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HOLOMORPHIC ALMOST MODULAR FORMS

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Abstract

Holomorphic almost modular forms are holomorphic functions of the complex upper half plane that can be approximated arbitrarily well (in a suitable sense) by modular forms of congruence subgroups of large index in $\mathrm{SL}(2,\mathbb{Z})$. It is proved that such functions have a rotation-invariant limit distribution when the argument approaches the real axis. An example of a holomorphic almost modular form is the logarithm of $\prod_{n=1}^{\infty}(1-\exp(2\pi\,\mathrm{i}\,n^2 z))$. The paper is motivated by the author's previous studies [*Int. Math. Res. Not.* 39 (2003) 2131–2151] on the connection between almost modular functions and the distribution of the sequence n^2x modulo one.

1. *Introduction*

Almost modular functions have recently been introduced in connection with the distribution of the sequence n^2x modulo one ($n=1,2,3,\ldots$). It is shown in [**5**] that, for every piecewise smooth periodic function $\psi:\mathbb{R}\longrightarrow\mathbb{C}$ of period one, and for x uniformly distributed in $[0,1)$, the error term

$$R_\psi^x(M)=\frac{1}{\sqrt{M}}\left(\sum_{n=1}^{M}\psi(n^2x)-M\int_0^1\psi(t)\,dt\right) \tag{1.1}$$

has a limit distribution as $M\to\infty$, which can be identified with the value distribution of a certain almost modular function. This observation resembles results published by Heath-Brown [**3**] and Bleher [**1**, **2**], who proved that error terms in lattice point problems for convex planar domains have limit distributions associated with *almost periodic functions* (in the sense of Besicovitch).

In the present work we draw attention to the holomorphic species of almost modular functions. A *holomorphic almost modular form* (HAMF) is defined as a holomorphic function of the complex upper half-plane $\mathfrak{H}$ to $\mathbb{C}$ that can be approximated arbitrarily well (in a sense to be made precise in Section 3) by modular forms of congruence subgroups $\Gamma_1(N)$ in $\mathrm{SL}(2,\mathbb{Z})$, as $N\to\infty$. An example of a HAMF is the logarithm of

$$\prod_{n=1}^{\infty}(1-e(n^2z)), \tag{1.2}$$

where $e(z):=\exp(2\pi\,\mathrm{i}\,z)$; see Section 3. As a consequence of the general limit theorem for almost modular functions [**5**, Theorem 8.2], we will see in Section 4 that, for $\operatorname{Re} z$ uniformly distributed in $[0,1)$,

$$(\operatorname{Im} z)^{1/4}\log\prod_{n=1}^{\infty}(1-e(n^2z)) \tag{1.3}$$

has a rotation-invariant limit distribution in $\mathbb{C}$, as $\operatorname{Im} z\to 0$.

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An explicit formula for the variance of the limit distribution is given in Section 5. The paper concludes with a short appendix (Section 6), containing background material from [**5**] on the definition of general almost modular functions and their limit theorems.

2. *Holomorphic modular forms*

For any integer x and any prime p, the standard quadratic residue symbol $(\frac{x}{p})$ is 1 if x is a square modulo p, and -1 otherwise. The *generalized quadratic residue symbol* $(\frac{a}{b})$ is, for any integer a and any odd integer b, characterized by the following properties [**4**, pp. 160–161].

(i) $(\frac{a}{b}) = 0$ if $(a,b) \neq 1$.
(ii) $(\frac{a}{-1}) = \operatorname{sgn} a$.
(iii) If $b > 0$, $b = \prod_i b_i$ and b_j are primes, not necessarily distinct, then $(\frac{a}{b}) = \prod_i (\frac{a}{b_i})$.
(iv) $(\frac{a}{-b}) = (\frac{a}{-1})(\frac{a}{b})$.
(v) $(\frac{0}{\pm 1}) = 1$.

It follows from these properties that the symbol is bimultiplicative:

$$\left(\frac{a_1 a_2}{b}\right) = \left(\frac{a_1}{b}\right)\left(\frac{a_2}{b}\right); \qquad \left(\frac{a}{b_1 b_2}\right) = \left(\frac{a}{b_1}\right)\left(\frac{a}{b_2}\right). \tag{2.1}$$

Furthermore, if $b > 0$, then $(\frac{\cdot}{b})$ defines a character modulo b; if $a \neq 0$, then $(\frac{a}{\cdot})$ defines a character modulo $4a$.

The action of $\gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{SL}(2,\mathbb{R})$ on the complex upper half plane $\mathfrak{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is defined by the fractional linear transformation $z \mapsto (az+b)/(cz+d)$. We are here interested in the congruence subgroups

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : a \equiv d \equiv 1,\ c \equiv 0 \bmod N \right\}. \tag{2.2}$$

Fix $\Gamma = \Gamma_1(N)$, with $4 \mid N$.

A *holomorphic modular form of weight* κ *for* Γ (with $\kappa \in \frac{1}{2}\mathbb{Z}$) is a holomorphic function $\mathfrak{H} \longrightarrow \mathbb{C}$ that satisfies the functional relation

$$f(\gamma z) = \left(\frac{c}{d}\right)^{2\kappa} (cz+d)^{\kappa} f(z) \tag{2.3}$$

for all $\gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma$, and that is holomorphic with respect to each cusp; see [**6**, Definition 1.3.3.]. This means that f has a Fourier expansion of the form

$$\sum_{m=0}^{\infty} \hat{f}_m^{(i)}\, e(m z_i) \tag{2.4}$$

for each cuspidal coordinate z_i (see the appendix, Section 6); hence f is bounded in each cusp. In (2.3), the square root $z^{1/2}$ is chosen such that $-\pi/2 < \arg z^{1/2} \leqslant \pi/2$, and $z^{m/2} := (z^{1/2})^m$, for $m \in \mathbb{Z}$.

Famous examples of holomorphic modular forms are the theta series

$$\theta(z) = \sum_{n \in \mathbb{Z}} e(n^2 z), \tag{2.5}$$

which is of weight $\kappa = \frac{1}{2}$, and Jacobi's Δ-function

$$\Delta(z) = e(z) \prod_{n=1}^{\infty} \big(1 - e(nz)\big)^{24}, \tag{2.6}$$

where $\kappa = 12$; see [**6**].

LEMMA 2.1. *Let $a_1, a_2, \ldots, a_K \in \mathbb{C}$. Then the function*

$$\xi^{(K)}(z) = \sum_{k=1}^{K} a_k\, \theta(kz) \tag{2.7}$$

is a modular form of weight $\frac{1}{2}$ for $\Gamma_1(N)$ with $N = 4\,\mathrm{lcm}(2,3,\ldots,K)$.

Proof. We have, for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(4k)$,

$$\begin{aligned} \theta\left(k\frac{az+b}{cz+d}\right) &= \theta\left(\frac{a(kz)+kb}{(c/k)(kz)+d}\right) \\ &= \left(\frac{(c/k)}{d}\right)(cz+d)^{1/2}\theta(kz), \end{aligned} \tag{2.8}$$

because

$$\begin{pmatrix} a & kb \\ c/k & d \end{pmatrix} \in \Gamma_1(4). \tag{2.9}$$

Since the generalized quadratic residue symbol is multiplicative,

$$\left(\frac{c}{d}\right) = \left(\frac{(c/k)}{d}\right)\left(\frac{k}{d}\right). \tag{2.10}$$

Furthermore, $(\frac{k}{\cdot})$ is a character mod $4k$, and hence, for $d \equiv 1 \bmod 4k$, we have

$$\left(\frac{k}{d}\right) = \left(\frac{k}{1}\right) = 1. \tag{2.11}$$

This shows that $\theta^{(k)}(z) := \theta(kz)$ is a modular form for $\Gamma_1(4k)$. The lemma now follows from the observation that

$$\Gamma_1(N) \subset \bigcap_{k=1}^{K} \Gamma_1(4k). \tag{2.12}$$

□

3. *Holomorphic almost modular forms*

DEFINITION 3.1. We call a holomorphic periodic function $\mathfrak{H} \longrightarrow \mathbb{C}$

$$\xi(z) = \sum_{m=0}^{\infty} \hat{\xi}_m\, e(mz) \tag{3.1}$$

a *holomorphic almost modular form* (HAMF) *of weight* $\frac{1}{2}$ if, for every $\varepsilon > 0$, there are an $N = N(\varepsilon)$ and a modular form

$$f_\varepsilon(z) = \sum_{m=0}^{\infty} \hat{f}_{\varepsilon,m}\, e(mz) \tag{3.2}$$

of weight $\frac{1}{2}$ for $\Gamma_1(N)$, such that

$$\limsup_{M\to\infty} \frac{1}{\sqrt{M}} \sum_{m=0}^{M} \left|\hat{\xi}_m - \hat{f}_{\varepsilon,m}\right|^2 < \varepsilon^2. \tag{3.3}$$

To construct examples of such functions, let $h : \mathfrak{H} \longrightarrow \mathbb{C}$ be a periodic holomorphic function of the form

$$h(z) = \sum_{k=1}^{\infty} \hat{h}_k \, e(kz), \tag{3.4}$$

with constants $C > 0$ and $\beta > \frac{1}{4}$ such that for all $k \in \mathbb{N}$,

$$|\hat{h}_k| \leqslant \frac{C}{k^\beta}. \tag{3.5}$$

THEOREM 3.1. *For h as in* (3.4) *and* (3.5), *the function*

$$\xi(z) = \sum_{n=1}^{\infty} h(n^2 z) \tag{3.6}$$

is a holomorphic almost modular function of weight $\frac{1}{2}$.

Proof. We choose as approximants, the modular forms (see Lemma 2.1)

$$\xi^{(K)}(z) := \frac{1}{2} \sum_{k=1}^{K} \hat{h}_k \, \theta(kz). \tag{3.7}$$

The Fourier coefficients of $\xi(z)$ are

$$\hat{\xi}_0 = 0, \qquad \hat{\xi}_m = \sum_{k=1}^{\infty} \sum_{\substack{n=1\\ m=n^2 k}}^{\infty} \hat{h}_k, \tag{3.8}$$

and those of $\xi^{(K)}(z)$ are

$$\hat{\xi}_0^{(K)} = \frac{1}{2} \sum_{k=1}^{K} \hat{h}_k, \qquad \hat{\xi}_m^{(K)} = \sum_{k=1}^{K} \sum_{\substack{n=1\\ m=n^2 k}}^{\infty} \hat{h}_k. \tag{3.9}$$

Therefore

$$\begin{aligned} \sum_{m=1}^{M} \left|\hat{\xi}_m - \hat{\xi}_m^{(K)}\right|^2 &= \sum_{k_1,k_2=K+1}^{\infty} \sum_{\substack{n_1,n_2=1\\ 1\leqslant n_1^2 k_1 = n_2^2 k_2 \leqslant M}}^{\infty} \hat{h}_{k_1} \overline{\hat{h}_{k_2}} \\ &= \sum_{(p,q,r,s)\in S_1} \hat{h}_{rp^2} \overline{\hat{h}_{rq^2}} \\ &\leqslant C^2 \sum_{(p,q,r,s)\in S_1} \frac{1}{r^{2\beta} p^{2\beta} q^{2\beta}}, \end{aligned} \tag{3.10}$$

where the sums are restricted to the set

$$S_1 = \{p,q,r,s \in \mathbb{N},\ \gcd(p,q) = 1,\ rp^2, rq^2 > K,\ 1 \leqslant rp^2 s^2 q^2 \leqslant M\}. \tag{3.11}$$

Thus

$$\sum_{m=1}^{M}\left|\hat{\xi}_m-\hat{\xi}_m^{(K)}\right|^2 \leqslant C^2\sqrt{M}\sum_{(p,q,r)\in S_2}\frac{1}{r^{1/2+2\beta}p^{1+2\beta}q^{1+2\beta}}+O(1)\sum_{(p,q,r)\in S_3}\frac{1}{r^{2\beta}p^{2\beta}q^{2\beta}}, \tag{3.12}$$

where

$$S_2=\{p,q,r\in\mathbb{N},\ \gcd(p,q)=1,\ rp^2,rq^2>K\}; \tag{3.13}$$

$$S_3=\{p,q,r\in\mathbb{N},\ \gcd(p,q)=1,\ 1\leqslant rp^2q^2\leqslant M\}. \tag{3.14}$$

The last sum in (3.12) is bounded (where we assume, without loss of generality, that $\frac{1}{4}<\beta<\frac{1}{2}$) by

$$\sum_{(p,q,r)\in S_3}\frac{1}{r^{2\beta}p^{2\beta}q^{2\beta}}=O(M^{1-2\beta})\sum_{\substack{p,q=1\\ 1\leqslant p^2q^2\leqslant M}}^{\infty}\frac{1}{p^{2-2\beta}q^{2-2\beta}}=O(M^{1-2\beta}). \tag{3.15}$$

Hence

$$\sum_{m=1}^{M}\left|\hat{\xi}_m-\hat{\xi}_m^{(K)}\right|^2 \leqslant C^2\sqrt{M}\sum_{(p,q,r)\in S_2}\frac{1}{r^{1/2+2\beta}p^{1+2\beta}q^{1+2\beta}}+O(M^{1-2\beta}). \tag{3.16}$$

For $\beta>\frac{1}{4}$, the sum in (3.16) converges and tends to zero for K large. The remainder in (3.16) is of sub-leading order; that is, $O(M^{1-2\beta})=o(\sqrt{M})$. So, given any $\varepsilon>0$, there is a large K such that

$$\limsup_{M\to\infty}\frac{1}{\sqrt{M}}\sum_{m=0}^{M}\left|\hat{\xi}_m-\hat{\xi}_m^{(K)}\right|^2<\varepsilon^2. \tag{3.17}$$

□

If we choose $h(z)=\log(1-e(z))$, we have $\hat{h}_k=-1/k$, and thus Theorem 3.1 implies that

$$\xi(z)=\log\prod_{n=1}^{\infty}\big(1-e(n^2z)\big) \tag{3.18}$$

is a holomorphic almost modular function.

4. *The limit theorem*

THEOREM 4.1. *Let $\xi(z)$ be a holomorphic almost modular function of weight $\frac{1}{2}$. Then, for $x:=\operatorname{Re}z$ uniformly distributed in $[0,1)$ with respect to Lebesgue measure, $(\operatorname{Im}z)^{1/4}\xi(z)$ has a limit distribution as $y:=\operatorname{Im}z\to 0$. That is, there exists a probability measure ν_ξ on $\mathbb{C}$ such that, for any bounded continuous function $g:\mathbb{C}\longrightarrow\mathbb{C}$, we have*

$$\lim_{y\to 0}\int_0^1 g\big(y^{1/4}\xi(x+\mathrm{i}\,y)\big)dx=\int_{\mathbb{C}}g(w)\,\nu_\xi(dw). \tag{4.1}$$

Furthermore, ν_ξ is invariant under rotations about the origin.

Proof. The following two lemmas show that Theorem 4.1 is a special case of the limit theorem for almost modular functions, Theorem 6.2. Rotational invariance of the limit distribution is proved at the end of this section. □

The manifold $\mathcal{M}_N = \Delta_1(N)\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and the function spaces $B_\sigma(\mathcal{M}_N)$, $\mathcal{B}^2$, which appear below, are defined in the appendix (Section 6).

LEMMA 4.2. *If $f(z)$ is a holomorphic modular form of weight κ for $\Gamma_1(N)$, then the function*

$$F(z,\phi) = (\operatorname{Im} z)^{\kappa/2} f(z)\, \mathrm{e}^{-\mathrm{i}\,\kappa\phi} \tag{4.2}$$

is a modular function of class $B_{\kappa/2}(\mathcal{M}_N)$.

Proof. For $[\gamma,\beta_\gamma]\in\Delta_1(N)$, we have

$$F\big([\gamma,\beta_\gamma](z,\phi)\big) = \big[\operatorname{Im}(\gamma z)\big]^{\kappa/2} f(\gamma z)\, \mathrm{e}^{-\mathrm{i}\,\kappa(\phi+\beta_\gamma)} = F(z,\phi) \tag{4.3}$$

because of (2.3) and

$$\big[\operatorname{Im}(\gamma z)\big]^{\kappa/2} = \frac{(\operatorname{Im} z)^{\kappa/2}}{|cz+d|^{\kappa}}, \qquad \mathrm{e}^{-\mathrm{i}\,\kappa\beta_\gamma} = j_\gamma(z)^{-2\kappa} = \left\{ \left(\frac{c}{d}\right) \left(\frac{cz+d}{|cz+d|}\right)^{1/2} \right\}^{-2\kappa}. \tag{4.4}$$

Hence F is a smooth function on $\mathcal{M}_N$. Because (by definition) f is bounded in each cusp, we have

$$F(z,\phi) = O(y_i^{\kappa/2}); \tag{4.5}$$

see condition (6.6) in the appendix (Section 6). ☐

LEMMA 4.3. *If $\xi(z)$ is a holomorphic almost modular function of weight $\frac{1}{2}$, then the function*

$$\Xi(z) = (\operatorname{Im} z)^{1/4}\xi(z) \tag{4.6}$$

is an almost modular function of class $\mathcal{B}^2$.

Proof. If f_ε are the approximants of ξ, we choose as approximants for Ξ the functions

$$F_\varepsilon(z,\phi) = (\operatorname{Im} z)^{1/4} f_\varepsilon(z)\, \mathrm{e}^{-\mathrm{i}\,\phi/2}. \tag{4.7}$$

Then, in view of (3.3),

$$\begin{aligned} \limsup_{y\to 0} \int_0^1 \big|\Xi(x+\mathrm{i}\,y) - F_\varepsilon(x+\mathrm{i}\,y,0)\big|^2 dx &= \limsup_{y\to 0} y^{1/2} \sum_{m=0}^{\infty} \big|\hat{\xi}_m - \hat{f}_{\varepsilon,m}\big|^2 \mathrm{e}^{-4\pi m y} \\ &= O(\varepsilon^2), \end{aligned} \tag{4.8}$$

and hence $\Xi\in\mathcal{B}^2$; see Definition 6.1. ☐

Proof of rotational invariance. The limit distribution for every approximant f_ε (see Theorem 6.1) is given by

$$\int_{\mathbb{C}} g(w)\, \nu_{f_\varepsilon}(dw) = \int_{\mathcal{M}_N} g\big(y^{1/4} f_\varepsilon(x+\mathrm{i}\,y)\, \mathrm{e}^{-\mathrm{i}\,\phi/2}\big) \frac{dx\, dy\, d\phi}{y^2}. \tag{4.9}$$

Substituting $\phi+2\omega$ for ϕ shows that $f_\varepsilon(z)$ and $f_\varepsilon(z)\,\mathrm{e}^{-\mathrm{i}\,\omega}$ have the same limit distribution for all $\omega\in[0,2\pi)$.

Hence $\xi(z)$ and $\xi(z)\,\mathrm{e}^{-\mathrm{i}\,\omega}$ share the same limit distribution. ☐

5. *The variance*

Let us return to the example introduced in Theorem 3.1, and derive an explicit formula for the variance of the limit distribution. By slightly modifying the steps in (3.10) and (3.12), one finds that for

$$\alpha(t) := \sum_{0 \leqslant m < t} \left|\hat{\xi}_m\right|^2 \tag{5.1}$$

and $t \to \infty$, we have

$$\alpha(t) \sim A\, t^{1/2}, \quad \text{with } A := \sum_{r=1}^{\infty} \sum_{\substack{p,q=1 \\ \gcd(p,q)=1}}^{\infty} \frac{\hat{h}_{rp^2} \overline{\hat{h}_{rq^2}}}{pq\sqrt{r}}. \tag{5.2}$$

Therefore, using Parseval's equality and a standard Abelian theorem for the Laplace transform [**7**, Chapter V],

$$\begin{aligned} \int_0^1 |\xi(x + \mathrm{i}\, y)|^2 dx &= \sum_{m=0}^{\infty} \left|\hat{\xi}_m\right|^2 \mathrm{e}^{-4\pi m y} \\ &= \int_0^{\infty} \mathrm{e}^{-4\pi y t}\, d\alpha(t) \\ &\sim A\, \Gamma(\tfrac{3}{2})\, (4\pi y)^{-1/2}, \end{aligned} \tag{5.3}$$

as $y \to 0$. Since Euler's function evaluates to $\sqrt{\pi}/2$, we have

$$\lim_{y \to 0} y^{1/2} \int_0^1 |\xi(x + \mathrm{i}\, y)|^2 dx = \frac{1}{4} \sum_{r=1}^{\infty} \sum_{\substack{p,q=1 \\ \gcd(p,q)=1}}^{\infty} \frac{\hat{h}_{rp^2} \overline{\hat{h}_{rq^2}}}{pq\sqrt{r}}. \tag{5.4}$$

6. *Appendix: Almost modular functions for* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$

This appendix provides some background material on the definition of almost modular functions and their limit theorems; the reader is referred to [**5**] for more detailed information.

Let us denote by $\mathrm{C}(\mathfrak{H})$ the space of continuous functions $\mathfrak{H} \longrightarrow \mathbb{C}$, and put $\varepsilon_g(z) = (cz + d)/|cz + d|$. The universal covering group of $\mathrm{SL}(2,\mathbb{R})$ is defined as the set

$$\widetilde{\mathrm{SL}}(2,\mathbb{R}) = \{[g, \beta_g] : g \in \mathrm{SL}(2,\mathbb{R}),\ \beta_g \in \mathrm{C}(\mathfrak{H}) \text{ such that } \mathrm{e}^{\mathrm{i}\,\beta_g(z)} = \varepsilon_g(z)\}, \tag{6.1}$$

with multiplication law

$$[g, \beta_g^1][h, \beta_h^2] = [gh, \beta_{gh}^3], \qquad \beta_{gh}^3(z) = \beta_g^1(hz) + \beta_h^2(z). \tag{6.2}$$

We may identify $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with $\mathfrak{H} \times \mathbb{R}$ via $[g, \beta_g] \mapsto (z, \phi) = (g\,\mathrm{i}, \beta_g(\mathrm{i}))$. The action of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ on $\mathfrak{H} \times \mathbb{R}$ is then $[g, \beta_g](z, \phi) = (gz, \phi + \beta_g(z))$. The Haar measure of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ reads, in this parametrization,

$$d\mu(g) = \frac{dx\, dy\, d\phi}{y^2}. \tag{6.3}$$

The group $\Delta_1(N)$ is the following discrete subgroup of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$:

$$\Delta_1(N) = \left\{[\gamma,\beta_\gamma] : \gamma \in \Gamma_1(N),\ \beta_\gamma \in \mathrm{C}(\mathfrak{H}) \text{ such that } \mathrm{e}^{\mathrm{i}\,\beta_\gamma(z)/2} = j_\gamma(z)\right\}, \quad (6.4)$$

where

$$j_\gamma(z) = \left(\frac{c}{d}\right)\left(\frac{cz+d}{|cz+d|}\right)^{1/2}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(4). \quad (6.5)$$

Here, $z^{1/2}$ denotes the principal branch of the square root of z; that is, the one for which $-\pi/2 < \arg z^{1/2} \leqslant \pi/2$.

The homogeneous space $\mathcal{M}_N = \Delta_1(N)\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R})$ has finite volume with respect to Haar measure (6.3). Also, $\mathcal{M}_N$ has a finite number of cusps, which are represented by the set $\eta_1,\ldots,\eta_\kappa \in \mathbb{Q}\cup\infty$ on the boundary of $\mathfrak{H}$. Let γ_i be a fractional linear transformation $\mathfrak{H} \longrightarrow \mathfrak{H}$ that maps the cusp at η_i to the standard cusp at ∞ of width one. Thus $(z_i,\phi_i) = \widetilde{\gamma}_i(z,\phi)$ yields a new set of coordinates, where the ith cusp appears as a cusp at ∞, which is invariant under $(z_i,\phi_i) \mapsto (z_i+1,\phi_i)$. The variable $y_i = \mathrm{Im}(\gamma_i z)$ measures the height into the ith cusp. For any $\sigma \geqslant 0$, we denote by $B_\sigma(\mathcal{M}_N)$ the class of functions $F \in \mathrm{C}(\mathcal{M}_N)$ such that, for all $i = 1,\ldots,\kappa$,

$$F(z,\phi) = O(y_i^\sigma), \qquad y_i \to \infty, \quad (6.6)$$

where the implied constant is independent of (z,ϕ). In view of the form of the invariant measure (6.3), we note that $B_\sigma(\mathcal{M}_N) \subset \mathrm{L}^p(\mathcal{M}_N,\mu)$ if $\sigma < 1/p$.

The following theorem [**5**, Theorem 6.1] states that the closed horocycles

$$\Delta_1(N)\{(x+\mathrm{i}\,y,0) : x \in [0,1)\} \quad (6.7)$$

are asymptotically equidistributed in $\mathcal{M}_N$, as $y \to 0$.

Theorem 6.1. *Let $0 \leqslant \sigma < 1$. Then, for every $F \in B_\sigma(\mathcal{M}_N)$, we have*

$$\lim_{y\to 0}\int_0^1 F(x+\mathrm{i}\,y,0)\,dx = \frac{1}{\mu(\mathcal{M}_N)}\int_{\mathcal{M}_N} F\,d\mu. \quad (6.8)$$

Let us now turn to the definition of *almost modular functions of class $\mathcal{B}^p$* or $\mathcal{H}$, respectively, as given in [**5**, Section 7]. In the following we will consider functions $\Xi : \mathfrak{H} \longrightarrow \mathbb{C}$ that are *periodic* – that is, for which $\Xi(z+1) = \Xi(z)$.

Definition 6.1. For any $p \geqslant 1$, let $\mathcal{B}^p$ be the class of periodic functions $\Xi : \mathfrak{H} \longrightarrow \mathbb{C}$ with the property that for every $\varepsilon > 0$ there are an integer $N = N(\varepsilon) > 0$ and a function $F_\varepsilon \in B_\sigma(\mathcal{M}_N)$ with $0 \leqslant \sigma < 1/p$ so that

$$\limsup_{y\to 0}\int_0^1 \left|\Xi(x+\mathrm{i}\,y) - F_\varepsilon(x+\mathrm{i}\,y,0)\right|^p dx < \varepsilon^p. \quad (6.9)$$

Definition 6.2. Let $\mathcal{H}$ be the class of periodic functions $\Xi : \mathfrak{H} \longrightarrow \mathbb{C}$ with the property that for every $\varepsilon > 0$ there are an integer $N = N(\varepsilon) > 0$ and a bounded continuous function $F_\varepsilon \in \mathrm{C}(\mathcal{M}_N)$ such that

$$\limsup_{y\to 0}\int_0^1 \min\left\{1, \left|\Xi(x+\mathrm{i}\,y) - F_\varepsilon(x+\mathrm{i}\,y,0)\right|\right\}dx < \varepsilon. \quad (6.10)$$

If $1 \leqslant q \leqslant p$, we have the inclusion $\mathcal{B}^p \subset \mathcal{B}^q \subset \mathcal{H}$; see [**5**, Proposition 7.3]. The central observation of [**5**] is the following limit theorem for almost modular functions [**5**, Theorem 8.2].

THEOREM 6.2. *Let $\Xi \in \mathcal{H}$. Then, for x uniformly distributed in $[0,1)$ with respect to Lebesgue measure, $\Xi(x+\mathrm{i}\,y)$ has a limit distribution as $y \to 0$. That is, there exists a probability measure ν_Ξ on $\mathbb{C}$ such that, for every bounded continuous function $g : \mathbb{C} \longrightarrow \mathbb{C}$,*

$$\lim_{y\to 0} \int_0^1 g\big(\Xi(x+\mathrm{i}\,y)\big)\,dx = \int_{\mathbb{C}} g(w)\,\nu_\Xi(dw). \tag{6.11}$$

The proof of this theorem closely follows the argument for almost periodic functions [**1**]. The main difference is that the equidistribution theorem for irrational Kronecker flows on multidimensional tori is here replaced by Theorem 6.1; see [**5**, Section 8].

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