Theorem 4 yields

$$\begin{aligned} \lim_{t\to\infty} \int_{\mathbb{T}^{d-1}} f\big(\boldsymbol{x}, n_-(\boldsymbol{x})\Phi^t\big) \widetilde{\chi}_\epsilon\big(n_-(\boldsymbol{x})\Phi^t\big)\, d\boldsymbol{x} &= \int_{\mathbb{T}^{d-1}\times\Gamma\setminus G} f(\boldsymbol{x}, M)\widetilde{\chi}_\epsilon(M)\, d\boldsymbol{x}\, d\mu(M) \\ &= \int_{\mathbb{T}^{d-1}\times\Gamma\setminus G} \widetilde{f}(\boldsymbol{x}, M)\chi_\epsilon(M)\, d\boldsymbol{x}\, d\mu(M). \end{aligned} \tag{3.36}$$

**Step 4: A volume computation.** To evaluate the right hand side of (3.36), we use (3.34):

$$\begin{aligned} \int_{\mathbb{T}^{d-1}\times\Gamma\setminus G} \widetilde{f}(\boldsymbol{x}, M)\chi_\epsilon(M)\, d\boldsymbol{x}\, d\mu(M) &= \int_{\mathbb{T}^{d-1}\times\Gamma_H\setminus G} \widetilde{f}(\boldsymbol{x}, M)\chi_\epsilon^1(M)\, d\boldsymbol{x}\, d\mu(M) \\ &= \int_{\mathbb{T}^{d-1}\times\Gamma_H\setminus\mathcal{H}_\epsilon^1} \widetilde{f}(\boldsymbol{x}, M)\, d\boldsymbol{x}\, d\mu(M). \end{aligned} \tag{3.37}$$

Given $\boldsymbol{y} \in \mathbb{R}^d$ we pick a matrix $M_{\boldsymbol{y}} \in G$ such that $(\mathbf{0}, 1)M_{\boldsymbol{y}} = \boldsymbol{y}$; recall (3.21) for an explicit choice of $M_{\boldsymbol{y}}$ for $y_d > 0$. The map

$$H \times \mathbb{R}^d \setminus \{\mathbf{0}\} \to G, \qquad (M, \boldsymbol{y}) \mapsto MM_{\boldsymbol{y}}, \tag{3.38}$$

provides a parametrization of $G$, where in view of (3.1)

$$d\mu = \zeta(d)^{-1} d\mu_H\, d\boldsymbol{y}. \tag{3.39}$$

Hence (3.37) equals

$$\frac{1}{\zeta(d)}\int_{\mathbb{T}^{d-1}\times\Gamma_H\backslash H\times\mathfrak{C}_\epsilon}\widetilde{f}\big(\boldsymbol{x},MM_{\boldsymbol{y}}\big)\,d\boldsymbol{x}\,d\mu_H(M)d\boldsymbol{y}. \tag{3.40}$$

For

$$D(y_d)=\begin{pmatrix} y_d^{-1/(d-1)}1_{d-1} & {}^{\mathrm{t}}\mathbf{0}\\ \mathbf{0} & y_d\end{pmatrix}, \tag{3.41}$$

we have

$$d\big(M_{\boldsymbol{y}},D(y_d)\big)=d\big(D(y_d)n_+(y_d^{-1}\boldsymbol{y}'),D(y_d)\big)=d\big(n_+(y_d^{-1}\boldsymbol{y}'),1_d\big)\le y_d^{-1}\|\boldsymbol{y}'\|. \tag{3.42}$$

We recall that $y_d^{-1}\|\boldsymbol{y}'\|<\epsilon$ for $\boldsymbol{y}\in\mathfrak{C}_\epsilon$. Therefore, with the choice of $\delta,\epsilon$ made in Steps 0 and 2, we have (note that (3.12) applies also to $\widetilde{f}$)

$$\left|(3.40)-\frac{1}{\zeta(d)}\int_{\mathbb{T}^{d-1}\times\Gamma_H\backslash H\times\mathfrak{C}_\epsilon}\widetilde{f}\big(\boldsymbol{x},MD(y_d)\big)\,d\boldsymbol{x}\,d\mu_H(M)d\boldsymbol{y}\right|<\frac{\delta}{\zeta(d)}\int_{\mathfrak{C}_\epsilon}d\boldsymbol{y}. \tag{3.43}$$

We have

$$\begin{aligned}\int_{\mathfrak{C}_\epsilon}\widetilde{f}\big(\boldsymbol{x},MD(y_d)\big)\,d\boldsymbol{y}&=\mathrm{vol}(\mathcal{B}_1^{d-1})\,\epsilon^{d-1}\int_\theta^1\widetilde{f}\big(\boldsymbol{x},MD(y_d)\big)\,y_d^{d-1}\,dy_d\\ &=(d-1)\,\mathrm{vol}(\mathcal{B}_1^{d-1})\,\epsilon^{d-1}\int_0^{|\log\theta|/(d-1)}\widetilde{f}\big(\boldsymbol{x},M\Phi^{-s}\big)\,\mathrm{e}^{-d(d-1)s}ds,\end{aligned} \tag{3.44}$$

and

$$\int_{\mathfrak{C}_\epsilon}d\boldsymbol{y}=\frac{1}{d}\,\mathrm{vol}(\mathcal{B}_1^{d-1})\,\epsilon^{d-1}(1-\theta^d), \tag{3.45}$$

where $\mathcal{B}_1^{d-1}$ denotes the unit ball in $\mathbb{R}^{d-1}$. So (3.43) becomes

(3.46)

$$\begin{aligned}\left|(3.40)-\frac{(d-1)\,\mathrm{vol}(\mathcal{B}_1^{d-1})\,\epsilon^{d-1}}{\zeta(d)}\int_0^{|\log\theta|/(d-1)}\int_{\mathbb{T}^{d-1}\times\Gamma_H\backslash H}\widetilde{f}\big(\boldsymbol{x},M\Phi^{-s}\big)\,d\boldsymbol{x}\,d\mu_H(M)\,\mathrm{e}^{-d(d-1)s}ds\right|\\ <\frac{\mathrm{vol}(\mathcal{B}_1^{d-1})\,\delta\,\epsilon^{d-1}}{d\,\zeta(d)}\,(1-\theta^d).\end{aligned}$$

**Step 5: Distance estimates.** Since (3.33) is a disjoint union, we have furthermore (this is in effect another way of writing (3.35) using (3.34))

$$\int_{\mathcal{F}_Q^\epsilon/\mathbb{Z}^{d-1}}f\big(\boldsymbol{x},n_-(\boldsymbol{x})\Phi^t\big)d\boldsymbol{x}=\sum_{\boldsymbol{r}\in\mathcal{F}_{Q,\theta}}\int_{\|\boldsymbol{x}-\boldsymbol{r}\|<\epsilon\mathrm{e}^{-dt}}f\big(\boldsymbol{x},n_-(\boldsymbol{x})\Phi^t\big)d\boldsymbol{x}. \tag{3.47}$$

Eq. (2.4) implies that

$$d\big(n_-(\boldsymbol{x})\Phi^t,n_-(\boldsymbol{r})\Phi^t\big)\le\mathrm{e}^{dt}\|\boldsymbol{x}-\boldsymbol{r}\|<\epsilon. \tag{3.48}$$

Because $f$ is uniformly continuous we therefore have, for the same $\delta,\epsilon$ as above:

$$\left|\int_{\|\boldsymbol{x}-\boldsymbol{r}\|<\epsilon\mathrm{e}^{-dt}}f\big(\boldsymbol{x},n_-(\boldsymbol{x})\Phi^t\big)d\boldsymbol{x}-\frac{\mathrm{vol}(\mathcal{B}_1^{d-1})\epsilon^{d-1}}{\mathrm{e}^{d(d-1)t}}f\big(\boldsymbol{r},n_-(\boldsymbol{r})\Phi^t\big)\right|<\frac{\mathrm{vol}(\mathcal{B}_1^{d-1})\,\delta\,\epsilon^{d-1}}{\mathrm{e}^{d(d-1)t}}, \tag{3.49}$$

uniformly for all $t\ge 0$.

**Step 6: Conclusion.** The approximations (3.46) and (3.49) hold uniformly for any $\delta>0$. Passing to the limit $\delta\to 0$, we obtain

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{\mathrm{e}^{d(d-1)t}}&\sum_{\boldsymbol{r}\in\mathcal{F}_{Q,\theta}}f\big(\boldsymbol{r},n_-(\boldsymbol{r})\Phi^t\big)\\ &=\frac{d-1}{\zeta(d)}\int_0^{|\log\theta|/(d-1)}\int_{\mathbb{T}^{d-1}\times\Gamma_H\backslash H}\widetilde{f}\big(\boldsymbol{x},M\Phi^{-s}\big)\,d\boldsymbol{x}\,d\mu_H(M)\,\mathrm{e}^{-d(d-1)s}ds.\end{aligned} \tag{3.50}$$

The asymptotics (3.9) show that

$$\limsup_{t\to\infty} \frac{|\mathcal{F}_Q \setminus \mathcal{F}_{Q,\theta}|}{\mathrm{e}^{d(d-1)t}} \leq \frac{\theta^d}{d\,\zeta(d)}, \tag{3.51}$$

which allows us to take the limit $\theta \to 0$ in (3.50). This concludes the proof for $\sigma = 0$ and $f$ compactly supported. For the general case, recall the remarks in Step 0. □

*Remark* 3.3. Let $(\boldsymbol{p}, q) \in \widehat{\mathbb{Z}}$. Using the bijection (3.18), choose $\gamma \in \Gamma$ such that $(\boldsymbol{p}, q)\gamma = (\mathbf{0}, 1)$. For $\boldsymbol{r} = \boldsymbol{p}/q \in \mathcal{F}_Q + \mathbb{Z}^{d-1}$, we then have

$$\gamma^{-1}\,{}^{\mathrm{t}}(n_-(\boldsymbol{r})D(q))^{-1} = \left( \begin{pmatrix} q^{-1/(d-1)}1_{d-1} & {}^{\mathrm{t}}\mathbf{0} \\ \boldsymbol{p} & q \end{pmatrix} \gamma \right)^{-1} \in H. \tag{3.52}$$

That is,

$$\Gamma\,{}^{\mathrm{t}}(n_-(\boldsymbol{r})D(q))^{-1} \in \Gamma\backslash\Gamma H, \tag{3.53}$$

and thus, for $Q = \mathrm{e}^{(d-1)(t-\sigma)}$,

$$\Gamma\,{}^{\mathrm{t}}(n_-(\boldsymbol{r})\Phi^t)^{-1} \in \Gamma\backslash\Gamma H\{\Phi^{-s} : s \in \mathbb{R}_{\geq\sigma}\}. \tag{3.54}$$

**Lemma 2.** *The set* $\Gamma\backslash\Gamma H\{\Phi^{-s} : s \in \mathbb{R}_{\geq\sigma}\}$ *is a closed embedded submanifold of* $\Gamma\backslash G$.

*Proof.* The set

$$\Gamma\backslash\Gamma H\{\Phi^{-s} : s \in \mathbb{R}_{\geq\sigma}\} = \Gamma\backslash\Gamma H\{D(y_d) : y_d \in (0, c]\}, \qquad c = \mathrm{e}^{-(d-1)\sigma}, \tag{3.55}$$

is the image of the immersion map

$$i : \mathcal{H}_0 \to \Gamma\backslash G, \qquad \Gamma_H M \mapsto \Gamma M, \tag{3.56}$$

$$\mathcal{H}_0 := \Gamma_H\backslash H\{D(y_d) : y_d \in (0, c]\}, \tag{3.57}$$

and is thus an immersed submanifold of $\Gamma\backslash G$. To show that it is in fact a closed embedded submanifold, we need to establish that $i$ is a proper map, i.e., every compact $\mathcal{K} \subset \Gamma\backslash G$ has a compact pre-image $i^{-1}(\mathcal{K})$; see e.g. [6, Chapter III]. Since $i$ is continuous, $i^{-1}(\mathcal{K})$ is closed. It therefore suffices to show that $i^{-1}(\mathcal{K})$ is contained in a compact subset of $\mathcal{H}_0$.

For $M \in G$, let $I(M) = \inf\{\|\boldsymbol{m}M\| : \boldsymbol{m} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}\}$. By Mahler's criterion, there is $\theta > 0$ such that $I(M) \geq \theta$ for all $M \in G$ with $\Gamma M \in \mathcal{K}$. If $\Gamma_H M \in i^{-1}(\mathcal{K})$, then $I(M) \geq \theta$ with $M = hD(y_d)$, $h \in H$. Thus $(\mathbf{0}, 1)M = y_d$ and therefore $y_d \geq \theta$. This implies that, for any $h \in H$,

$$i(\Gamma_H h) = \Gamma h \in \mathcal{K}' := \mathcal{K}\{D(y_d)^{-1} : \theta \leq y_d \leq c\}, \tag{3.58}$$

where $\mathcal{K}'$ is a compact subset of $\Gamma\backslash G$.

It is a basic fact that, since $H$ is a closed subgroup of $G$ and $\Gamma_H = \Gamma \cap H$ is a lattice in $H$, the set $\Gamma\backslash\Gamma H$ is a closed embedded submanifold of $\Gamma\backslash G$ [12, Theorem 1.13]. We denote by $j : \Gamma_H\backslash H \to \Gamma\backslash\Gamma H$ the immersion map. Thus $j^{-1}(\mathcal{K}')$ is a compact subset of $\Gamma_H\backslash H$, and $i^{-1}(\mathcal{K})$ is contained in the compact subset $j^{-1}(\mathcal{K}')\{D(y_d) : \theta \leq y_d \leq c\}$ of $\mathcal{H}_0$. □

The significance of (3.54) and Lemma 2 is that it allows us reduce the continuity hypotheses of Theorem 6 and Remark 3.2 to continuity of $\widetilde{f}$ restricted to the closed embedded submanifold

$$\mathbb{T}^{d-1} \times \Gamma\backslash\Gamma H\{\Phi^{-s} : s \in \mathbb{R}_{\geq\sigma}\}. \tag{3.59}$$

We will exploit this fact in the proof of Theorem 8.

## 4. A variant of Theorem 6

The following variant of Theorem 6 will be key in the proof of Theorem 1. Recall the definition of $\widehat{\boldsymbol{a}}$ and $D(T)$ in (2.22) and (3.41), respectively.

**Theorem 7.** *Let $\mathcal{D} \subset \{\boldsymbol{x} \in \mathbb{R}^d : 0 < x_1, \ldots, x_{d-1} \leq x_d\}$ be bounded with boundary of Lebesgue measure zero, and $f : \overline{\mathcal{D}} \times \Gamma\backslash G \to \mathbb{R}$ bounded continuous. Then*

$$\lim_{T\to\infty} \frac{1}{T^d} \sum_{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d \cap T\mathcal{D}} f\Big(\frac{\boldsymbol{a}}{T}, n_-(\widehat{\boldsymbol{a}})D(T)\Big) = \frac{1}{\zeta(d)} \int_{\mathcal{D}\times\Gamma_H\backslash H} \widetilde{f}\big(\boldsymbol{y}, MD(y_d)\big)\, d\boldsymbol{y}\, d\mu_H(M) \tag{4.1}$$

*with $\widetilde{f}(\boldsymbol{x}, M) := f(\boldsymbol{x}, {}^{\mathrm{t}}M^{-1})$.*

*Proof.* Let $g : \mathbb{R}^{d-1} \times \Gamma\backslash G \to \mathbb{R}$ be a bounded continuous function. We apply Theorem 6 with $T = \mathrm{e}^{(d-1)t}$, $c = \mathrm{e}^{-(d-1)\sigma}$, and the test function

$$f(\boldsymbol{x}, M) = \sum_{\boldsymbol{n}\in\mathbb{Z}^{d-1}} g(\boldsymbol{x}+\boldsymbol{n}, M)\chi_{[0,1]^{d-1}}(\boldsymbol{x}+\boldsymbol{n}). \tag{4.2}$$

Note that this sum has at most $2^{d-1}$ non-zero terms. The function $f(\boldsymbol{x}, M)$ is bounded everywhere, and continuous on $[(0,1)^{d-1} + \mathbb{Z}^{d-1}] \times \Gamma\backslash G$; hence Remark 3.2, together with the asymptotics (3.9), yield

$$\begin{aligned} &\lim_{T\to\infty} \frac{\zeta(d)}{T^d} \sum_{\substack{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d \\ 1\leq a_1,\ldots,a_{d-1}\leq a_d \\ a_d\leq cT}} g\big(\widehat{\boldsymbol{a}}, n_-(\widehat{\boldsymbol{a}})D(T)\big) \\ &= (d-1)\int_\sigma^\infty \int_{[0,1]^{d-1}\times\Gamma_H\backslash H} \widetilde{g}(\boldsymbol{x}, M\Phi^{-s})\, d\boldsymbol{x}\, d\mu_H(M)\, \mathrm{e}^{-d(d-1)s} ds \\ &= \int_0^c \int_{[0,1]^{d-1}\times\Gamma_H\backslash H} \widetilde{g}\big(\boldsymbol{x}, MD(y_d)\big)\, d\boldsymbol{x}\, d\mu_H(M)\, y_d^{d-1} dy_d \end{aligned} \tag{4.3}$$

where we have substituted in the last step $y_d = \mathrm{e}^{-(d-1)s}$. So for any $0 \leq b < c$ we have

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T^d} \sum_{\substack{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d \\ 1\leq a_1,\ldots,a_{d-1}\leq a_d \\ bT<a_d\leq cT}} g\big(\widehat{\boldsymbol{a}}, n_-(\widehat{\boldsymbol{a}})D(T)\big) \\ &= \frac{1}{\zeta(d)} \int_b^c \int_{[0,1]^{d-1}\times\Gamma_H\backslash H} \widetilde{g}\big(\boldsymbol{x}, MD(y_d)\big)\, d\boldsymbol{x}\, d\mu_H(M)\, y_d^{d-1} dy_d, \end{aligned} \tag{4.4}$$

and hence for $h : \mathbb{R}^{d-1} \times \mathbb{R} \times \Gamma\backslash G \to \mathbb{R}$ continuous with support in $\mathbb{R}^{d-1} \times \mathcal{I} \times \Gamma\backslash G$ and $\mathcal{I} \subset \mathbb{R}_{\geq 0}$ bounded, we have

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T^d} \sum_{\substack{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d \\ 1\leq a_1,\ldots,a_{d-1}\leq a_d}} h\Big(\widehat{\boldsymbol{a}}, \frac{a_d}{T}, n_-(\widehat{\boldsymbol{a}})D(T)\Big) \\ &= \frac{1}{\zeta(d)} \int_{[0,1]^{d-1}\times\mathcal{I}\times\Gamma_H\backslash H} \widetilde{h}\big(\boldsymbol{x}, y_d, MD(y_d)\big)\, d\boldsymbol{x}\, y_d^{d-1} dy_d\, d\mu_H(M). \end{aligned} \tag{4.5}$$

We now take $h(\boldsymbol{x}, y_d, M) = \chi_{\mathcal{D}}(\boldsymbol{x}y_d, y_d)\, f((\boldsymbol{x}y_d, y_d), M)$ with $f$ as in Theorem 7, and substitute $\boldsymbol{y}' = \boldsymbol{x}y_d$. Note that with this choice $h$ is no longer continuous; but $\mathcal{D}$ has boundary of measure zero and thus Remark 3.2 applies. □

Remark 3.3 and Theorem 7 now imply the following theorem. Given a bounded subset $\mathcal{D} \subset \mathbb{R}^d_{\geq 0}$, define

$$\mathcal{M}_{\mathcal{D}} = \big\{(\boldsymbol{y}, \Gamma\, {}^{\mathrm{t}}M^{-1}D(y_d)^{-1}) : (\boldsymbol{y}, \Gamma M) \in \overline{\mathcal{D}} \times \Gamma\backslash\Gamma H\big\}, \tag{4.6}$$

which, in view of Lemma 2, is a closed embedded submanifold of $\mathbb{R}^d \times \Gamma\backslash G$. The bijection

$$\overline{\mathcal{D}} \times \Gamma_H\backslash H \to \mathcal{M}_{\mathcal{D}}, \qquad (\boldsymbol{y}, \Gamma_H M) \mapsto (\boldsymbol{y}, \Gamma\, {}^{\mathrm{t}}M^{-1} D(y_d)^{-1}), \tag{4.7}$$

allows us to define a natural measure $\nu$ on $\mathcal{M}_{\mathcal{D}}$ as the pushforward of $\mathrm{vol} \times \mu_H$, where vol is Lebesgue measure on $\mathbb{R}^d$ and $\mu_H$ as defined in (3.4). In the following we understand the interior and closure of subsets of $\mathcal{M}_{\mathcal{D}}$ with respect to the topology of $\mathcal{M}_{\mathcal{D}}$.

Since $n_-(\widehat{\boldsymbol{a}})D(T) = n_-(\widehat{\boldsymbol{a}})D(a_d)D(a_d/T)^{-1}$, eq. (3.53) implies that

$$\left(\frac{\boldsymbol{a}}{T}, \Gamma n_-(\widehat{\boldsymbol{a}})D(T)\right) \subset \mathcal{M}_{\mathcal{D}}. \tag{4.8}$$

**Theorem 8.** *Let $\mathcal{D} \subset \{\boldsymbol{x} \in \mathbb{R}^d : 0 < x_1, \ldots, x_{d-1} \leq x_d\}$ be bounded with boundary of Lebesgue measure zero, and $\mathcal{A} \subset \mathcal{M}_{\mathcal{D}}$. Then*

$$\liminf_{T\to\infty} \frac{1}{T^d} \#\left\{\boldsymbol{a} \in \widehat{\mathbb{Z}}^d : \left(\frac{\boldsymbol{a}}{T}, \Gamma n_-(\widehat{\boldsymbol{a}})D(T)\right) \in \mathcal{A}\right\} \geq \frac{\nu(\mathcal{A}^\circ)}{\zeta(d)} \tag{4.9}$$

*and*

$$\limsup_{T\to\infty} \frac{1}{T^d} \#\left\{\boldsymbol{a} \in \widehat{\mathbb{Z}}^d : \left(\frac{\boldsymbol{a}}{T}, \Gamma n_-(\widehat{\boldsymbol{a}})D(T)\right) \in \mathcal{A}\right\} \leq \frac{\nu(\overline{\mathcal{A}})}{\zeta(d)}. \tag{4.10}$$

*Proof.* The inclusion (4.8) shows that the limit relation (4.1) in Theorem 7 holds for any bounded continuous function $f : \mathcal{M}_{\mathcal{D}} \to \mathbb{R}$. We can thus once more apply the above probabilistic argument [20, Chapter III] (used in the justification of Theorem 5) to prove (4.9) and (4.10). □

## 5. Upper and lower limits

Let us first of all note that we may assume in Theorem 1 without loss of generality that $\mathcal{D} \subset [0,1]^d$. Secondly, due to the symmetry of $F(\boldsymbol{a})$ under any permutation of the coefficients $a_i$, we may assume that $\mathcal{D} \subset \{\boldsymbol{x} \in \mathbb{R}^d : 0 \leq x_1, \ldots, x_{d-1} \leq x_d\}$. Thirdly, it is sufficient to prove Theorem 1 for all bounded subsets of $\{\boldsymbol{x} \in \mathbb{R}^d : \eta \leq x_1, \ldots, x_{d-1} \leq x_d\}$, for any fixed $\eta > 0$. This is due to the fact that for any bounded set $\mathcal{D} \subset [0,1]^d$ with boundary of measure zero,

$$\lim_{T\to\infty} \frac{1}{T^d} \#\{\boldsymbol{a} \in \widehat{\mathbb{Z}}^d \cap T(\mathcal{D} \setminus \mathbb{R}^d_{\geq\eta})\} = \frac{\mathrm{vol}\,(\mathcal{D} \setminus \mathbb{R}^d_{\geq\eta})}{\zeta(d)} \leq \frac{d\,\eta}{\zeta(d)}. \tag{5.1}$$

We will therefore assume in the remainder of this section that, in addition to the assumptions of Theorem 1,

$$\mathcal{D} \subset \{\boldsymbol{x} \in \mathbb{R}^d : \eta \leq x_1, \ldots, x_{d-1} \leq x_d \leq 1\}, \tag{5.2}$$

for arbitrary fixed $\eta > 0$.

The following is an immediate corollary of Theorem 3 (set $T = \mathrm{e}^{(d-1)t}$ and recall that $W_\delta(\lambda\boldsymbol{\alpha}, M) = \lambda W_\delta(\boldsymbol{\alpha}, M)$ for any $\lambda > 0$).

**Lemma 3.** *Let $\boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2} \cap T\mathcal{D}$ with $\mathcal{D}$ as in (5.2), and $0 < \delta \leq \frac{1}{2}$. Then*

$$\left| \frac{F(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} - \frac{W_\delta(\boldsymbol{y}', n_-(\widehat{\boldsymbol{a}})D(T))}{(y_1 \cdots y_d)^{1/(d-1)}} \right| \leq \frac{d}{\eta\, T^{1/(d-1)}}, \tag{5.3}$$

*where $\boldsymbol{y} = T^{-1}\boldsymbol{a}$.*

In view of this lemma, the plan is thus to apply Theorem 8 with the set

$$\mathcal{A} = \mathcal{A}_R = \left\{(\boldsymbol{y}, \Gamma\, {}^{\mathrm{t}}M^{-1}D(y_d)^{-1}) : \boldsymbol{y} \in \mathcal{D},\ M \in \Gamma\backslash\Gamma H,\ \frac{W_\delta(\boldsymbol{y}',\, {}^{\mathrm{t}}M^{-1}D(y_d)^{-1})}{(y_1 \cdots y_d)^{1/(d-1)}} > R\right\}. \tag{5.4}$$

In the following, let

$$M = \begin{pmatrix} A & {}^{\mathrm{t}}\boldsymbol{b} \\ \boldsymbol{0} & 1 \end{pmatrix} \in H, \tag{5.5}$$

where $A \in G_0$, $\boldsymbol{b} \in \mathbb{R}^{d-1}$. Then

$$ {}^{\mathrm{t}}M^{-1} = \begin{pmatrix} {}^{\mathrm{t}}A^{-1} & {}^{\mathrm{t}}\mathbf{0} \\ -\boldsymbol{b}\,{}^{\mathrm{t}}A^{-1} & 1 \end{pmatrix}, \tag{5.6} $$

and

$$ (\boldsymbol{m}+\boldsymbol{\xi})\,{}^{\mathrm{t}}M^{-1}D(y_d)^{-1} = \big((\boldsymbol{m}'+\boldsymbol{\xi}'-(m_d+\xi_d)\boldsymbol{b})\,{}^{\mathrm{t}}A^{-1}y_d^{1/(d-1)}, (m_d+\xi_d)y_d^{-1}\big). \tag{5.7} $$

Assuming $\xi_d \in (-\frac{1}{2}, \frac{1}{2}]$, we deduce that, for all $0 < \delta \le \frac{1}{2}$, the statement $(m_d+\xi_d)y_d^{-1} \in (-\delta, \delta)$ implies $m_d = 0$ since $0 < y_d \le 1$. Therefore,

$$ \begin{aligned} &W_\delta(\boldsymbol{\alpha}, {}^{\mathrm{t}}M^{-1}D(y_d)^{-1}) \\ &= \sup_{\boldsymbol{\xi}\in\mathbb{T}^d} \min\nolimits_+ \big\{(\boldsymbol{m}+\boldsymbol{\xi})\,{}^{\mathrm{t}}M^{-1}D(y_d)^{-1}\cdot(\boldsymbol{\alpha},0) : \boldsymbol{m}\in\mathbb{Z}^d,\ (\boldsymbol{m}+\boldsymbol{\xi})\,{}^{\mathrm{t}}M^{-1}D(y_d)^{-1} \in \mathcal{R}_\delta\big\} \\ &= y_d^{1/(d-1)} \sup_{\substack{\boldsymbol{\xi}'\in\mathbb{T}^{d-1} \\ \xi_d\in(-\delta y_d,\delta y_d)}} \min\nolimits_+ \big\{(\boldsymbol{m}'+\boldsymbol{\xi}'-\xi_d\boldsymbol{b})\,{}^{\mathrm{t}}A^{-1}\cdot\boldsymbol{\alpha} : \boldsymbol{m}'\in\mathbb{Z}^{d-1},\ (\boldsymbol{m}'+\boldsymbol{\xi}'-\xi_d\boldsymbol{b})\,{}^{\mathrm{t}}A^{-1} \in \mathbb{R}^{d-1}_{\ge 0}\big\}. \end{aligned} \tag{5.8} $$

The substitution $\boldsymbol{\xi}' \mapsto \boldsymbol{\xi}' + \xi_d\boldsymbol{b}$ explains that the above supremum is independent of $\boldsymbol{b}$. So

$$ \begin{aligned} &W_\delta(\boldsymbol{\alpha}, {}^{\mathrm{t}}M^{-1}D(y_d)^{-1}) \\ &= y_d^{1/(d-1)} \sup_{\boldsymbol{\xi}'\in\mathbb{T}^{d-1}} \min\nolimits_+ \big\{(\boldsymbol{m}'+\boldsymbol{\xi}')\,{}^{\mathrm{t}}A^{-1}\cdot\boldsymbol{\alpha} : \boldsymbol{m}'\in\mathbb{Z}^{d-1},\ (\boldsymbol{m}'+\boldsymbol{\xi}')\,{}^{\mathrm{t}}A^{-1} \in \mathbb{R}^{d-1}_{\ge 0}\big\} \\ &= y_d^{1/(d-1)} V(\boldsymbol{\alpha}, {}^{\mathrm{t}}A^{-1}), \end{aligned} \tag{5.9} $$

where

$$ \begin{aligned} V(\boldsymbol{\alpha}, A) &= \sup_{\boldsymbol{\zeta}\in\mathbb{T}^{d-1}} \min\nolimits_+ \big\{(\boldsymbol{n}+\boldsymbol{\zeta})A\cdot\boldsymbol{\alpha} : \boldsymbol{n}\in\mathbb{Z}^{d-1},\ (\boldsymbol{n}+\boldsymbol{\zeta})A \in \mathbb{R}^{d-1}_{\ge 0}\big\} \\ &= \sup_{\boldsymbol{\zeta}\in\mathbb{T}^{d-1}} \min\nolimits_+ \big((\mathbb{Z}^{d-1}+\boldsymbol{\zeta})A \cap \mathbb{R}^{d-1}_{\ge 0}\big)\cdot\boldsymbol{\alpha}. \end{aligned} \tag{5.10} $$

Now set $\boldsymbol{\alpha} = \boldsymbol{y}' = (y_1, \ldots, y_{d-1})$, and

$$ Y = (y_1 \cdots y_{d-1})^{-1/(d-1)} \operatorname{diag}(y_1, \ldots, y_{d-1}) \in G_0, \tag{5.11} $$

so that $\boldsymbol{y}' = (y_1 \cdots y_{d-1})^{1/(d-1)}\boldsymbol{e}Y$. Then

$$ V(\boldsymbol{y}', A) = (y_1 \cdots y_{d-1})^{1/(d-1)} V(\boldsymbol{e}, AY) \tag{5.12} $$

and hence

$$ \frac{W_\delta(\boldsymbol{y}', {}^{\mathrm{t}}M^{-1}D(y_d)^{-1})}{(y_1 \cdots y_d)^{1/(d-1)}} = V(\boldsymbol{e}, {}^{\mathrm{t}}A^{-1}Y). \tag{5.13} $$

Set

$$ V(A) := V(\boldsymbol{e}, A) = \sup_{\boldsymbol{\zeta}\in\mathbb{T}^{d-1}} \min\big((\mathbb{Z}^{d-1}+\boldsymbol{\zeta})A \cap \mathbb{R}^{d-1}_{\ge 0}\big)\cdot\boldsymbol{e}. \tag{5.14} $$

We conclude that

$$ \mathcal{A}_R = \Big\{\big(\boldsymbol{y}, \Gamma\,{}^{\mathrm{t}}M^{-1}D(y_d)^{-1}\big) : (\boldsymbol{y}, M) \in \mathcal{D}\times\Gamma\backslash\Gamma H,\ V({}^{\mathrm{t}}A^{-1}Y) > R\Big\}. \tag{5.15} $$

**Lemma 4.** *$V(A)$ is a continuous function on $\Gamma_0\backslash G_0$.*

*Proof.* We have $V(\gamma A) = V(A)$ for all $\gamma \in \Gamma_0$ by the same argument as in (2.19), and hence $V(A)$ is a function on $\Gamma_0\backslash G_0$. It is sufficient to establish the continuity of $V(A)$ on compact subsets of $G_0$. Let us thus fix a compact set $\mathcal{C} \subset G_0$, and define

$$ K = \{\boldsymbol{\zeta}A : \boldsymbol{\zeta} \in [0,1]^{d-1},\ A \in \mathcal{C}\}, \tag{5.16} $$

which is a compact subset of $\mathbb{R}^{d-1}$. Then, for all $A \in \mathcal{C}$,

$$V(A) = \sup_{\boldsymbol{x}\in L} \min \big( (\mathbb{Z}^{d-1} A + \boldsymbol{x}) \cap \mathbb{R}^{d-1}_{\geq 0} \big) \cdot \boldsymbol{e}, \tag{5.17}$$

where $L$ is any set containing $K$. Clearly $V(A)$ is bounded on $\mathcal{C}$, i.e., there is $R > 0$ such that $V(A) \leq R$ for all $A \in \mathcal{C}$. Thus

$$V(A) = \sup_{\boldsymbol{x}\in L} \min \big( (\mathbb{Z}^{d-1} A + \boldsymbol{x}) \cap R\Delta \big) \cdot \boldsymbol{e}, \tag{5.18}$$

where $\Delta$ is the simplex (1.5). For $K' = K + [-1,1]\boldsymbol{e}$,

$$S = \mathbb{Z}^{d-1} \cap \bigcup_{A\in\mathcal{C}} \bigcup_{\boldsymbol{x}\in K'} \big( (R\Delta - \boldsymbol{x}) A^{-1} \big) \tag{5.19}$$

is a finite subset of $\mathbb{Z}^{d-1}$, and we have

$$V(A) = \sup_{\boldsymbol{x}\in K'} \min_{\boldsymbol{m}\in S} \big( (\boldsymbol{m}A + \boldsymbol{x}) \cap \mathbb{R}^{d-1}_{\geq 0} \big) \cdot \boldsymbol{e} \tag{5.20}$$

for all $A \in \mathcal{C}$. (The reason why we use $K'$ rather than $K$ in the definition of $S$ will become clear below.)

Fix $\epsilon \in (0, 1)$. Then there exists $\delta > 0$ such that, for all $A, A' \in \mathcal{C}$ with $d(A, A') < \delta$, we have

$$\|\boldsymbol{m}A - \boldsymbol{m}A'\| < \epsilon \qquad \text{for all } \boldsymbol{m} \in S. \tag{5.21}$$

Thus, for any $\boldsymbol{m} \in S$ we have

$$\boldsymbol{m}A' + \boldsymbol{x} - \epsilon\boldsymbol{e} \in \mathbb{R}^{d-1}_{\geq 0} \quad \text{implies} \quad \boldsymbol{m}A + \boldsymbol{x} \in \mathbb{R}^{d-1}_{\geq 0}, \tag{5.22}$$

and secondly

$$\begin{aligned} (\boldsymbol{m}A' + \boldsymbol{x} - \epsilon\boldsymbol{e}) \cdot \boldsymbol{e} &= (\boldsymbol{m}A' + \boldsymbol{x}) \cdot \boldsymbol{e} - d\epsilon \\ &\geq (\boldsymbol{m}A + \boldsymbol{x}) \cdot \boldsymbol{e} - (\sqrt{d} + d)\epsilon. \end{aligned} \tag{5.23}$$

Now choose $\boldsymbol{x} \in K$ such that

$$\min \big( (\mathbb{Z}^{d-1} A + \boldsymbol{x}) \cap \mathbb{R}^{d-1}_{\geq 0} \big) \cdot \boldsymbol{e} \geq V(A) - \epsilon. \tag{5.24}$$

Then (5.22) and (5.23) yield

$$\min_{\boldsymbol{m}\in S} \big( (\boldsymbol{m}A' + \boldsymbol{x} - \epsilon\boldsymbol{e}) \cap \mathbb{R}^{d-1}_{\geq 0} \big) \cdot \boldsymbol{e} \geq V(A) - (1 + \sqrt{d} + d)\epsilon. \tag{5.25}$$

Since $\boldsymbol{x} - \epsilon\boldsymbol{e} \in K'$ (because $\boldsymbol{x} \in K$ and $0 < \epsilon < 1$), the left hand side is at most $V(A')$. That is, $V(A') \geq V(A) - (1 + \sqrt{d} + d)\epsilon$. We conclude by interchanging $A$ and $A'$ that

$$|V(A') - V(A)| \leq (1 + \sqrt{d} + d)\epsilon. \tag{5.26}$$

for all $A, A' \in \mathcal{C}$ with $d(A, A') < \delta$. □

Since $V(A)$ is continuous, we have for any $\epsilon \in (0, R]$,

$$\mathcal{A}_{R+\epsilon} \subset \mathcal{A}_R^\circ \subset \overline{\mathcal{A}}_R \subset \mathcal{A}_{R-\epsilon}. \tag{5.27}$$

Define the function $\Psi_d : \mathbb{R}_{\geq 0} \to [0, 1]$ by

$$\Psi_d(R) := \mu_0\big( \{ A \in \Gamma_0\backslash G_0 : V(A) > R \} \big), \tag{5.28}$$

which is non-increasing. Note that by the invariance of $\mu_0$ under the right $G_0$-action and under $A \mapsto {}^{\mathrm{t}}A^{-1}$, we have

$$\Psi_d(R) = \mu_0\big( \{ A \in \Gamma_0\backslash G_0 : V({}^{\mathrm{t}}A^{-1} Y) > R \} \big). \tag{5.29}$$

As to the right hand sides of (4.9) and (4.10), the above calculations show that for any $\epsilon \in (0, R]$,

$$\nu\big(\mathcal{A}_R^\circ\big) \geq \operatorname{vol}(\mathcal{D})\, \Psi_d(R + \epsilon) \tag{5.30}$$

and

$$\nu\big(\overline{\mathcal{A}}_R\big) \leq \operatorname{vol}(\mathcal{D})\, \Psi_d(R-\epsilon). \tag{5.31}$$

Thus, combining these inequalities with Theorem 8 and Lemma 3, we obtain the following.

**Lemma 5.** *Let $R > 0$. For any $\epsilon \in (0, R]$,*

$$\liminf_{T\to\infty} \frac{1}{T^d} \# \bigg\{ \boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2} \cap T\mathcal{D} : \frac{F(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} > R \bigg\} \geq \frac{\operatorname{vol}(\mathcal{D})}{\zeta(d)}\, \Psi_d(R+\epsilon), \tag{5.32}$$

$$\limsup_{T\to\infty} \frac{1}{T^d} \# \bigg\{ \boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2} \cap T\mathcal{D} : \frac{F(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} > R \bigg\} \leq \frac{\operatorname{vol}(\mathcal{D})}{\zeta(d)}\, \Psi_d(R-\epsilon). \tag{5.33}$$

With this lemma, the proof of Theorem 1 is complete if we can show that $\Psi_d(R)$ is continuous (since then the lim sup and lim inf must coincide). This will be proved in Section 7.

## 6. Lattice free domains and covering radii

We denote the standard basis vectors in $\mathbb{R}^{d-1}$ by $\boldsymbol{e}_1 = (1, 0, \ldots, 0), \ldots, \boldsymbol{e}_{d-1} = (0, \ldots, 0, 1)$. Consider the simplex (1.5) and denote the face perpendicular to $\boldsymbol{e}_i$ by $\Delta_i$ ($i = 1, \ldots, d-1$), and by $\Delta_d$ the face perpendicular to $\boldsymbol{e}$.

Recall from the previous section:

$$V(A) = \sup_{\boldsymbol{\zeta}\in\mathbb{T}^{d-1}} \min\big( (\mathbb{Z}^{d-1} + \boldsymbol{\zeta})A \cap \mathbb{R}^{d-1}_{\geq 0} \big) \cdot \boldsymbol{e}. \tag{6.1}$$

The following lemma states, that the simplex $\Delta$, enlarged by a factor of $V(A)$ and suitably translated, is a maximal lattice free domain; cf. also [16].

**Lemma 6.** *If $V(A) = R$ for some $R > 0$, then there is a vector $\boldsymbol{\zeta} \in \mathbb{R}^{d-1}$ such that*

(i) $\mathbb{Z}^{d-1}A \cap (R\Delta^\circ + \boldsymbol{\zeta}) = \emptyset$;

(ii) $\mathbb{Z}^{d-1}A \cap (R\Delta_i^\circ + \boldsymbol{\zeta}) \neq \emptyset$ *for all* $i = 1, \ldots, d$.

*On the other hand, if* (i) *and* (ii) *hold for some $R > 0$, $\boldsymbol{\zeta} \in \mathbb{R}^{d-1}$, then $R \leq V(A)$.*

*Proof.* If $\mathbb{Z}^{d-1}A \cap (R\Delta^\circ + \boldsymbol{\zeta}) \neq \emptyset$ for all $\boldsymbol{\zeta}$, then $V(A) < R$, contradicting our assumption $V(A) = R$. Hence there exists $\boldsymbol{\zeta}$ such that (i) holds. If $\mathbb{Z}^{d-1}A \cap (R\Delta_i^\circ + \boldsymbol{\zeta}) = \emptyset$ for some $i$, then there exists a larger translate $R'\Delta^\circ + \boldsymbol{\zeta}'$ (for some $R' > R$, $\boldsymbol{\zeta}' \in \mathbb{R}^{d-1}$) which is lattice free, and hence $V(A) \geq R' > R$. This proves (ii), and the final statement is evident. □

**Theorem 9.** *Denote by $\rho(A)$ the covering radius of the simplex $\Delta$ with respect to the lattice $\mathbb{Z}^{d-1}A$. Then*

$$\rho(A) = V(A). \tag{6.2}$$

*Proof.* (We adapt the argument of [16, Theorem 2].) Let $V(A) = R$ and assume $\mathbb{Z}^{d-1}A + R\Delta \neq \mathbb{R}^{d-1}$. Then there is $\boldsymbol{\xi} \in \mathbb{R}^{d-1}$ such that $\boldsymbol{\xi} + \boldsymbol{v} \notin R\Delta$ for all $\boldsymbol{v} \in \mathbb{Z}^{d-1}A$. Hence $\mathbb{Z}^{d-1}A \cap (R\Delta - \boldsymbol{\xi}) = \emptyset$, and, by Lemma 6, $V(A) > R$; a contradiction. This shows $\rho(A) \leq V(A)$.

On the other hand, again by Lemma 6, for any $R' < R = V(A)$ there exists $\boldsymbol{\zeta} \in \mathbb{R}^{d-1}$ such that $\mathbb{Z}^{d-1}A \cap (R'\Delta + \boldsymbol{\zeta}) = \emptyset$, and hence no element of $\mathbb{Z}^{d-1}A$ is covered by the translates of $R'\Delta + \boldsymbol{\zeta}$. This proves $\rho(A) > R'$ and hence $\rho(A) = V(A)$. □

## 7. Continuity of the limit distribution

The following lemma shows that $\Psi_d(R)$ is continuous.

**Lemma 7.** *For every $R > 0$,*

$$\mu_0\big(\{A \in \Gamma_0\backslash G_0 : V(A) = R\}\big) = 0. \tag{7.1}$$

*Proof.* By Lemma 6 (ii), the set $\{A \in G_0 : V(A) = R\}$ is a subset of

(7.2)
$$\bigcup_{\boldsymbol{n}_1,\ldots,\boldsymbol{n}_d \in \mathbb{Z}^{d-1}} \big\{A \in G_0 : \text{there exists } \boldsymbol{\zeta} \in \mathbb{R}^{d-1} \text{ such that } \boldsymbol{n}_i A \cap (R\Delta_i^\circ + \boldsymbol{\zeta}) \neq \emptyset \ (i = 1, \ldots, d)\big\}.$$

We therefore need to show that each set in the above union has $\mu_0$-measure zero. Since the sets $R\Delta_i^\circ$ are contained in the respective hyperplanes $\boldsymbol{e}_i \cdot \boldsymbol{y} = 0$ (for $i = 1, \ldots, d-1$) and $\boldsymbol{e} \cdot \boldsymbol{y} = R$ (for $i = d$), it suffices to show that

$$\big\{A \in G_0 : \text{there exists } \boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_{d-1}) \in \mathbb{R}^{d-1} \text{ such that } \boldsymbol{e}_i \cdot \boldsymbol{n}_i A = \zeta_i \ (i = 1, \ldots, d-1),\ \boldsymbol{e} \cdot \boldsymbol{n}_d A = R + \boldsymbol{e} \cdot \boldsymbol{\zeta}\big\} \tag{7.3}$$

has measure zero. Evidently (7.3) equals

$$\Big\{A \in G_0 : \boldsymbol{e} \cdot \boldsymbol{n}_d A = R + \sum_{i=1}^{d-1} \boldsymbol{e}_i \cdot \boldsymbol{n}_i A\Big\} = \Big\{A \in G_0 : \operatorname{tr}(LA) = R\Big\}, \tag{7.4}$$

with the matrix

$$L = \begin{pmatrix} \boldsymbol{n}_d - \boldsymbol{n}_1 \\ \vdots \\ \boldsymbol{n}_d - \boldsymbol{n}_{d-1} \end{pmatrix}. \tag{7.5}$$

If $L = 0$ the set (7.4) is empty (since $R > 0$) and hence has measure zero. If $L \neq 0$ then the set (7.4) is a submanifold of codimension one; note that the map $G_0 \to \mathbb{R}$, $A \mapsto \operatorname{tr}(LA)$, has non-vanishing differential except at the (at most two) points $A \in G_0$ for which $LA$ is proportional to the identity matrix. Hence the set (7.4) has measure zero also in this case and the proof is complete. □


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## Appendix A. The distribution of sublattices

Sections 3 and 4 establish the equidistribution of Farey sequences embedded in large horospheres. These results provide an alternative perspective on Schmidt's work on the distribution of sublattices of $\mathbb{Z}^d$ [17]. In the present appendix, we will reformulate Theorems 7 and 8 in a form that clarifies the relationship between the two approaches.

Let us fix a piecewise continuous map $K : \mathrm{S}_1^{d-1} \to G$ of the unit sphere $\mathrm{S}_1^{d-1}$ such that $\boldsymbol{y}K(\boldsymbol{y}) = (\boldsymbol{0}, 1)$. By *piecewise continuous* we mean here: there is a partition of $\mathrm{S}_1^{d-1}$ by subsets $\mathcal{P}_i$ with boundary of Lebesgue measure zero, so that $K$ restricted to $\mathcal{P}_i$ can be extended to a continuous map on the closure $\overline{\mathcal{P}}_i$.

We extend the definition of $K$ to $\mathbb{R}^d \setminus \{\boldsymbol{0}\} \to G$ by setting

$$K(\boldsymbol{y}) = K(\mathring{\boldsymbol{y}}) D(\|\boldsymbol{y}\|)^{-1} \tag{A.1}$$

with $D$ as in (3.41) and $\mathring{\boldsymbol{y}} := \boldsymbol{y}/\|\boldsymbol{y}\|$. The extended map still satisfies $\boldsymbol{y}K(\boldsymbol{y}) = (\boldsymbol{0}, 1)$.

As in Remark 3.3, we choose $\gamma \in \Gamma$ such that $\boldsymbol{a}\gamma = (\boldsymbol{0}, 1)$. Then $(\boldsymbol{0}, 1)\gamma^{-1}K(\boldsymbol{a}) = (\boldsymbol{0}, 1)$, which implies $\gamma^{-1}K(\boldsymbol{a}) \in H$, and hence $\Gamma K(\boldsymbol{a}) \in \Gamma\backslash\Gamma H$.

**Theorem 10.** *Fix a piecewise continuous embedding $K : \mathbb{R}^d \setminus \{\boldsymbol{0}\} \to G$ as defined above. Let $\mathcal{D} \subset \mathbb{R}^d$ be bounded with boundary of Lebesgue measure zero, and $f : \overline{\mathcal{D}} \times \Gamma\backslash\Gamma H \to \mathbb{R}$ bounded continuous. Then*

$$\lim_{T\to\infty} \frac{1}{T^d} \sum_{\boldsymbol{a} \in \widehat{\mathbb{Z}}^d \cap T\mathcal{D}} f\Big(\frac{\boldsymbol{a}}{T}, K(\boldsymbol{a})\Big) = \frac{1}{\zeta(d)} \int_{\mathcal{D} \times \Gamma_H\backslash H} f\big(\boldsymbol{y}, M\big)\, d\boldsymbol{y}\, d\mu_H(M). \tag{A.2}$$

*Proof.* In view of the fact that $\Gamma\backslash\Gamma H$ is a closed embedded submanifold of $\Gamma\backslash G$, it suffices to prove that, for $f : \overline{\mathcal{D}} \times \Gamma\backslash G \to \mathbb{R}$ bounded continuous,

$$\lim_{T\to\infty} \frac{1}{T^d} \sum_{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d\cap T\mathcal{D}} f\Big(\frac{\boldsymbol{a}}{T}, {}^{\mathrm{t}}K(\boldsymbol{a})^{-1}\Big) = \frac{1}{\zeta(d)} \int_{\mathcal{D}\times\Gamma_H\backslash H} \widetilde{f}\big(\boldsymbol{y}, M\big)\, d\boldsymbol{y}\, d\mu_H(M). \tag{A.3}$$

We may assume without loss of generality that $f$ has compact support (cf. Step 0 of the proof of Theorem 6), and that $\mathcal{D} \subset \{\boldsymbol{x} \in \mathbb{R}^d : \eta \le x_1, \ldots, x_{d-1} \le x_d\} \cap \mathbb{R}_{>0}\mathcal{P}_i$ for some fixed $\eta > 0$ and $\mathcal{P}_i$ as defined in the second paragraph of this appendix.

If $\boldsymbol{y} \in \mathcal{D}$, then $y_d \ge \eta$, and we may expand

$$K(\mathring{\boldsymbol{y}})^{-1} = \begin{pmatrix} A(\mathring{\boldsymbol{y}}) & {}^{\mathrm{t}}\boldsymbol{b}(\mathring{\boldsymbol{y}}) \\ \boldsymbol{0} & 1 \end{pmatrix} M_{\mathring{\boldsymbol{y}}}, \tag{A.4}$$

with $M_{\boldsymbol{y}}$ as in (3.21). The maps $A$, $\boldsymbol{b}$ are continuous on $\overline{\mathcal{P}}_i \cap \mathbb{R}^d_{\ge\eta}$, and hence bounded. A short calculation shows that

$$\begin{aligned} K(\boldsymbol{y})^{-1} &= \begin{pmatrix} A(\mathring{\boldsymbol{y}}) & {}^{\mathrm{t}}\boldsymbol{b}(\mathring{\boldsymbol{y}})\|\boldsymbol{y}\|^{-d/(d-1)} \\ \boldsymbol{0} & 1 \end{pmatrix} M_{\boldsymbol{y}} \\ &= \begin{pmatrix} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{b}(\mathring{\boldsymbol{y}})\|\boldsymbol{y}\|^{-d/(d-1)} \\ \boldsymbol{0} & 1 \end{pmatrix} \begin{pmatrix} A(\mathring{\boldsymbol{y}}) & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix} M_{\boldsymbol{y}}. \end{aligned} \tag{A.5}$$

Set

$$K_0(\boldsymbol{y})^{-1} = \begin{pmatrix} A(\mathring{\boldsymbol{y}}) & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix} M_{\boldsymbol{y}}. \tag{A.6}$$

Because $\|\boldsymbol{a}\| \ge \sqrt{d}\,\eta T$, we have

$$d({}^{\mathrm{t}}K(\boldsymbol{a})^{-1}, {}^{\mathrm{t}}K_0(\boldsymbol{a})^{-1}) \le \sup_{\boldsymbol{y}\in\mathcal{D}} \big\|\boldsymbol{b}(\mathring{\boldsymbol{y}})\big\| \,\big(\sqrt{d}\,\eta T\big)^{-d/(d-1)}, \tag{A.7}$$

where the supremum is finite by the continuity of $\boldsymbol{b}$. Since $f$ is uniformly continuous, it therefore suffices to establish (A.3) with $K(\boldsymbol{a})^{-1}$ replaced by $K_0(\boldsymbol{a})^{-1}$. We now apply Theorem 7 with the test function

$$f_0(\boldsymbol{y}, M) = f\bigg(\boldsymbol{y}, MD(y_d) \begin{pmatrix} {}^{\mathrm{t}}A(\mathring{\boldsymbol{y}}) & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}\bigg), \tag{A.8}$$

which is bounded continuous on $\overline{\mathcal{D}} \times \Gamma\backslash G$ (under the above assumptions on $f$ and $\mathcal{D}$). With this choice,

$$\begin{aligned} f_0\Big(\frac{\boldsymbol{a}}{T}, n_-(\widehat{\boldsymbol{a}})D(T)\Big) &= f\bigg(\frac{\boldsymbol{a}}{T}, n_-(\widehat{\boldsymbol{a}})D(T)D(a_d/T) \begin{pmatrix} {}^{\mathrm{t}}A(\mathring{\boldsymbol{a}}) & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}\bigg) \\ &= f\Big(\frac{\boldsymbol{a}}{T}, {}^{\mathrm{t}}K_0(\boldsymbol{a})^{-1}\Big). \end{aligned} \tag{A.9}$$

As to the right hand side of (4.1), we have

$$\begin{aligned} \widetilde{f}_0(\boldsymbol{y}, MD(y_d)) &= f_0\big(\boldsymbol{y}, {}^{\mathrm{t}}M^{-1}D(y_d)^{-1}\big) \\ &= f\bigg(\boldsymbol{y}, {}^{\mathrm{t}}M^{-1}D(y_d)^{-1}D(y_d) \begin{pmatrix} {}^{\mathrm{t}}A(\mathring{\boldsymbol{y}}) & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}\bigg) \\ &= \widetilde{f}\bigg(\boldsymbol{y}, M \begin{pmatrix} A(\mathring{\boldsymbol{y}})^{-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}\bigg). \end{aligned} \tag{A.10}$$

Eq. (A.3) now follows from the right $H$-invariance of $\mu_H$. □

The following theorem is a corollary of Theorem 10; the proof is analogous to that of Theorem 8.

**Theorem 11.** *Fix a piecewise continuous embedding $K:\mathbb{R}^d\setminus\{\mathbf{0}\}\to G$ as defined above. Let $\mathcal{D}\subset\mathbb{R}^d$ be bounded with boundary of Lebesgue measure zero, and $\mathcal{A}\subset\overline{\mathcal{D}}\times\Gamma\backslash\Gamma H$. Then*

$$\liminf_{T\to\infty}\frac{1}{T^d}\#\Big\{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d:\Big(\frac{\boldsymbol{a}}{T},\Gamma K(\boldsymbol{a})\Big)\in\mathcal{A}\Big\}\geq\frac{(\mathrm{vol}\times\mu_H)\big(\mathcal{A}^\circ\big)}{\zeta(d)} \tag{A.11}$$

*and*

$$\limsup_{T\to\infty}\frac{1}{T^d}\#\Big\{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d:\Big(\frac{\boldsymbol{a}}{T},\Gamma K(\boldsymbol{a})\Big)\in\mathcal{A}\Big\}\leq\frac{(\mathrm{vol}\times\mu_H)\big(\overline{\mathcal{A}}\big)}{\zeta(d)}. \tag{A.12}$$

Let us now explain how the above statements are related to Schmidt's results on the distribution of primitive sublattices [17].

Two lattices $\Lambda,\Lambda'\subset\mathbb{R}^d$ of rank $m$ are called *similar*, if there is an invertible angle-preserving linear transformation $R$ (that is, $R\in\mathbb{R}_{>0}\,\mathrm{O}(d)$), such that $\Lambda'=\Lambda R$.

Let us denote by $\mathrm{Gr}_m(\mathbb{R}^d)$ the Grassmannian of $m$-dimensional linear subspaces of $\mathbb{R}^d$. The map

$$\widehat{\mathbb{Z}}^d\to\mathrm{Gr}_{d-1}(\mathbb{R}^d),\qquad \boldsymbol{a}\mapsto\boldsymbol{a}^\perp:=\{\boldsymbol{x}\in\mathbb{R}^d:\boldsymbol{x}\cdot\boldsymbol{a}=0\} \tag{A.13}$$

gives a one-to-one correspondence between primitive lattice points and rational subspaces of dimension $d-1$. A *primitive sublattice of $\mathbb{Z}^d$ of rank $d-1$* is defined as

$$\Lambda_{\boldsymbol{a}}=\mathbb{Z}^d\cap\boldsymbol{a}^\perp, \tag{A.14}$$

and hence there is a one-to-one correspondence between primitive lattice points and primitive sublattices of rank $d-1$. The covolume of $\Lambda_{\boldsymbol{a}}$ equals $\|\boldsymbol{a}\|$. Note that

$$\boldsymbol{a}^\perp\,{}^{\mathrm{t}}K(\boldsymbol{a})^{-1}=(\mathbf{0},1)^\perp=\mathbb{R}^{d-1}\times\{0\}, \tag{A.15}$$

with $K(\boldsymbol{a})$ as in (A.1). Hence

$$\Lambda_{\boldsymbol{a}}\,{}^{\mathrm{t}}K(\boldsymbol{a})^{-1}=\mathbb{Z}^d\,{}^{\mathrm{t}}K(\boldsymbol{a})^{-1}\cap(\mathbb{R}^{d-1}\times\{0\}) \tag{A.16}$$

and

$$\Lambda_{\boldsymbol{a}}\,{}^{\mathrm{t}}K(\boldsymbol{a})^{-1}=\|\boldsymbol{a}\|^{-1/(d-1)}\Lambda_{\boldsymbol{a}}\,{}^{\mathrm{t}}K(\mathring{\boldsymbol{a}})^{-1}. \tag{A.17}$$

We now choose the above embedding $K$ such that $K(\mathring{\boldsymbol{y}})\in\mathrm{SO}(d)$; see e.g. [11, Section 4.2, footnote 3] for an explicit construction. The map

$$\Lambda_{\boldsymbol{a}}\mapsto\Lambda'_{\boldsymbol{a}}:=\Lambda_{\boldsymbol{a}}\,{}^{\mathrm{t}}K(\boldsymbol{a})^{-1} \tag{A.18}$$

maps primitive sublattices of $\mathbb{Z}^d$ of rank $d-1$ to lattices in $\mathbb{R}^{d-1}$. Eq. (A.17) shows $\Lambda_{\boldsymbol{a}}$ and $\Lambda'_{\boldsymbol{a}}$ are similar; it furthermore implies that $\Lambda'_{\boldsymbol{a}}$ has covolume one.

In [17] Schmidt proves that, as $T\to\infty$, the set $\{\Lambda'_{\boldsymbol{a}}:\|\boldsymbol{a}\|\leq T\}$ becomes uniformly distributed in the space of lattices of covolume one, $\Gamma_0\backslash G_0$, with respect to the right $G_0$-invariant measure $\mu_0$. In particular, Theorem 3 in [17] (adapted to the case of primitive lattices of rank $d-1$) follows from our Theorem 11, if we set

$$\mathcal{A}=\left\{\left(\boldsymbol{y},\Gamma\begin{pmatrix}A&{}^{\mathrm{t}}\boldsymbol{b}\\ \mathbf{0}&1\end{pmatrix}\right):\boldsymbol{y}\in\mathcal{D},\ A\in\mathcal{A}_0,\ \boldsymbol{b}\in\mathbb{R}^{d-1}\right\}\subset\overline{\mathcal{D}}\times\Gamma\backslash\Gamma H, \tag{A.19}$$

where $\mathcal{D}\subset\mathbb{R}^d$ has boundary of Lebesgue measure zero, and $\mathcal{A}_0\subset\Gamma_0\backslash G_0$ is arbitrary. Theorem 2 in [17] is obtained when $\mathcal{D}$ is taken to be the unit ball.

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