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PAIR CORRELATION DENSITIES OF INHOMOGENEOUS QUADRATIC FORMS, II

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Abstract
Denote by $\| \cdot \|$ the Euclidean norm in $\mathbb{R}^k$. We prove that the local pair correlation density of the sequence $\|\boldsymbol{m} - \boldsymbol{\alpha}\|^k$, $\boldsymbol{m} \in \mathbb{Z}^k$, is that of a Poisson process, under Diophantine conditions on the fixed vector $\boldsymbol{\alpha} \in \mathbb{R}^k$: in dimension two, vectors $\boldsymbol{\alpha}$ of any Diophantine type are admissible; in higher dimensions ($k > 2$), Poisson statistics are observed only for Diophantine vectors of type $\kappa < (k-1)/(k-2)$. Our findings support a conjecture of M. Berry and M. Tabor on the Poisson nature of spectral correlations in quantized integrable systems.

1. Introduction

1.1
Berry and Tabor [1] have conjectured that the local correlations of quantum energy levels of integrable systems are those of independent random numbers from a Poisson process. We present here a proof of this conjecture for the two-point correlations of the sequence

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty,$$

given by the values of

$$\|\boldsymbol{m} - \boldsymbol{\alpha}\|^2 = (m_1 - \alpha_1)^2 + \cdots + (m_k - \alpha_k)^2$$

at lattice points $\boldsymbol{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k$, for fixed $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$. These numbers represent the eigenvalues of the Laplacian

$$-\Delta = -\frac{\partial^2}{\partial x_1^2} - \ldots - \frac{\partial^2}{\partial x_k^2}$$

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on the flat torus $\mathbb{T}^k$ with quasi-periodicity conditions

$$\varphi(\boldsymbol{x}+\boldsymbol{l}) = \mathrm{e}^{-2\pi \mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{l}}\varphi(\boldsymbol{x}), \quad \boldsymbol{l} \in \mathbb{Z}^k,$$

and may therefore be viewed as energy levels of the quantized geodesic flow. Statistical properties of the above sequence were first studied by Z. Cheng, J. Lebowitz, and P. Major [3], [4] in dimension $k = 2$. We extend here our studies [11], [12] to dimensions $k \geq 2$.

Previous results on the Berry-Tabor conjecture for flat tori include [6], [8], [13] in dimension $k = 2$ and [18], [17], [19] for $k > 2$. For more details and references, see [2], [8], [10], [14].

1.2

We are interested in the local correlations between the λ_j on the scale of the mean spacing. Because the mean density is increasing as $\lambda \to \infty$, that is,

$$\frac{1}{\lambda}\#\{j : \lambda_j \leq \lambda\} = \frac{1}{\lambda}\#\{\boldsymbol{m} \in \mathbb{Z}^k : \|\boldsymbol{m}-\boldsymbol{\alpha}\|^2 \leq \lambda\} \sim B_k \lambda^{k/2-1},$$

where B_k is the volume of the unit ball, it is necessary to rescale the sequence by setting

$$X_j = \lambda_j^{k/2}.$$

Then

$$\frac{1}{X}\#\{j : X_j \leq X\} = \frac{1}{X}\#\{\boldsymbol{m} \in \mathbb{Z}^k : \|\boldsymbol{m}-\boldsymbol{\alpha}\|^k \leq X\} \to B_k$$

for $X \to \infty$, and hence the mean spacing is constant, as required.

1.3

The *pair correlation density* of a sequence with constant mean density D is defined as

$$R_2[a,b](X) = \frac{1}{DX}\#\{i \neq j : X_i, X_j \in [X, 2X],\ X_i - X_j \in [a,b]\}.$$

We recall the following classical result.

THEOREM 1.4

If the X_j come from a Poisson process with mean density D, one has

$$\lim_{X\to\infty} R_2[a,b](X) = D(b-a)$$

almost surely.

1.5

We prove here a similar result for the *deterministic* sequence in Section 1.1, which holds, however, only under Diophantine conditions on $\boldsymbol{\alpha}$. The vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$ is said to be *Diophantine of type* κ if there exists a constant C such that

$$\max_j \left| \alpha_j - \frac{m_j}{q} \right| > \frac{C}{q^\kappa}$$

for all $m_1, \ldots, m_k, q \in \mathbb{Z}$, $q > 0$. The smallest possible value for κ is $\kappa = 1 + 1/k$. In this case, $\boldsymbol{\alpha}$ is called *badly approximable*.

THEOREM 1.6

Suppose that $\boldsymbol{\alpha}$ is Diophantine of type $\kappa < (k-1)/(k-2)$ and that the components of the vector $(\boldsymbol{\alpha}, 1) \in \mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$. Then

$$\lim_{X\to\infty} R_2[a,b](X) = B_k(b-a).$$

The condition in Theorem 1.6 is satisfied if, for instance, the components of $(\boldsymbol{\alpha}, 1)$ form a basis of a real algebraic number field of degree $k+1$. In this case, $\kappa = 1 + 1/k$ (see [15]).

The condition $\kappa < (k-1)/(k-2)$ in Theorem 1.6 is sharp.

THEOREM 1.7

Let $k > 2$. For any $a > 0$, there exists a set $C \subset \mathbb{T}^k$ of second Baire category for which the following holds.

(i) *All $\boldsymbol{\alpha} \in C$ are Diophantine of type $\kappa = (k-1)/(k-2)$, and the components of the vector $(\boldsymbol{\alpha}, 1) \in \mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$.*

(ii) *For $\boldsymbol{\alpha} \in C$, we find arbitrarily large X such that*

$$R_2[-a,a](X) \geq \frac{\log X}{\log\log\log X}.$$

(iii) *For $\boldsymbol{\alpha} \in C$, there exists an infinite sequence $L_1 < L_2 < \cdots \to \infty$ such that*

$$\lim_{j\to\infty} R_2[-a,a](L_j) = 2\pi a.$$

In Theorem 1.7(ii), $\log\log\log X$ may be replaced by any slowly increasing positive function $\nu(X) \leq \log\log\log X$ with $\nu(X) \to \infty$ as $X \to \infty$.

Without imposing any Diophantine condition, the rate of divergence may be even worse.

THEOREM 1.8
For any $a > 0$, there exists a set $C \subset \mathbb{T}^k$ of second Baire category for which the following holds.

(i) *For $\boldsymbol{\alpha} \in C$, the components of the vector $(\boldsymbol{\alpha}, 1) \in \mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$.*

(ii) *For $\boldsymbol{\alpha} \in C$, we find arbitrarily large X such that*

$$R_2[-a, a](X) \geq \begin{cases} \frac{\log X}{\log\log\log X} & (k = 2), \\ \frac{X^{(k-2)/k}}{\log\log\log X} & (k > 2). \end{cases}$$

(iii) *For $\boldsymbol{\alpha} \in C$, there exists an infinite sequence $L_1 < L_2 < \cdots \to \infty$ such that*

$$\lim_{j\to\infty} R_2[-a, a](L_j) = 2\pi a.$$

Again, $\log\log\log X$ may be replaced by any slowly increasing positive function $\nu(X) \leq \log\log\log X$ with $\nu(X) \to \infty$ as $X \to \infty$.

Theorems 1.7 and 1.8 are proved in Section 8.

2. Rescaling

2.1
We see in this section how Theorem 1.6, which is the central result of this paper, follows as a straightforward corollary from the asymptotics of the generalized pair correlation function

$$R_2(\psi, \lambda) = \frac{1}{B_k \lambda^{k/2}} \sum_{i,j=1}^{\infty} \psi\Big(\frac{\lambda_i}{\lambda}, \frac{\lambda_j}{\lambda}, \lambda^{k/2-1}(\lambda_i - \lambda_j)\Big)$$

with $\psi \in \mathrm{C}_0(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R})$, that is, continuous and of compact support.

THEOREM 2.2
Let $\psi \in \mathrm{C}_0(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R})$. Suppose that the components of $(\boldsymbol{\alpha}, 1) \in \mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$, and assume that $\boldsymbol{\alpha}$ is Diophantine of type $\kappa < (k-1)/(k-2)$. Then

$$\lim_{\lambda\to\infty} R_2(\psi, \lambda) = \frac{k}{2} \int_0^\infty \psi(r, r, 0) r^{k/2-1}\, dr + \frac{k^2}{4} B_k \int_{\mathbb{R}} \int_0^\infty \psi(r, r, s) r^{k-2}\, dr\, ds.$$

2.3. Theorem 2.2 ⇒ Theorem 1.6.

Let us now show how Theorem 2.2 implies Theorem 1.6. For $\psi_1, \psi_2 \in \mathrm{C}_0(\mathbb{R}^+)$ with support in the compact interval I not containing the origin zero, and $\sigma \in \mathrm{C}_0(\mathbb{R})$, we define

$$\psi(r_1, r_2, s) = \psi_1(r_1^{k/2})\psi_2(r_2^{k/2})\sigma\big(\rho(r_1, r_2)s\big)$$

with

$$\rho(r_1, r_2) = \frac{r_1^{k/2} - r_2^{k/2}}{r_1 - r_2} = \begin{cases} \sum_{\nu=1}^{k/2} r_1^{k/2-\nu} r_2^{\nu-1} & (k \text{ even}), \\ \frac{1}{r_1^{1/2}+r_2^{1/2}} \sum_{\nu=1}^{k} r_1^{(k-\nu)/2} r_2^{(\nu-1)/2} & (k \text{ odd}). \end{cases}$$

It is evident that we can find a constant $\delta > 0$ such that

$$\delta < \rho(r_1, r_2) < \frac{1}{\delta}$$

uniformly for all $r_1, r_2 \in I$.

The assumptions on ψ in Theorem 2.2 are therefore satisfied, giving

$$\begin{aligned} \lim_{\lambda\to\infty} \frac{1}{B_k \lambda^{k/2}} \sum_{i,j=1}^{\infty} \psi_1\Big(\frac{\lambda_i^{k/2}}{\lambda^{k/2}}\Big)\psi_2\Big(\frac{\lambda_j^{k/2}}{\lambda^{k/2}}\Big)\sigma(\lambda_i^{k/2} - \lambda_j^{k/2}) \\ = \frac{k}{2}\sigma(0) \int_0^\infty \psi_1(r^{k/2})\psi_2(r^{k/2}) r^{k/2-1}\, dr \\ + \frac{k^2}{4} B_k \int_{\mathbb{R}} \int_0^\infty \psi_1(r^{k/2})\psi_2(r^{k/2})\sigma\big(\rho(r, r)s\big) r^{k-2}\, dr\, ds. \end{aligned}$$

With $\rho(r, r) = (k/2) r^{k/2-1}$ and the substitutions $X = \lambda^{k/2}$, $x = r^{k/2}$, and $s \mapsto s/\rho(r, r)$, we finally have

$$\begin{aligned} \lim_{X\to\infty} \frac{1}{B_k X} \sum_{i,j=1}^{\infty} \psi_1\Big(\frac{X_i}{X}\Big)\psi_2\Big(\frac{X_j}{X}\Big)\sigma(X_i - X_j) \\ = \sigma(0) \int_0^\infty \psi_1(x)\psi_2(x)\, dx + B_k \int_{\mathbb{R}} \sigma(s)\, ds \int_0^\infty \psi_1(x)\psi_2(x)\, dx. \end{aligned}$$

The first term on the right-hand side comes obviously from the diagonal terms $X_i = X_j$ (use the asymptotics in Sec. 1.2), so

$$\lim_{X\to\infty} \frac{1}{B_k X} \sum_{i\neq j} \psi_1\Big(\frac{X_i}{X}\Big)\psi_2\Big(\frac{X_j}{X}\Big)\sigma(X_i - X_j) = B_k \int_{\mathbb{R}} \sigma(s)\, ds \int_0^\infty \psi_1(x)\psi_2(x)\, dx,$$

which is a smoothed version of Theorem 1.6. We complete the proof by quoting a standard density argument (cf. [12, proof of Th. 1.8]) in which the characteristic functions of the intervals $[1, 2]$, $[1, 2]$, and $[a, b]$ are approximated from above and below by smooth functions ψ_1, ψ_2, and σ, respectively. □

2.4
It is sufficient to restrict our attention to the following special case of Theorem 2.2. Put

$$R_2(\psi_1,\psi_2,h,\lambda)=\frac{1}{B_k\lambda^{k/2}}\sum_{i,j=1}^{\infty}\psi_1\Big(\frac{\lambda_i}{\lambda}\Big)\psi_2\Big(\frac{\lambda_j}{\lambda}\Big)\hat{h}\big(\lambda^{k/2-1}(\lambda_i-\lambda_j)\big).$$

Here $\psi_1,\psi_2\in\mathscr{S}(\mathbb{R}_+)$ are real valued, and $\mathscr{S}(\mathbb{R}_+)$ denotes the Schwartz class of infinitely differentiable functions of the half-line $\mathbb{R}_+$ (including the origin) which, as well as their derivatives, decrease rapidly at $+\infty$; $\hat{h}$ is the Fourier transform of a compactly supported function $h\in\mathrm{C}_0(\mathbb{R})$,

$$\hat{h}(s)=\int_{\mathbb{R}}h(u)e\Big(\frac{1}{2}us\Big)\,du,$$

with the shorthand $e(z):=\mathrm{e}^{2\pi\mathrm{i}z}$.

We prove the following in Section 7.

THEOREM 2.5
Let $\psi_1,\psi_2\in\mathscr{S}(\mathbb{R}_+)$, and let $h\in\mathrm{C}_0(\mathbb{R})$. Suppose that the components of $(\boldsymbol{\alpha},1)\in\mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$, and assume that $\boldsymbol{\alpha}$ is Diophantine of type $\kappa<(k-1)/(k-2)$. Then

$$\begin{aligned}\lim_{\lambda\to\infty}R_2(\psi_1,\psi_2,h,\lambda)&=\frac{k}{2}\hat{h}(0)\int_0^{\infty}\psi_1(r)\psi_2(r)r^{k/2-1}\,dr\\&\quad+\frac{k^2}{4}B_k\int\hat{h}(s)\,ds\int_0^{\infty}\psi_1(r)\psi_2(r)r^{k-2}\,dr.\end{aligned}$$

2.6. Theorem 2.5 ⇒ Theorem 2.2.
For any fixed $\epsilon>0$, we find finite linear combinations (cf. [12, Sec. 8.6])

$$\psi^{\pm}(r_1,r_2,s)=\sum_{\nu}\psi_{1,\nu}^{\pm}(r_1)\psi_{2,\nu}^{\pm}(r_2)\hat{h}_{\nu}^{\pm}(s)$$

of functions satisfying the conditions of Theorem 2.5 such that

$$\psi^{-}(r_1,r_2,s)\le\psi(r_1,r_2,s)\le\psi^{+}(r_1,r_2,s)$$

and

$$\iint\big(\psi^{+}(r,r,s)-\psi^{-}(r,r,s)\big)r^{k-2}\,dr\,ds<\epsilon.$$

Theorem 2.5 tells us that

$$\lim_{\lambda\to\infty}\frac{1}{B_k\lambda^{k/2}}\sum_{i\ne j}\psi^{\pm}\Big(\frac{\lambda_i}{\lambda},\frac{\lambda_j}{\lambda},\lambda^{k/2-1}(\lambda_i-\lambda_j)\Big)=\frac{k^2}{4}B_k\iint\psi^{\pm}(r,r,s)r^{k-2}\,dr\,ds.$$

(Recall that the first term in Th. 2.5 comes trivially from the diagonal terms $i = j$.) This implies

$$\limsup_{\lambda\to\infty} \frac{1}{B_k \lambda^{k/2}} \sum_{i\neq j} \psi\Big(\frac{\lambda_i}{\lambda}, \frac{\lambda_j}{\lambda}, \lambda^{k/2-1}(\lambda_i - \lambda_j)\Big) \leq \frac{k^2}{4} B_k \Big(\iint \psi(r,r,s) r^{k-2}\, dr\, ds + \epsilon\Big)$$

and

$$\liminf_{\lambda\to\infty} \frac{1}{B_k \lambda^{k/2}} \sum_{i\neq j} \psi\Big(\frac{\lambda_i}{\lambda}, \frac{\lambda_j}{\lambda}, \lambda^{k/2-1}(\lambda_i - \lambda_j)\Big) \geq \frac{k^2}{4} B_k \Big(\iint \psi(r,r,s) r^{k-2} dr\, ds - \epsilon\Big).$$

Because these inequalities hold for arbitrarily small $\epsilon > 0$, Theorem 2.2 must be true. □

3. Outline of the proof of Theorem 2.5

Using the Fourier transform, we may write

$$\begin{aligned} R_2(\psi_1, \psi_2, h, \lambda) &= \frac{1}{B_k} \int \Big(\frac{1}{\lambda^{k/4}} \sum_j \psi_1\Big(\frac{\lambda_j}{\lambda}\Big) e\Big(\frac{1}{2}\lambda_j \lambda^{k/2-1} u\Big)\Big) \\ &\quad\times \overline{\Big(\frac{1}{\lambda^{k/4}} \sum_j \psi_2\Big(\frac{\lambda_j}{\lambda}\Big) e\Big(\frac{1}{2}\lambda_j \lambda^{k/2-1} u\Big)\Big)} h(u)\, du \\ &= \frac{1}{B_k \lambda^{k/2-1}} \int \Big(\frac{1}{\lambda^{k/4}} \sum_j \psi_1\Big(\frac{\lambda_j}{\lambda}\Big) e\Big(\frac{1}{2}\lambda_j u\Big)\Big) \\ &\quad\times \overline{\Big(\frac{1}{\lambda^{k/4}} \sum_j \psi_2\Big(\frac{\lambda_j}{\lambda}\Big) e\Big(\frac{1}{2}\lambda_j u\Big)\Big)} h(\lambda^{-(k/2-1)} u)\, du. \end{aligned}$$

The sum

$$\theta_\psi(u, \lambda) = \frac{1}{\lambda^{k/4}} \sum_j \psi\Big(\frac{\lambda_j}{\lambda}\Big) e\Big(\frac{1}{2}\lambda_j u\Big)$$

is identified as a Jacobi theta sum living on a certain noncompact but finite-volume manifold Σ (see Sec. 4). The integration in

$$R_2(\psi_1, \psi_2, h, \lambda) = \frac{1}{B_k} \lambda^{-(k/2-1)} \int \theta_{\psi_1}(u, \lambda) \overline{\theta_{\psi_2}(u, \lambda)} h(\lambda^{-(k/2-1)} u)\, du$$

amounts to averaging along a unipotent orbit on Σ, which becomes equidistributed as $\lambda \to \infty$ (see Sec. 5). Diophantine conditions on $\boldsymbol{\alpha}$ are necessary to secure the convergence of the limit (see Sec. 6).

The equidistribution theorem then yields

$$\frac{1}{\mu(\Sigma)} \int_\Sigma \theta_{\psi_1} \overline{\theta_{\psi_2}}\, d\mu \int h(u)\, du,$$

where μ is the invariant measure. The first integral can be calculated quite easily (see Sec. 7), and we see that

$$\frac{1}{\mu(\Sigma)} \int_\Sigma \theta_{\psi_1} \overline{\theta_{\psi_2}}\, d\mu \int h(u)\, du = \frac{k}{2} B_k \int \psi_1(r)\psi_2(r)\, r^{k/2-1}\, dr \int h(u)\, du,$$

which finally yields

$$\frac{k}{2} B_k \hat{h}(0) \int \psi_1(r)\psi_2(r)\, r^{k/2-1}\, dr$$

(cf. the first term in Th. 2.5).

An additional contribution comes from an arc of the orbit, which vanishes into the cusp. Even though the length of that arc tends to zero, the average over the unbounded theta function gives a nonvanishing contribution

$$\frac{k^2}{2} B_k^2 h(0) \int \psi_1(r)\psi_2(r)\, r^{k-2}\, dr = \frac{k^2}{4} B_k^2 \int \hat{h}(u)\, du \int \psi_1(r)\psi_2(r)\, r^{k-2}\, dr,$$

which corresponds to the second term in Theorem 2.5.

4. Theta sums

4.1

Consider the semidirect product group $G^k = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^{2k}$ with multiplication law

$$(M; \boldsymbol{\xi})(M'; \boldsymbol{\xi}') = (MM'; \boldsymbol{\xi} + M\boldsymbol{\xi}'),$$

where $M, M' \in \mathrm{SL}(2,\mathbb{R})$ and $\boldsymbol{\xi}, \boldsymbol{\xi}' \in \mathbb{R}^{2k}$; the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^{2k}$ is defined canonically as

$$M\boldsymbol{\xi} = \begin{pmatrix} a\boldsymbol{x} + b\boldsymbol{y} \\ c\boldsymbol{x} + d\boldsymbol{y} \end{pmatrix}, \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix},$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^k$. A convenient parametrization of $\mathrm{SL}(2,\mathbb{R})$ can be obtained by means of the Iwasawa decomposition

$$M = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

which is unique for $\tau = u + \mathrm{i}v \in \mathfrak{H}$, $\phi \in [0, 2\pi)$, where $\mathfrak{H}$ denotes the upper half-plane $\mathfrak{H} = \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}$.

4.2

For any Schwartz function $f \in \mathscr{S}(\mathbb{R}^k)$, we define the *Jacobi theta sum* Θ_f by

$$\Theta_f(\tau, \phi; \boldsymbol{\xi}) = v^{k/4} \sum_{\boldsymbol{m} \in \mathbb{Z}^k} f_\phi\big((\boldsymbol{m} - \boldsymbol{y})v^{1/2}\big)\, e\Big(\frac{1}{2}\|\boldsymbol{m} - \boldsymbol{y}\|^2 u + \boldsymbol{m} \cdot \boldsymbol{x}\Big),$$

where

$$f_\phi(\boldsymbol{w}) = \int_{\mathbb{R}^k} G_\phi(\boldsymbol{w}, \boldsymbol{w}')\, f(\boldsymbol{w}')\, dw'$$

with the integral kernel

$$G_\phi(\boldsymbol{w}, \boldsymbol{w}') = e\Big(-\frac{k\sigma_\phi}{8}\Big)|\sin\phi|^{-k/2} e\Big[\frac{(1/2)(\|\boldsymbol{w}\|^2 + \|\boldsymbol{w}'\|^2)\cos\phi - \boldsymbol{w} \cdot \boldsymbol{w}'}{\sin\phi}\Big],$$

where $\sigma_\phi = 2\nu + 1$ when $\nu\pi < \phi < (\nu + 1)\pi$, $\nu \in \mathbb{Z}$. The operators $U^\phi : f \mapsto f_\phi$ are unitary (see [7], [9] for details). Note, in particular, $U^0 = \mathrm{id}$.

The proofs of the remaining statements in this section are found in [12, Sec. 4].

LEMMA 4.3

Let $f_\phi = U^\phi f$ with $f \in \mathscr{S}(\mathbb{R}^k)$. Then, for any $R > 1$, there is a constant c_R such that for all $\boldsymbol{w} \in \mathbb{R}^k$, $\phi \in \mathbb{R}$, we have

$$|f_\phi(\boldsymbol{w})| \le c_R(1 + \|\boldsymbol{w}\|)^{-R}.$$

4.4

Let us consider the following discrete subgroup in G^k:

$$\Gamma^k = \left\{\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \begin{pmatrix} ab\boldsymbol{s} \\ cd\boldsymbol{s} \end{pmatrix} + \boldsymbol{m}\right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}),\ \boldsymbol{m} \in \mathbb{Z}^{2k}\right\} \subset G^k$$

with $\boldsymbol{s} = (1/2, 1/2, \ldots, 1/2) \in \mathbb{R}^k$.

LEMMA 4.5

Γ^k is generated by the elements

$$\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \boldsymbol{0}\right), \qquad \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{0} \end{pmatrix}\right), \qquad \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \boldsymbol{m}\right), \quad \boldsymbol{m} \in \mathbb{Z}^{2k}.$$

PROPOSITION 4.6

The left action of the group Γ^k on G^k is properly discontinuous. A fundamental domain of Γ^k in G^k is given by

$$\mathscr{F}_{\Gamma^k} = \mathscr{F}_{\mathrm{SL}(2,\mathbb{Z})} \times \{\phi \in [0, \pi)\} \times \left\{\boldsymbol{\xi} \in \Big[-\frac{1}{2}, \frac{1}{2}\Big)^{2k}\right\}.$$

where $\mathscr{F}_{\mathrm{SL}(2,\mathbb{Z})}$ is the fundamental domain in $\mathfrak{H}$ of the modular group $\mathrm{SL}(2,\mathbb{Z})$, given by $\{\tau \in \mathfrak{H} : u \in [-1/2, 1/2), |\tau| > 1\}$.

PROPOSITION 4.7
For $f, g \in \mathscr{S}(\mathbb{R}^k)$, $\Theta_f(\tau,\phi;\boldsymbol{\xi})\overline{\Theta_g(\tau,\phi;\boldsymbol{\xi})}$ is invariant under the left action of Γ^k.

PROPOSITION 4.8
Let $f, g \in \mathscr{S}(\mathbb{R}^k)$. For any $R > 1$, we have

$$\Theta_f\left(\tau,\phi;\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\right)\overline{\Theta_g\left(\tau,\phi;\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\right)} = v^{k/2}\sum_{\boldsymbol{m}\in\mathbb{Z}^k} f_\phi\big((\boldsymbol{m}-\boldsymbol{y})v^{1/2}\big)\overline{g_\phi\big((\boldsymbol{m}-\boldsymbol{y})v^{1/2}\big)} + O_R(v^{-R})$$

uniformly for all $(\tau,\phi;\boldsymbol{\xi}) \in G^k$ with $v > 1/2$. In addition,

$$\Theta_f\left(\tau,\phi;\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\right)\overline{\Theta_g\left(\tau,\phi;\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\right)} = v^{k/2} f_\phi\big((\boldsymbol{n}-\boldsymbol{y})v^{1/2}\big)\overline{g_\phi\big((\boldsymbol{n}-\boldsymbol{y})v^{1/2}\big)} + O_R(v^{-R})$$

uniformly for all $(\tau,\phi;\boldsymbol{\xi}) \in G^k$ with $v > 1/2$, $\boldsymbol{y} \in \boldsymbol{n} + [-1/2, 1/2]^k$, and $\boldsymbol{n} \in \mathbb{Z}^k$.

LEMMA 4.9
The subgroup

$$\Gamma_\theta \ltimes \mathbb{Z}^{2k},$$

where

$$\Gamma_\theta = \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) :\ ab \equiv cd \equiv 0 \bmod 2\right\}$$

is the theta group, is of index three in Γ^k.

LEMMA 4.10
Γ^k is of finite index in $\mathrm{SL}(2,\mathbb{Z}) \ltimes ((1/2)\mathbb{Z})^{2k}$.

4.11
Note: The theta sum defined in this section is related to the sum $\theta_{\psi_1}(u,\lambda)$ in Section 3 by

$$\theta_{\psi_1}(u,\lambda)\overline{\theta_{\psi_2}(u,\lambda)} = \Theta_f\left(u + \mathrm{i}\frac{1}{\lambda}, 0; \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{\alpha}\end{pmatrix}\right)\overline{\Theta_g\left(u + \mathrm{i}\frac{1}{\lambda}, 0; \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{\alpha}\end{pmatrix}\right)}$$

with

$$f(\boldsymbol{w}) = \psi_1(\|\boldsymbol{w}\|^2), \qquad g(\boldsymbol{w}) = \psi_2(\|\boldsymbol{w}\|^2).$$

5. Equidistribution

THEOREM 5.1

Let Γ be a subgroup of $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$ of finite index, and assume that the components of the vector $(\boldsymbol{y}, 1) \in \mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$. Let h be a continuous function $\mathbb{R} \to \mathbb{R}_+$ with compact support. Then, for any bounded continuous function F on $\Gamma\backslash G^k$ and any $\sigma \geq 0$, we have

$$\lim_{v\to 0} v^\sigma \int_{\mathbb{R}} F\left(u + \mathrm{i}v, 0; \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{y} \end{pmatrix}\right) h(v^\sigma u)\, du = \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} F\, d\mu \int h(w)\, dw,$$

where μ is the Haar measure of G^k.

Proof

For $\sigma = 0$, the above statement is proved in [12, Th. 5.7] (see also N. Shah's more general [16, Th. 1.4]). The case where $\sigma > 0$ is easier and, in fact, follows from the result for $\sigma = 0$ since the translate of the unipotent orbit is expanding at a faster rate.

As in [12, Sec. 5], we define the unipotent flow $\Psi^t : \Gamma\backslash G^k \to \Gamma\backslash G^k$ by right translation with

$$\Psi_0^t = \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}; \boldsymbol{0}\right),$$

and, furthermore, we define the flow $\Phi^t : \Gamma\backslash G^k \to \Gamma\backslash G^k$ by right translation with

$$\Phi_0^t = \left(\begin{pmatrix} \mathrm{e}^{-t/2} & 0 \\ 0 & \mathrm{e}^{t/2} \end{pmatrix}; \boldsymbol{0}\right).$$

By [12, Th. 5.7], the orbit segment

$$\Gamma\left\{\left(u + \mathrm{i}\mathrm{e}^{-t}, 0; \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{y} \end{pmatrix}\right) : u \in [-1, 1]\right\}$$

is dense in $\Gamma\backslash G^k$ in the limit $t \to \infty$. Hence we find a sequence $\{u_t\}_{t\in\mathbb{R}_+}$ with $u_t \in [-1, 1]$ such that

$$\Gamma g_t := \Gamma\left(u_t + \mathrm{i}\mathrm{e}^{-t}, 0; \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{y} \end{pmatrix}\right) = \Gamma\left(1; \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{y} \end{pmatrix}\right) \Psi^{u_t} \Phi^t$$

converges in the limit $t \to \infty$ to a generic point in $\Gamma\backslash G^k$. Theorem 2 in [5] implies then that for any constant $B \neq 0$,

$$\frac{1}{B\mathrm{e}^{\sigma t}} \int_0^{B\mathrm{e}^{\sigma t}} F\left(u_t + u + \mathrm{i}\mathrm{e}^{-t}, 0; \begin{pmatrix} \mathbf{0} \\ \boldsymbol{y} \end{pmatrix}\right) du = \frac{1}{B\mathrm{e}^{(1+\sigma)t}} \int_0^{B\mathrm{e}^{(1+\sigma)t}} F(g_t \Psi^u)\, du$$
$$\to \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} F\, d\mu$$

as $t \to \infty$. Because F is bounded and u_t is contained in a compact interval, note that

$$\begin{aligned} &\frac{1}{B\mathrm{e}^{\sigma t}} \int_0^{B\mathrm{e}^{\sigma t}} F\left(u_t + u + \mathrm{i}\mathrm{e}^{-t}, 0; \begin{pmatrix} \mathbf{0} \\ \boldsymbol{y} \end{pmatrix}\right) du \\ &\qquad = \frac{1}{B\mathrm{e}^{\sigma t}} \int_{u_t}^{B\mathrm{e}^{\sigma t}+u_t} F\left(u + \mathrm{i}\mathrm{e}^{-t}, 0; \begin{pmatrix} \mathbf{0} \\ \boldsymbol{y} \end{pmatrix}\right) du \\ &\qquad = \frac{1}{B\mathrm{e}^{\sigma t}} \int_0^{B\mathrm{e}^{\sigma t}} F\left(u + \mathrm{i}\mathrm{e}^{-t}, 0; \begin{pmatrix} \mathbf{0} \\ \boldsymbol{y} \end{pmatrix}\right) du + O(\mathrm{e}^{-\sigma t}). \end{aligned}$$

Therefore, for any constants $-\infty < A < B < \infty$,

$$\lim_{t\to\infty} \frac{1}{\mathrm{e}^{\sigma t}} \int_{A\mathrm{e}^{\sigma t}}^{B\mathrm{e}^{\sigma t}} F\left(u + \mathrm{i}\mathrm{e}^{-t}, 0; \begin{pmatrix} \mathbf{0} \\ \boldsymbol{y} \end{pmatrix}\right) du = \frac{(B-A)}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} F\, d\mu.$$

The theorem now follows from a standard approximation argument. (Approximate h from above and below by step functions.) $\square$

6. Diophantine conditions

6.1

In order to extend the equidistribution results to unbounded test functions such as $\Theta_f \overline{\Theta_g}$, let us study the following model functions, whose asymptotics in the cusp is similar to that of $\Theta_f \overline{\Theta_g}$. Let $G = G^k$, and let $\Gamma = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$. Define, furthermore, the subgroup

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\} \subset \mathrm{SL}(2,\mathbb{Z}),$$

put

$$v_\gamma := \mathrm{Im}(\gamma\tau) = \frac{v}{|c\tau + d|^2} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and put $\boldsymbol{y}_\gamma := c\boldsymbol{x} + d\boldsymbol{y}$. Let χ_R be the characteristic function of the interval $[R, \infty)$,

$$\chi_R(t) = \begin{cases} 1 & (t \geq R), \\ 0 & (t < R). \end{cases}$$

For any $f \in \mathrm{C}(\mathbb{R}^k)$ of rapid decay (i.e., $f(\boldsymbol{w})$ decays rapidly for $\|w\| \to \infty$), the function

$$F_R(\tau;\boldsymbol{\xi}) = \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}(2,\mathbb{Z})} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f\big((\boldsymbol{y}_\gamma+\boldsymbol{m})v_\gamma^{1/2}\big)\, v_\gamma^\beta\, \chi_R(v_\gamma), \qquad R>1,$$

is invariant under the action of Γ. If τ lies in the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$, given by $\mathscr{F}_{\mathrm{SL}(2,\mathbb{Z})} = \{\tau \in \mathfrak{H} : u \in [-1/2, 1/2), |\tau| > 1\}$, then $F_R(\tau;\boldsymbol{\xi})$ has the representation

$$F_R(\tau;\boldsymbol{\xi}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^k} \big\{ f\big((\boldsymbol{y}+\boldsymbol{m})v^{1/2}\big) + f\big((-\boldsymbol{y}+\boldsymbol{m})v^{1/2}\big)\big\} v^\beta \chi_R(v).$$

The remaining sum over $\boldsymbol{m}$ is rapidly converging since f is of rapid decay.

6.2

The L^1-norm of F_R over $\Gamma\backslash G$ is, for $f \geq 0$,

$$\mu(F_R) = \int_{\Gamma\backslash G} F_R(\tau;\boldsymbol{\xi})\, d\mu(\tau,\phi;\boldsymbol{\xi})$$

with Haar measure

$$d\mu(\tau,\phi;\boldsymbol{\xi}) = \frac{du\, dv\, d\phi\, dx\, dy}{v^2}.$$

We therefore have

$$\mu(F_R) = 2\pi \int_{\mathbb{R}^k} f(\boldsymbol{w})\, dw \int_R^\infty v^{\beta-k/2-2}\, dv = 2\pi \frac{R^{-(k/2+1-\beta)}}{k/2+1-\beta} \int_{\mathbb{R}^k} f(\boldsymbol{w})\, dw$$

for $\beta < k/2+1$, and $\mu(F_R) = \infty$ otherwise. In the following we are especially interested in $\beta = k/2$, for which

$$\mu(F_R) = 2\pi R^{-1} \int_{\mathbb{R}^k} f(\boldsymbol{w})\, dw.$$

6.3

As in [12, Sec. 6.4], we may write the sum in $F_R(\tau;\boldsymbol{\xi})$ explicitly as

$$\begin{aligned}
&F_R(\tau;\boldsymbol{\xi})\\
&\quad= \sum_{\boldsymbol{m}\in\mathbb{Z}^k} \big\{ f\big((\boldsymbol{y}+\boldsymbol{m})v^{1/2}\big) + f\big((-\boldsymbol{y}+\boldsymbol{m})v^{1/2}\big)\big\} v^\beta \chi_R(v)\\
&\qquad+ \sum_{\boldsymbol{m}\in\mathbb{Z}^k} \Big\{ f\Big((\boldsymbol{x}+\boldsymbol{m})\frac{v^{1/2}}{|\tau|}\Big) + f\Big((-\boldsymbol{x}+\boldsymbol{m})\frac{v^{1/2}}{|\tau|}\Big)\Big\} \frac{v^\beta}{|\tau|^{2\beta}} \chi_R\Big(\frac{v}{|\tau|^2}\Big)\\
&\qquad+ \sum_{\substack{(c,d)\in\mathbb{Z}^2\\ \gcd(c,d)=1\\ c,d\neq 0}} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f\Big((c\boldsymbol{x}+d\boldsymbol{y}+\boldsymbol{m})\frac{v^{1/2}}{|c\tau+d|}\Big) \frac{v^\beta}{|c\tau+d|^{2\beta}} \chi_R\Big(\frac{v}{|c\tau+d|^2}\Big).
\end{aligned}$$

In what follows we restrict our attention to the case where $\beta = k/2$ and $\boldsymbol{\xi} = \binom{\mathbf{0}}{\boldsymbol{y}}$.

PROPOSITION 6.4

Let $\boldsymbol{y}$ be Diophantine of type κ. Then, for any ϵ, ϵ' with $0 < \epsilon < 1$ and $0 < \epsilon' < 1/(\kappa - 1)$,

$$\limsup_{v \to 0} v^{k/2-1} \int_{|u|>v^{1-\epsilon}} F_R\left(u + \mathrm{i}v; \begin{pmatrix} \mathbf{0} \\ \boldsymbol{y} \end{pmatrix}\right) h(v^{k/2-1}u)\, du \\ \ll_{\epsilon,\epsilon'} R^{-(1/(\kappa-1)-k+2)/2} + R^{-\epsilon'/2}.$$

Note that the above expression vanishes, for $R \to \infty$, when $\kappa < (k-1)/(k-2)$. The second term is obviously relevant only in dimension $k = 2$ since for $k > 2$ we may choose ϵ' in such a way that $1/(\kappa - 1) < \epsilon' + k - 2$.

The key ingredient in the proof is the following lemma.

LEMMA 6.5

Let $\boldsymbol{\alpha}$ be Diophantine of type κ, and let $f \in \mathrm{C}(\mathbb{R}^k)$ of rapid decay. Then, for any fixed $A > 1$ and $\epsilon > 0$ with $\epsilon < 1/(\kappa - 1)$,

$$\sum_{d=1}^{D} \sum_{\boldsymbol{m} \in \mathbb{Z}^k} f\big(T(d\boldsymbol{\alpha} + \boldsymbol{m})\big) \ll \begin{cases} T^{-A} & (D \le T^{\epsilon}), \\ 1 & (T^{\epsilon} \le D \le T^{1/(\kappa-1)}), \\ D\, T^{-1/(\kappa-1)} & (D \ge T^{1/(\kappa-1)}), \end{cases}$$

uniformly for all $D, T > 1$.

Proof

Let us divide the sum over d into blocks of the form

$$\sum_{0 \le d \le T^{1/(\kappa-1)}} \sum_{\boldsymbol{m} \in \mathbb{Z}^k} f\big(T((b + d)\boldsymbol{\alpha} + \boldsymbol{m})\big).$$

The number of such blocks is $\ll DT^{-1/(\kappa-1)} + 1$. Since $\boldsymbol{\alpha}$ is of type κ, there is a constant C such that for all $0 < |q| \le T^{1/(\kappa-1)}$, we have

$$\frac{C}{|q|T} \le \frac{C}{|q|^{\kappa}} \le \max_j \left| \alpha_j - \frac{m_j}{q} \right|,$$

and thus

$$\max_j |q\alpha_j - m_j| \ge \frac{C}{T}.$$

For b fixed, the minimal distance between the points $(b+d)\boldsymbol{\alpha}+\boldsymbol{m}$ $(0 \le d \le T^{1/(\kappa-1)}$, $\boldsymbol{m} \in \mathbb{Z}^k)$ is bounded from below by

$$\min_{0<|q|\le T^{1/(\kappa-1)},\boldsymbol{m}\in\mathbb{Z}^k} \|q\boldsymbol{\alpha}+\boldsymbol{m}\| \ge \min_{0<|q|\le T^{1/(\kappa-1)},\boldsymbol{m}\in\mathbb{Z}^k} \max_j |q\alpha_j - m_j| \ge \frac{C}{T}.$$

Hence any rectangular box with sides $\ll 1/T$ contains at most a bounded number of points. Because f is rapidly decreasing, we therefore find

$$\sum_{0\le d\le T^{\frac{1}{\kappa-1}}} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f\big(T((b+d)\boldsymbol{\alpha}+\boldsymbol{m})\big) \ll 1,$$

independently of b. This explains the second and third bound. The first bound is obtained from

$$\|d\boldsymbol{\alpha}+\boldsymbol{m}\| \ge \max_j |d\alpha_j - m_j| \ge \frac{C}{d^{\kappa-1}} \ge \frac{C}{D^{\kappa-1}},$$

which holds for all $d = 1, \dots, D$. Since f is rapidly decreasing, we have

$$\sum_{d=1}^{D} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f\big(T(d\boldsymbol{\alpha}+\boldsymbol{m})\big) \ll D\Big(\frac{D^{\kappa-1}}{T}\Big)^B$$

for any $B > 1$. □

6.6. Proof of Proposition 6.4

Let us assume, without loss of generality, that f is positive and even, that is, that $f \ge 0$, $f(-\boldsymbol{w}) = f(\boldsymbol{w})$.

It follows from the expansion in Section 6.3 that for $v < 1$ the first term involving $\chi_R(v)$ vanishes, and hence we are left with

$$\begin{aligned} F_R\left(\tau; \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{y}\end{pmatrix}\right) &= 2\sum_{\boldsymbol{m}\in\mathbb{Z}^k} f\Big(\boldsymbol{m}\frac{v^{1/2}}{|\tau|}\Big)\frac{v^{k/2}}{|\tau|^k}\chi_R\Big(\frac{v}{|\tau|^2}\Big) \\ &\quad + 2\sum_{\substack{(c,d)\in\mathbb{Z}^2\\ \gcd(c,d)=1\\ c>0,d\neq 0}} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f\Big((d\boldsymbol{y}+\boldsymbol{m})\frac{v^{1/2}}{|c\tau+d|}\Big)\frac{v^{k/2}}{|c\tau+d|^k}\chi_R\Big(\frac{v}{|c\tau+d|^2}\Big). \end{aligned}$$

6.6.1
With regard to the first term in the above expansion, a change of variable $u = vt$ yields

$$\begin{aligned}
&v^{k/2-1}\int_{|u|>v^{1-\epsilon}} 2\sum_{\boldsymbol{m}\in\mathbb{Z}^k} f\Big(\boldsymbol{m}\frac{v^{1/2}}{|\tau|}\Big)\frac{v^{k/2}}{|\tau|^k}\chi_R\Big(\frac{v}{|\tau|^2}\Big)h(v^{k/2-1}u)\,du \\
&\quad = 2\sum_{\boldsymbol{m}\in\mathbb{Z}^k}\int_{|t|>v^{-\epsilon}} f\Big(\frac{\boldsymbol{m}}{v^{1/2}(t^2+1)^{1/2}}\Big)\frac{1}{(t^2+1)^{k/2}}\chi_R\Big(\frac{1}{v(t^2+1)}\Big)h(v^{k/2}t)\,dt \\
&\quad \sim 2\,f(0)\,h(0)\int_{|t|>v^{-\epsilon}}\frac{dt}{(t^2+1)^{k/2}} \to 0
\end{aligned}$$

as $v \to 0$.

6.6.2
An upper bound for the remaining terms is obtained by dropping the condition $|u| > v^{1-\epsilon}$ in the integral. We then need to estimate

$$S(v) = \sum_{\substack{(c,d)\in\mathbb{Z}^2\\ \gcd(c,d)=1\\ c>0,d\neq 0}}\ \sum_{\boldsymbol{m}\in\mathbb{Z}^k} J(v,c,d,\boldsymbol{m})$$

with

$$\begin{aligned}
&J(v,c,d,\boldsymbol{m}) \\
&\quad = v^{k/2-1}\int_{\mathbb{R}} f\Big((d\boldsymbol{y}+\boldsymbol{m})\frac{v^{1/2}}{|c\tau+d|}\Big)\frac{v^{k/2}}{|c\tau+d|^k}\chi_R\Big(\frac{v}{|c\tau+d|^2}\Big)\ h(v^{k/2-1}u)\,du.
\end{aligned}$$

Replacing u by $t = v^{-1}(u + d/c)$ gives

$$\begin{aligned}
&\frac{1}{c^k}\int_{\mathbb{R}} f\Big((d\boldsymbol{y}+\boldsymbol{m})\frac{1}{\sqrt{c^2 v(t^2+1)}}\Big)\frac{1}{(t^2+1)^{k/2}} \\
&\qquad\qquad \times\chi_R\Big(\frac{1}{c^2 v(t^2+1)}\Big)\ h\Big(v^{k/2-1}\Big(vt-\frac{d}{c}\Big)\Big)\,dt.
\end{aligned}$$

The range of integration is bounded by

$$R < \frac{1}{c^2 v(t^2+1)}, \qquad \text{that is,} \qquad |t| \ll \frac{1}{c\sqrt{vR}}.$$

Therefore $|vt| \ll v^{1/2}c^{-1}R^{-1/2}$ is uniformly close to zero, and hence, because of the compact support of h, we find $|d| \ll cv^{-(k/2-1)}$. So

$$S(v) \ll \sum_{c=1}^{\infty}\ \sum_{0<|d|\ll cv^{-(k/2-1)}}\ \sum_{\boldsymbol{m}\in\mathbb{Z}^k} K(v,c,d,\boldsymbol{m})$$

with

$$K(v,c,d,\boldsymbol{m}) = \frac{1}{c^k}\int_{\mathbb{R}} f\Big((d\boldsymbol{y}+\boldsymbol{m})\frac{1}{\sqrt{c^2v(t^2+1)}}\Big)\frac{1}{(t^2+1)^{k/2}}\chi_R\Big(\frac{1}{c^2v(t^2+1)}\Big)\,dt.$$

6.6.3

To apply Lemma 6.5 with $D = cv^{-(k/2-1)}$, $T = (c^2v(t^2+1))^{-1/2} > \sqrt{R} > 1$, split the range of integration into the ranges

$$\begin{aligned}
&(1):\quad cv^{-(k/2-1)} \le \big(c^2v(t^2+1)\big)^{-\epsilon/2},\\
&(2):\quad \big(c^2v(t^2+1)\big)^{-\epsilon/2} \le cv^{-(k/2-1)} \le \big(c^2v(t^2+1)\big)^{-\delta/2},\\
&(3):\quad cv^{-(k/2-1)} \ge \big(c^2v(t^2+1)\big)^{-\delta/2},
\end{aligned}$$

with $\delta = 1/(\kappa-1)$. Denote the corresponding integrals by $K_1(v,c,d,\boldsymbol{m})$, $K_2(v,c,d,\boldsymbol{m})$, and $K_3(v,c,d,\boldsymbol{m})$, respectively.

6.6.4

Because $R^{-1/2} \ge T^{-1}$, we find in the first range, $D \le T^{\epsilon}$, that

$$\begin{aligned}
\sum_{c>0}\sum_{d\ll cv^{-(k/2-1)}}\sum_{\boldsymbol{m}\in\mathbb{Z}^k} K_1(v,c,d,\boldsymbol{m})
&\ll R^{-A/2}\sum_{c>0}\frac{1}{c^k}\int_{(1)}\frac{1}{(t^2+1)^{k/2}}\chi_R\Big(\frac{1}{c^2v(t^2+1)}\Big)\,dt\\
&\ll R^{-A/2}\sum_{c>0}\frac{1}{c^k}\int_{\mathbb{R}}\frac{1}{(t^2+1)^{k/2}}\,dt\\
&\ll R^{-A/2}.
\end{aligned}$$

6.6.5

For an upper bound, the second range, $T^{\epsilon} \le D \le T^{\delta}$, may be extended to $T^{\epsilon} \le D$, that is,

$$c^{1+\epsilon}(t^2+1)^{\epsilon/2} \ge v^{k/2-1-\epsilon/2}.$$

We therefore have

$$\begin{aligned}\sum_{c>0} \sum_{d\ll cv^{-(k/2-1)}} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} & K_2(v,c,d,\boldsymbol{m}) \\ & \ll \sum_{c>0} \frac{1}{c^k} \int_{(2)} \frac{dt}{(t^2+1)^{k/2}} \\ & \ll \sum_{c>0} \frac{1}{c^k} \big\{c^{1+\epsilon} v^{-(k/2-1-\epsilon/2)}\big\}^{(k/2-1)2/\epsilon} \int_{\mathbb{R}} \frac{dt}{t^2+1} \\ & \ll v^A \sum_{c>0} c^{-B}\end{aligned}$$

with

$$A = -\Big(\frac{k}{2} - 1 - \frac{\epsilon}{2}\Big)\Big(\frac{k}{2} - 1\Big)\frac{2}{\epsilon}$$

and

$$B = -\Big(\frac{k}{2} - 1 - \epsilon\Big)\frac{2}{\epsilon} = 1 - \Big(\frac{k}{2} - 1 - \frac{\epsilon}{2}\Big)\frac{2}{\epsilon}.$$

If we choose ϵ in such a way that $k-2 < \epsilon < \delta = 1/(\kappa-1)$, we find that for $k>2$ we have $A>0$ and $B>1$. Hence

$$\sum_{c>0} \sum_{d\ll cv^{-(k/2-1)}} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} K_2(v,c,d,\boldsymbol{m}) \to 0$$

for small v. In the case where $k=2$, we exploit the inclusion $R^{\epsilon/2} < T^{\epsilon} \le D \ll c$, which yields

$$\sum_{c>0} \sum_{d\ll c} \sum_{\boldsymbol{m}\in\mathbb{Z}^2} K_2(v,c,d,\boldsymbol{m}) \ll \sum_{c>R^{\epsilon/2}} c^{-2} \int \frac{dt}{t^2+1} \ll R^{-\epsilon/2}$$

(cf. [12]).

6.6.6

In the third range, we have for v sufficiently small,

$$\begin{aligned}
&\sum_{c>0} \sum_{d \ll cv^{-(k/2-1)}} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} K_3(v,c,d,\boldsymbol{m}) \\
&\quad \ll \sum_{c>0} \frac{1}{c^k} cv^{-(k/2-1)} \int_{(3)} c^\delta v^{\delta/2}(t^2+1)^{(\delta-k)/2} \chi_R\Big(\frac{1}{c^2 v(t^2+1)}\Big)\,dt \\
&\quad = v^{(\delta-k)/2+1} \sum_{c>0} c^{\delta-k+1} \int_{(3)} (t^2+1)^{(\delta-k)/2} \chi_R\Big(\frac{1}{c^2 v(t^2+1)}\Big)\,dt \\
&\quad \ll v^{(\delta-k)/2+1} \int_{\mathbb{R}} \Big\{ \sum_{c=1}^{\infty} c^{\delta-k+1} \chi_R\Big(\frac{1}{c^2 v(t^2+1)}\Big)\Big\} (t^2+1)^{(\delta-k)/2}\,dt \\
&\quad < v^{(\delta-k)/2+1} \int_{\mathbb{R}} \Big\{ \int_0^\infty x^{\delta-k+1} \chi_R\Big(\frac{1}{x^2 v(t^2+1)}\Big)\,dx\Big\} (t^2+1)^{(\delta-k)/2}\,dt \\
&\quad = \int_{\mathbb{R}} \Big\{ \int_0^\infty x^{\delta-k+1} \chi_R\Big(\frac{1}{x^2}\Big)\,dx\Big\} (t^2+1)^{-1}\,dt \\
&\quad = \int_{\mathbb{R}} \Big\{ \frac{x^{\delta-k+2}}{\delta-k+2}\Big\}_0^{R^{-1/2}} (t^2+1)^{-1}\,dt \\
&\quad = \pi \frac{R^{-(\delta-k+2)/2}}{\delta-k+2}.
\end{aligned}$$

The proof of Proposition 6.4 is complete. □

6.7

Let us define the characteristic function on $\Gamma\backslash G^k$:

$$X_R(\tau) = \sum_{\gamma\in\{\Gamma_\infty\cup(-1)\Gamma_\infty\}\backslash \mathrm{SL}(2,\mathbb{Z})} \chi_R(v_\gamma),$$

where χ_R is the characteristic function of $[R,\infty)$. Proposition 6.4 allows us now to extend the equidistribution theorem, Theorem 5.1, to unbounded functions that are dominated by F_R; that is, for some fixed constant $L > 1$ we have

$$|F(\tau,\phi;\boldsymbol{\xi})| X_R(\tau) \le L + F_R(\tau;\boldsymbol{\xi})$$

for all sufficiently large $R > 1$, uniformly for all $(\tau,\phi;\boldsymbol{\xi}) \in G^k$.

THEOREM 6.8

Let Γ be a subgroup of $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$ of finite index. Let h be a continuous function $\mathbb{R} \to \mathbb{R}_+$ with compact support. Suppose the continuous function $F \ge 0$ is dominated by F_R. Fix some $\boldsymbol{y} \in \mathbb{T}^k$ such that the components of the vector $(\boldsymbol{y}, 1) \in \mathbb{R}^{k+1}$ are *linearly independent over* $\mathbb{Q}$. *Then, for any* ϵ *with* $0 < \epsilon < 1$,

$$\liminf_{v\to 0} v^{k/2-1} \int_{|u|>v^{1-\epsilon}} F\left(u+\mathrm{i}v, 0; \begin{pmatrix}\mathbf{0}\\ \mathbf{y}\end{pmatrix}\right) h(v^{k/2-1}u)\, du \geq \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} F\, d\mu \int h.$$

Assume, furthermore, that $\mathbf{y}$ *is Diophantine of type* $\kappa < (k-1)/(k-2)$. *Then, for any* $\epsilon > 0$,

$$\limsup_{v\to 0} v^{k/2-1} \int_{|u|>v^{1-\epsilon}} F\left(u+\mathrm{i}v, 0; \begin{pmatrix}\mathbf{0}\\ \mathbf{y}\end{pmatrix}\right) h(v^{k/2-1}u)\, du \leq \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} F\, d\mu \int h.$$

Proof

The theorem follows from Theorem 5.1 and Proposition 6.4 in the same manner as [12, Th. 7.3]. □

6.9

The subgroup $\Gamma = \Gamma^k$ is a subgroup of finite index in $\mathrm{SL}(2,\mathbb{Z}) \ltimes ((1/2)\mathbb{Z})^{2k}$ rather than $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$ (see Lem. 4.10). We therefore need to rephrase Theorem 6.8 slightly. Define the dominating function $\hat{F}_R$ on $\Gamma\backslash G^k$ by $\hat{F}_R(\tau;\boldsymbol{\xi}) = F_R(\tau; 2\boldsymbol{\xi})$, with F_R as in Section 6.7.

COROLLARY 6.10

Let Γ *be a subgroup of* $\mathrm{SL}(2,\mathbb{Z}) \ltimes ((1/2)\mathbb{Z})^{2k}$ *of finite index, let* h, $\mathbf{y}$ *be as in Theorem 6.8, and let* $F : \Gamma\backslash G^k \to \mathbb{C}$ *be a continuous function that is dominated by* $\hat{F}_R$. *If* $\mathbf{y}$ *is Diophantine of type* $\kappa < (k-1)/(k-2)$, *then, for any* ϵ *with* $0 < \epsilon < 1$,

$$\lim_{v\to 0} v^{k/2-1} \int_{|u|>v^{1-\epsilon}} F\left(u+\mathrm{i}v, 0; \begin{pmatrix}\mathbf{0}\\ \mathbf{y}\end{pmatrix}\right) h(v^{k/2-1}u)\, du = \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} F\, d\mu \int h.$$

Proof

The proof is analogous to that of [12, Cor. 7.6]. □

7. The main theorem

THEOREM 7.1 (Main theorem)

Suppose $f(\mathbf{w}) = \psi_1(\|\mathbf{w}\|^2)$ *and* $g(\mathbf{w}) = \psi_2(\|\mathbf{w}\|^2)$, *with* $\psi_1, \psi_2 \in \mathscr{S}(\mathbb{R}_+)$ *real val-*

ued. Let h be a continuous function $\mathbb{R} \to \mathbb{C}$ with compact support. Assume that the components of $(\boldsymbol{y}, 1) \in \mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$ and that $\boldsymbol{y}$ is Diophantine of type $\kappa < (k-1)/(k-2)$. Then

$$\begin{aligned}\lim_{v\to 0} v^{k/2-1} \int_{\mathbb{R}} \Theta_f \left(u + \mathrm{i}v, 0; \begin{pmatrix}\mathbf{0}\\ \boldsymbol{y}\end{pmatrix}\right) \overline{\Theta_g \left(u + \mathrm{i}v, 0; \begin{pmatrix}\mathbf{0}\\ \boldsymbol{y}\end{pmatrix}\right)}\, h(v^{k/2-1}u)\, du \\ = \frac{k^2}{2} B_k^2 h(0) \int_0^\infty \psi_1(r)\psi_2(r) r^{k-2}\, dr \\ + \frac{k}{2} B_k \int_{\mathbb{R}} h(u)\, du \int_0^\infty \psi_1(r)\psi_2(r) r^{k/2-1}\, dr,\end{aligned}$$

where B_k is the volume of the k-dimensional unit ball.

We need the following two lemmas.

LEMMA 7.2
We have

$$\frac{1}{\mu(\Gamma^k\backslash G^k)} \int_{\Gamma^k\backslash G^k} \Theta_f(\tau, \phi; \boldsymbol{\xi}) \overline{\Theta_g(\tau, \phi; \boldsymbol{\xi})}\, d\mu = \int_{\mathbb{R}^k} f(\boldsymbol{w}) \overline{g(\boldsymbol{w})}\, dw.$$

Note that if $f(\boldsymbol{w}) = \psi_1(\|\boldsymbol{w}\|^2)$ and $g(\boldsymbol{w}) = \psi_2(\|\boldsymbol{w}\|^2)$, we have

$$\int f(\boldsymbol{w}) \overline{g(\boldsymbol{w})}\, dw = \frac{k}{2} B_k \int_0^\infty \psi_1(r)\psi_2(r) r^{k/2-1}\, dr.$$

Proof
A short calculation shows that

$$\int_{\mathbb{T}^{2k}} \Theta_f(\tau, \phi; \boldsymbol{\xi}) \overline{\Theta_g(\tau, \phi; \boldsymbol{\xi})}\, d\xi = \int f_\phi(\boldsymbol{w}) \overline{g_\phi(\boldsymbol{w})}\, dw.$$

Since $f_\phi = U^\phi f$ with U^ϕ unitary, we have

$$\int f_\phi(\boldsymbol{w}) \overline{g_\phi(\boldsymbol{w})}\, dw = \int f(\boldsymbol{w}) \overline{g(\boldsymbol{w})}\, dw. \qquad \square$$

LEMMA 7.3
Suppose $f(\boldsymbol{w}) = \psi_1(\|\boldsymbol{w}\|^2)$ *and* $g(\boldsymbol{w}) = \psi_2(\|\boldsymbol{w}\|^2)$. *For any* $1/2 < \gamma < 1$, *we have*

$$\lim_{v\to 0} v^{k/2-1} \int_{|u|<v^\gamma} \Theta_f\left(u+\mathrm{i}v, 0; \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{y}\end{pmatrix}\right) \overline{\Theta_g\left(u+\mathrm{i}v, 0; \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{y}\end{pmatrix}\right)} h(v^{k/2-1}u)\, du$$
$$= \frac{k^2}{2} B_k^2 h(0) \int_0^\infty \psi_1(r)\psi_2(r)\, r^{k-2}\, dr.$$

Proof
From Proposition 4.8 we know that

$$\Theta_f\left(-\frac{1}{\tau}, \arg\tau; \begin{pmatrix}-\boldsymbol{y}\\ \boldsymbol{0}\end{pmatrix}\right) \overline{\Theta_g\left(-\frac{1}{\tau}, \arg\tau; \begin{pmatrix}-\boldsymbol{y}\\ \boldsymbol{0}\end{pmatrix}\right)}$$
$$= \frac{v^{k/2}}{|\tau|^k} f_{\arg\tau}(\boldsymbol{0})\overline{g_{\arg\tau}(\boldsymbol{0})} + O_R\left(\left(\frac{v}{|\tau|^2}\right)^{-R}\right)$$

holds uniformly for $|u| < v^{1/2} < 1$. The remainder vanishes for $|u| < v^\gamma < 1$. Now

$$\begin{aligned} & f_{\arg\tau}(\boldsymbol{0})\overline{g_{\arg\tau}(\boldsymbol{0})} \\ &\quad = \frac{|\tau|^k}{v^k}\left\{\int e\Big(\frac{1}{2}\|\boldsymbol{w}\|^2\frac{u}{v}\Big)\, f(\boldsymbol{w})\, dw\right\}\overline{\left\{\int e\Big(\frac{1}{2}\|\boldsymbol{w}\|^2\frac{u}{v}\Big)\, g(\boldsymbol{w})\, dw\right\}} \\ &\quad = \frac{|\tau|^k}{v^k}\,\frac{k^2}{4} B_k^2 \int_0^\infty e\Big(\frac{(r_1-r_2)u}{2v}\Big)\psi_1(r_1)\psi_2(r_2)\, r_1^{k/2-1}\, dr_1\, r_2^{k/2-1}\, dr_2 \end{aligned}$$

(replace $\boldsymbol{w}$ by polar coordinates), and so, as $v\to\infty$,

$$\begin{aligned} & \int_{|u|<v^\gamma} v^{k/2-1}\Theta_f\left(u+\mathrm{i}v, 0; \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{y}\end{pmatrix}\right) \overline{\Theta_g\left(u+\mathrm{i}v, 0; \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{y}\end{pmatrix}\right)} h(v^{k/2-1}u)\, du \\ &\quad \sim v^{-1}\frac{k^2}{4} B_k^2 \int_{|u|<v^\gamma}\int_0^\infty e\Big(\frac{(r_1-r_2)u}{2v}\Big)\psi_1(r_1)\psi_2(r_2) \\ &\qquad \times r_1^{k/2-1}\, dr_1\, r_2^{k/2-1}\, dr_2 h(v^{k/2-1}u)\, du \\ &\quad \sim \frac{k^2}{2} B_k^2 h(0) \int_{2|u|<v^{\gamma-1}}\int_0^\infty e\big((r_1-r_2)u\big) \\ &\qquad \times \psi_1(r_1)\psi_2(r_2)\, r_1^{k/2-1}\, dr_1\, r_2^{k/2-1}\, dr_2\, du \\ &\quad \sim \frac{k^2}{2} B_k^2 h(0) \int_{\mathbb{R}}\int_0^\infty e\big((r_1-r_2)u\big)\psi_1(r_1)\psi_2(r_2) \\ &\qquad \times r_1^{k/2-1}\, dr_1\, r_2^{k/2-1}\, dr_2\, du \\ &\quad = \frac{k^2}{2} B_k^2 h(0) \int_0^\infty \psi_1(r)\psi_2(r)\, r^{k-2}\, dr \end{aligned}$$

by Parseval's equality. □

Proof of the main theorem
We may assume, without loss of generality, that in Theorem 7.1 h is positive. Split the integration on the left-hand side of Theorem 7.1 into

$$\int_{\mathbb{R}} = \int_{|u|<v^{1-\epsilon}} + \int_{|u|>v^{1-\epsilon}}$$

for some small $\epsilon > 0$. The first integral gives, by virtue of Lemma 7.3, the contribution

$$\frac{k^2}{2} B_k^2 h(0) \int_0^\infty \psi_1(r)\psi_2(r)\, r^{k-2}\, dr.$$

Corollary 6.10, together with Lemma 7.2, yields the second term on the right-hand side of Theorem 7.1 (cf. [12, Sec. 8.4] for more details). □

Proof of Theorem 2.5
By construction, we have

$$\begin{aligned} & R_2(\psi_1, \psi_2, h, \lambda) \\ &= \frac{1}{B_k} v^{k/2-1} \int_{\mathbb{R}} \Theta_f \left(u + \mathrm{i}\frac{1}{\lambda}, 0; \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\alpha} \end{pmatrix}\right) \overline{\Theta_g \left(u + \mathrm{i}\frac{1}{\lambda}, 0; \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\alpha} \end{pmatrix}\right)}\, h(v^{k/2-1}u)\, du \end{aligned}$$

with $v = \lambda^{-1}$. Recall that $2h(0) = \int \hat{h}(s)\, ds$ by Fourier inversion; and thus we have finally $\int h(u)\, du = \hat{h}(0)$. □

8. Counter examples

We assume throughout this section that $k > 2$. The case where $k = 2$ is studied in [12, Sec. 9].

8.1
Suppose that α_{k-1}, α_k are both rational, and suppose that $(\alpha_1, \dots, \alpha_{k-2})$ is a badly approximable $(k-2)$-tuple. In this case, we find a constant C such that

$$\max_{1\le j\le k} \left|\alpha_j - \frac{m_j}{q}\right| \ge \max_{1\le j\le k-2} \left|\alpha_j - \frac{m_j}{q}\right| > \frac{C}{q^{1+1/(k-2)}}$$

for all $m_1, \dots, m_j, q \in \mathbb{Z}$, $q > 0$, and so $\boldsymbol{\alpha}$ is of type $\kappa = (k-1)/(k-2)$. On the other hand, we have

$$\begin{aligned}
&\#\big\{(\boldsymbol{m},\boldsymbol{n}) \in \mathbb{Z}^k \times \mathbb{Z}^k : \ \boldsymbol{m} \neq \boldsymbol{n}, \\
&\qquad\qquad \|\boldsymbol{m}-\boldsymbol{\alpha}\|^k \le X, \|\boldsymbol{n}-\boldsymbol{\alpha}\|^k \le X,\ \|\boldsymbol{m}-\boldsymbol{\alpha}\|^k = \|\boldsymbol{n}-\boldsymbol{\alpha}\|^k\big\} \\
\ge\ &\#\big\{(\boldsymbol{m},\boldsymbol{n}) \in \mathbb{Z}^k \times \mathbb{Z}^k : \ \boldsymbol{m} \neq \boldsymbol{n},\ (m_1,\dots,m_{k-2}) = (n_1,\dots,n_{k-2}), \\
&\qquad\qquad \|\boldsymbol{m}-\boldsymbol{\alpha}\|^k \le X, \|\boldsymbol{n}-\boldsymbol{\alpha}\|^k \le X,\ \|\boldsymbol{m}-\boldsymbol{\alpha}\|^2 = \|\boldsymbol{n}-\boldsymbol{\alpha}\|^2\big\}.
\end{aligned}$$

This is easily seen to be bounded from below by

$$\begin{aligned}
\gg X^{(k-2)/k} \#\big\{&(m_{k-1}, m_k, n_{k-1}, n_k) \in \mathbb{Z}^4 : \\
&|m_{k-1}|, |m_k|, |n_{k-1}|, |n_k| \ll X^{1/k},\ (m_{k-1}, m_k) \neq (n_{k-1}, n_k), \\
&(m_{k-1}-\alpha_{k-1})^2 + (m_k-\alpha_k)^2 = (n_{k-1}-\alpha_{k-1})^2 + (n_k-\alpha_k)^2\big\} \\
&\qquad\qquad \sim X^{(k-2)/k} \times \tilde{c}_{\boldsymbol{\alpha}} X^{2/k} \log X,
\end{aligned}$$

as $X \to \infty$, for some constant $\tilde{c}_{\boldsymbol{\alpha}} > 0$ (cf. [12, Sec. 9]). We conclude that, for X large enough,

$$\begin{aligned}
\frac{1}{X}\#\big\{(\boldsymbol{m},\boldsymbol{n}) \in \mathbb{Z}^k \times \mathbb{Z}^k :\ \boldsymbol{m} \neq \boldsymbol{n},\ \|\boldsymbol{m}-\boldsymbol{\alpha}\|^k \le X, \|\boldsymbol{n}-\boldsymbol{\alpha}\|^k \le X, \\
\|\boldsymbol{m}-\boldsymbol{\alpha}\|^k = \|\boldsymbol{n}-\boldsymbol{\alpha}\|^k\big\} \ge c_{\boldsymbol{\alpha}} \log X
\end{aligned}$$

for some constant $c_{\boldsymbol{\alpha}} > 0$.

8.2
By a similar argument, one has for $\boldsymbol{\alpha} \in \mathbb{Q}^k$,

$$\begin{aligned}
\frac{1}{X}\#\big\{(\boldsymbol{m},\boldsymbol{n}) \in \mathbb{Z}^k \times \mathbb{Z}^k :\ \boldsymbol{m} \neq \boldsymbol{n},\ \|\boldsymbol{m}-\boldsymbol{\alpha}\|^k \le X, \|\boldsymbol{n}-\boldsymbol{\alpha}\|^k \le X, \\
\|\boldsymbol{m}-\boldsymbol{\alpha}\|^k = \|\boldsymbol{n}-\boldsymbol{\alpha}\|^k\big\} \sim c_{\boldsymbol{\alpha}} X^{(k-2)/k}
\end{aligned}$$

for $X \to \infty$. This can be derived, for example, in the case where $\boldsymbol{\alpha} = \boldsymbol{0}$, from the asymptotics

$$\int_0^1 \Theta_f\Big(u + \mathrm{i}\frac{1}{\lambda}, 0; \boldsymbol{0}\Big)\overline{\Theta_g\Big(u + \mathrm{i}\frac{1}{\lambda}, 0; \boldsymbol{0}\Big)}\, du \sim b\lambda^{k/2-1}$$

(cf., e.g., [9, Th. 6.1]).

Proof of Theorem 1.7
Let $\mathscr{B}$ be a countable dense set of badly approximable $(k-2)$-tuples. Enumerate the

quadratic forms $\|\boldsymbol{x}-\boldsymbol{\alpha}_j\|^2$ with $\boldsymbol{\alpha}_j \in \mathscr{B} \times \mathbb{Q}^2$ as $P_1, P_2, P_3, \ldots$. Because of the bound derived in Section 8.1, given any $X > 1$, there exists an $M_j > X$ such that

$$R_2^{\boldsymbol{\alpha}_j}[0,0](M_j) \geq \frac{\log M_j}{\log\log\log M_j}.$$

We find a small $\epsilon_j = \epsilon_j(M_j) > 0$ such that

$$R_2^{\boldsymbol{\alpha}}[-a,a](M_j) \geq R_2^{\boldsymbol{\alpha}_j}[0,0](M_j)$$

for all $\boldsymbol{\alpha} \in B_j$, where B_j is the open set of all $\boldsymbol{\alpha}$ with $\|\boldsymbol{\alpha}-\boldsymbol{\alpha}_j\| < \epsilon_j$. Individually, the sets B_j shrink to a point as $X \to \infty$, but the union

$$\bigcup_{j:M_j \geq X} B_j$$

is open and dense in $\mathbb{T}^k$. Therefore

$$B = \bigcap_{X=1}^{\infty} \bigcup_{j:M_j \geq X} B_j$$

is of second Baire category. So if $\boldsymbol{\alpha} \in B$, then, given any X, there exists some $M \geq X$, such that

$$R_2^{\boldsymbol{\alpha}}[-a,a](M) \geq \frac{\log M}{\log\log\log M}.$$

Note that the proof remains valid if $\log\log\log$ is replaced by any slowly increasing positive function $\nu \leq \log\log\log$ with $\nu(X) \to \infty$ $(X \to \infty)$.

Property (iii) follows from Theorem 1.6 by the same string of arguments used in [12, Sec. 9.3]. □

Proof of Theorem 1.8

The proof follows from the relation in Section 8.2. It is otherwise identical to the proof of [12, Th. 1.13]. □

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