Energy Level Statistics, Lattice Point Problems, and Almost Modular Functions

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Summary. One of the central aims in quantum chaos is to classify quantum systems according to universal statistical properties. It has been conjectured that the energy levels of generic integrable quantum systems have the same statistical properties as random numbers from a Poisson process (Berry & Tabor 1977), and chaotic quantum systems the same as eigenvalues of random matrices from suitably chosen ensembles (Bohigas, Giannoni & Schmit 1984). I review some recent developments concerning simple classes of integrable systems, where the study of eigenvalue correlations leads to subtle lattice point counting problems which, in some instances, can be solved by ergodic theoretic techniques. In a special example (the so-called "boxed oscillator") energy level statistics are related to the statistical distribution of the fractional parts of the sequence $n^2\alpha$. We will see that the error term of this distribution can be identified with an *almost modular function*, and that the fluctuations of the error term are governed by a general limit theorem for such functions.

1 Introduction

The classification of quantum systems according to universal statistical properties is one of the central objectives in quantum chaos. The topic is discussed in detail in Eugene Bogomolny's lectures [7] and I will here concentrate on a special class of quantum systems whose level statistics can be understood in terms of lattice point counting problems. Let us consider a Hamiltonian with discrete energy spectrum $\lambda_1 \leq \lambda_2 \leq \ldots \to \infty$. We assume that the number of levels (counted with multiplicity) grows asymptotically as

$$\#\{j : \lambda_j \leq \lambda\} \sim \overline{N}(\lambda) \quad (\lambda \to \infty) \tag{1.1}$$

where $\overline{N}(\lambda) = c\lambda^\gamma$ with constants $c > 0$, $\gamma \geq 1$. To investigate its statistical properties it is convenient to rescale the sequence by setting $X_j = \overline{N}(\lambda_j)$ which yields mean density $= 1$, i.e.,

$$\#\{j : X_j \leq X\} \sim X \quad (X \to \infty). \tag{1.2}$$

The central conjecture, put forward by Berry and Tabor in 1977 [1], is that if the Hamiltonian is classically integrable (and sufficiently "generic") then the X_j have the same local statistical properties as independent random variables from a Poisson process. This means that

$$\mathcal{N}(T, L) := \#\{j : T \leq X_j \leq T + L\}, \tag{1.3}$$

the number of X_j in a randomly shifted interval $[T, T+L]$ of fixed length L, is distributed according to the Poisson law $\frac{L^k}{k!}\,\mathrm{e}^{-L}$. More precisely, let $\rho : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ be a continuous probability density with compact support, and define the family of probability densities ρ_X with $X \in \mathbb{R}_{\geq 1}$ by $\rho_X(T) = X^{-1}\rho(TX^{-1})$. The assertion is now that $\mathcal{N}(T, L)$ has a Poisson limit distribution, if T is distributed according to ρ_X and $X \to \infty$. That is, for any bounded function $g : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ we have

$$\int_0^\infty g\big(\mathcal{N}(T, L)\big)\, \rho_X(T)\, dT \to \sum_{k=0}^\infty g(k) \frac{L^k}{k!}\, \mathrm{e}^{-L}\,. \tag{1.4}$$

This is in contrast to chaotic systems where the spectral statistics are expected to follow those of random matrix ensembles.

The central idea behind the Berry-Tabor conjecture is that the energy levels of an integrable Hamiltonian are in semiclassical approximation given by the EBK quantization

$$\lambda_j(\hbar) \sim H(\hbar(\boldsymbol{m} + \boldsymbol{\alpha})), \qquad \hbar \to 0, \tag{1.5}$$

where $H(\boldsymbol{I})$ is the classical Hamiltonian in the action variables; $\boldsymbol{m}$ runs over integer lattice points and $\boldsymbol{\alpha}$ is a fixed vector determined by topological data such as Maslov indices. One case where this approximation can be controlled sufficiently well to study spectral correlations is when H is the negative Laplacian $-\Delta$ on surfaces with integrable geodesic flow. For examples in the case of surfaces of revolution (with some technical assumptions) one has [10, 11]

$$\lambda_j = F(m_1, m_2 + \tfrac{1}{2}), \quad (m_1, m_2) \in \mathbb{Z}^2, \quad |m_1| \leq m_2, \tag{1.6}$$

where $F(\boldsymbol{x}) = F_2(\boldsymbol{x}) + F_0(\boldsymbol{x}) + O(\|\boldsymbol{x}\|^{-1})$, $\|\boldsymbol{x}\| \to \infty$, and F_2, F_0 are smooth homogeneous functions of degree 2 and 0, respectively. Note that in the case of the Laplacian the semiclassical limit $\hbar \to 0$ is equivalent to the high energy limit $j \to \infty$.

Sinai [42] and Major [17] proved the Poisson limit theorem (1.4) for generic F in a certain function space. A "generic" function has, however, level curves $F(\boldsymbol{x}) = 1$ which are not twice differentiable. Advances towards a proof of the Poisson conjecture for systems with analytic F, such as the Laplacian on surfaces with integrable geodesic flow, have been made only recently. Sarnak [38] showed that the pair correlation statistics are Poisson for the eigenvalues of tori with a generic flat metric (we shall see below that pair correlation or two-point statistics correspond to the variance of the distribution of $\mathcal{N}(T, L)$). The eigenvalues of a flat torus are given by positive definite binary quadratic forms $\alpha m^2 + \beta mn + \gamma n^2$ ($m, n \in \mathbb{Z}$), and "generic" refers to a choice of (α, β, γ) in a set of full Lebesgue measure. These results were extended by VanderKam to tori of arbitrary dimension [43] and also to higher-order correlation functions [44]. Eskin, Margulis and Mozes [12] strengthened considerably Sarnak's result by giving explicit diophantine conditions on (α, β, γ) under which the pair correlation statistics of two-dimensional flat tori is Poisson. It is interesting to note, however, that the fluctuations of the spectral form factor (the Fourier transform of the pair correlation density) are in this case not consistent with the Poisson model [18].

Berry and Tabor point out that there are many examples of integrable systems which violate their general conjecture, and that hence the Poisson distribution should only be expected for "generic" systems. One of the most interesting counter examples is the multi-dimensional harmonic oscillator whose eigenvalues are given by the values of the linear form $\boldsymbol{\omega} \cdot \boldsymbol{m}$ at lattice points $\boldsymbol{m} \in \mathbb{N}^k$; see Berry and Tabor's original work [1], and subsequent papers by Pandey, Bohigas, Giannoni and Ramaswamy [30, 31], Bleher [2, 3], Mazel and Sinai [29], Greenman [13, 14], and myself [21].

In the present paper we focus on two special classes of integrable systems. The first example is the k-dimensional standard torus $\mathbb{T}^k$ threaded by flux lines, where the question of energy level statistics corresponds to counting lattice points in thin spherical shells centered at $\boldsymbol{\alpha}$. It was first studied in connection with the Berry-Tabor conjecture by Cheng, Lebowitz and Major [8, 9]. In Sects. 2 and 3 I will review recent results on the pair correlation statistics [24, 25], which were announced in [22, 23].

The second example is the "boxed oscillator", i.e., a particle constrained by a box in x-direction and by a harmonic potential in the y-direction, so that

$H = -\partial_x^2 - \partial_y^2 + \omega^2 y^2$. In this case the eigenvalue correlations are closely related with the local statistics of the fractional parts of the sequence $n^2\alpha$, which were studied by Sinai [41], Pellegrinotti [32], Rudnick, Sarnak and Zaharescu [35, 36, 45], and Zelditch [46]. In Sects. 4 and 5 I will discuss joint work with Strömbergsson [28], which relates the pair correlation problem for $n^2\alpha$ to a natural equidistribution problem in hyperbolic geometry.

It is crucial in the Poisson limit theorem (1.4) that L is kept fixed. If L increases (sufficiently slowly) with T then the left-hand-side is expected to converge to a Gaussian distribution, see Bleher's review [5] for a detailed discussion. (In a recent paper [16], Hughes and Rudnick prove a central limit theorem for lattice points in annuli.) If, on the other hand, L grows sufficiently fast with T (e.g. $L = T$) the limiting distribution (provided it exists) is typically non-universal. In the case when the eigenvalues are given by values of positive definite binary quadratic forms (or more general functions homogeneous of degree two) the work of Heath-Brown [15] and Bleher [4, 5] shows that the limit distribution can be described in terms of almost periodic functions. Bleher and Bourgain obtained a similar result for the multidimensional torus threaded by flux lines, under certain diophantine conditions on the flux strength [6].

In the case of the boxed oscillator, we will see in Sect. 6 that, rather than almost periodic functions, almost *modular* functions will describe the distribution of the error term. This last section is based on the papers [26, 27].

2 Torus Threaded by Flux Lines and Lattice Points in Thin Spherical Shells

The quantum mechanics of a free particle on a k-dimensional torus threaded by flux lines of strength $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ is described by the Hamiltonian

$$H = \sum_j \left(\frac{1}{2\pi \mathrm{i}} \frac{\partial}{\partial x_j} - \alpha_j \right)^2 \tag{2.1}$$

acting on periodic functions φ, i.e., $\varphi(\boldsymbol{x} + \boldsymbol{l}) = \varphi(\boldsymbol{x})$, for all $\boldsymbol{l} \in \mathbb{Z}^k$. The eigenfunctions of H are $\varphi_{\boldsymbol{m}}(\boldsymbol{x}) = e(\boldsymbol{m} \cdot \boldsymbol{x})$, where $\boldsymbol{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k$, and its eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \cdots \to \infty$ are given by

$$\|\boldsymbol{m} - \boldsymbol{\alpha}\|^2 = (m_1 - \alpha_1)^2 + \cdots + (m_k - \alpha_k)^2. \tag{2.2}$$

Geometrically, the eigenvalues of H thus correspond to squared radii of spheres with center $\boldsymbol{\alpha}$ which contain at least one lattice point $\boldsymbol{m} \in \mathbb{Z}^k$; the multiplicity of the eigenvalue corresponds to the number of lattice points on the sphere. Since the number of lattice points in a ball of large radius is approximately its volume, we find that (1.1) holds with $\overline{N}(\lambda) = B_k \lambda^{k/2}$ where B_k is the volume of the unit ball. According to the Berry-Tabor conjecture we expect the rescaled sequence $X_j = B_k\|\boldsymbol{m}-\boldsymbol{\alpha}\|^k$ to satisfy the Poisson limit theorem (1.4), at least for "generic" choices of $\boldsymbol{\alpha}$. Hence, in geometric terms, the conjecture says that the number of lattice points inside a random spherical shell with fixed volume L, whose inner sphere encloses a ball of volume T (randomly distributed with law ρ_X), has a Poisson limit distribution as $X \to \infty$.

As a first step towards a proof of the conjecture we shall here show that the second moment of $\mathcal{N}(T, L)$, the *number variance*

$$\Sigma^2(X, L) := \frac{1}{X}\int_0^\infty \{\mathcal{N}(T, L) - L\}^2 \rho\left(\frac{T}{X}\right) dT, \tag{2.3}$$

converges indeed to the variance of the Poisson distribution, which is L. Note in the above definition of $\Sigma^2(X, L)$ that, in view of (1.2), the expectation value of $\mathcal{N}(T, L)$ is asymptotically

$$\frac{1}{X}\int_0^\infty \mathcal{N}(T, L)\,\rho\left(\frac{T}{X}\right) dT \to L. \tag{2.4}$$

As we shall see the set of "generic" $\boldsymbol{\alpha}$ can be characterized by an explicit diophantine condition which is in fact satisfied by a set of $\boldsymbol{\alpha}$ of full Lebesgue measure.

The vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$ is said to be *diophantine of type* κ, if there exists a constant $C > 0$ such that

$$\max_j \left|\alpha_j - \frac{m_j}{q}\right| > \frac{C}{q^\kappa} \tag{2.5}$$

for all $m_1, \ldots, m_k, q \in \mathbb{Z}$, $q > 0$. The smallest possible value for κ is $\kappa = 1+\frac{1}{k}$. In this case $\boldsymbol{\alpha}$ is called *badly approximable*. Examples of badly approximable vectors are $\boldsymbol{\alpha}$ such that the components of $(\boldsymbol{\alpha}, 1)$ form a basis of a real algebraic number field of the degree $k + 1$ ([39], Theorem 6F). On the other hand, for any $\kappa > 1 + \frac{1}{k}$, the set of diophantine vectors of type κ is of full Lebesgue measure ([39], Theorem 6G).

Theorem 1 (Poisson limit of the number variance). *Suppose $\boldsymbol{\alpha}$ is diophantine of type $\kappa < \frac{k-1}{k-2}$ and the components of the vector $(\boldsymbol{\alpha}, 1) \in \mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$. Then, for every $L > 0$,*

$$\lim_{X\to\infty} \Sigma^2(X, L) = L. \tag{2.6}$$

This theorem is a corollary of a more general statement on the convergence of the *pair correlation density* of the X_j, which is proved in [24, 25]. For any $\psi \in \mathrm{C}_0(\mathbb{R}_{>0}\times\mathbb{R}_{>0}\times\mathbb{R})$ (i.e., continuous and of compact support) let us define the pair correlation function

$$R_2(\psi, \lambda) = \frac{1}{B_k\lambda^{k/2}} \sum_{i,j=1}^\infty \psi\left(\frac{\lambda_i}{\lambda}, \frac{\lambda_j}{\lambda}, \lambda^{k/2-1}(\lambda_i - \lambda_j)\right). \tag{2.7}$$

We then have the following statement (Theorem 2.2, [25]).

Theorem 2 (Poisson limit of pair correlation). *Let* $\psi \in \mathrm{C}_0(\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R})$*. Suppose the components of* $(\boldsymbol{\alpha}, 1) \in \mathbb{R}^{k+1}$ *are linearly independent over* $\mathbb{Q}$*, and assume* $\boldsymbol{\alpha}$ *is diophantine of type* $\kappa < \frac{k-1}{k-2}$*. Then*

$$\lim_{\lambda \to \infty} R_2(\psi, \lambda) = \frac{k}{2} \int_0^\infty \psi(r, r, 0) r^{k/2-1} dr + \frac{k^2}{4} B_k \int_{\mathbb{R}} \int_0^\infty \psi(r, r, s) r^{k-2} dr\, ds. \tag{2.8}$$

To see more clearly what this theorem says about the distribution of the rescaled sequence X_j, let us put

$$\tilde{R}_2(\psi, X) = \frac{1}{X} \sum_{i,j=1}^{\infty} \psi\left(\frac{X_i}{X}, \frac{X_j}{X}, X_i - X_j \right). \tag{2.9}$$

The map

$$\omega : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}, \qquad \begin{pmatrix} r_1 \\ r_2 \\ s \end{pmatrix} \mapsto \begin{pmatrix} B_k r_1^{k/2} \\ B_k r_2^{k/2} \\ B_k R(r_1, r_2) s \end{pmatrix} \tag{2.10}$$

with $R(r_1, r_2) = (r_1^{k/2} - r_2^{k/2})/(r_1 - r_2)$ is invertible, continuous and in particular maps compact sets to compact sets. We may therefore choose as a suitable test function in Theorem 2 the function $\psi = \tilde{\psi} \circ \omega$, for any $\tilde{\psi} \in \mathrm{C}_0(\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R})$. So

$$\psi(r_1, r_2, s) = \tilde{\psi}(B_k r_1^{k/2}, B_k r_2^{k/2}, B_k R(r_1, r_2) s). \tag{2.11}$$

After a simple change of variables this shows that Theorem 2 is equivalent to the statement that (under the same conditions on $\boldsymbol{\alpha}$) *for any* $\tilde{\psi} \in \mathrm{C}_0(\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R})$ *we have*

$$\lim_{X \to \infty} \tilde{R}_2(\tilde{\psi}, X) = \int_0^\infty \tilde{\psi}(r, r, 0)\, dr + \int_{\mathbb{R}} \int_0^\infty \tilde{\psi}(r, r, s)\, dr\, ds. \tag{2.12}$$

The first term represents the asymptotic contribution of the diagonal terms ($i = j$) in the sum, while the second asserts that the spacings $X_i - X_j$ (for $i \neq j$) are uniformly distributed, as one would expect from independent random variables with constant mean spacing. We will show in Appendix A that Theorem 1 follows in fact from (2.12) for a special choice of test function ψ.

The diophantine conditions in the above theorems are in fact sharp; there are diophantine vectors $\boldsymbol{\alpha}$ of type $\kappa = \frac{k-1}{k-2}$ such that the components of $(\boldsymbol{\alpha}, 1) \in \mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$, for which the conclusion of the theorems do not hold. Such $\boldsymbol{\alpha}$ are of the form $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ where $(\alpha_1, \ldots, \alpha_{k-2}) \in \mathbb{R}^{k-2}$ is badly approximable by rationals (i.e., diophantine of type $\frac{k-1}{k-2} = 1+\frac{1}{k-2}$) and $(\alpha_{k-1}, \alpha_k) \in \mathbb{R}^2$ are very well approximable vectors which form a set of second Baire category in $\mathbb{R}^2$. (A set of second category is a set which cannot be represented as a countable union of nowhere dense sets.) The idea here is that the pair correlation function diverges at a logarithmic rate for $\boldsymbol{\alpha}$ with $(\alpha_{k-1}, \alpha_k) \in \mathbb{Q}^2$, which is still felt by well approximable (α_{k-1}, α_k); see [24, 25] for details. (Note that the set C in Theorem 1.7 [25] is wrongly characterized as a second category subset in $\mathbb{R}^k$, since we impose diophantine conditions. C is only a dense subset in $\mathbb{R}^k$.)

3 Theta Sums and Unipotent Flows

Let us firstly note that it is sufficient (see [25] for details) to prove Theorem 2 for pair correlation functions of the form

$$R_2(\psi_1, \psi_2, h, \lambda) = \frac{1}{B_k \lambda^{k/2}} \sum_{i,j=1}^{\infty} \psi_1(\frac{\lambda_i}{\lambda}) \psi_2(\frac{\lambda_j}{\lambda}) \hat{h}\big(\lambda^{k/2-1}(\lambda_i - \lambda_j)\big), \tag{3.1}$$

Here $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}_{\geq 0})$ are real-valued, and $\mathcal{S}(\mathbb{R}_{\geq 0})$ denotes the Schwartz class of infinitely differentiable functions of the half line $\mathbb{R}_{\geq 0}$ which, as well as their derivatives, decrease rapidly at $+\infty$. $\hat{h}$ is the Fourier transform of a compactly supported function $h \in \mathrm{C}_0(\mathbb{R})$, $\hat{h}(s) = \int_{\mathbb{R}} h(u) e(\frac{1}{2} us)\, du$ with the shorthand $e(z) := \mathrm{e}^{2\pi \mathrm{i} z}$.

A short calculation shows that $R_2(\psi_1, \psi_2, h, \lambda)$ can be written as an integral over a product of theta sums,

$$\begin{aligned} R_2(\psi_1, \psi_2, h, \lambda) = \frac{1}{B_k} v^{k/2-1} \times \\ \times \int_{\mathbb{R}} \Theta_f\left(u + \mathrm{i}\frac{1}{\lambda}, 0; \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\alpha} \end{pmatrix}\right) \overline{\Theta_g\left(u + \mathrm{i}\frac{1}{\lambda}, 0; \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\alpha} \end{pmatrix}\right)}\, h(v^{k/2-1} u)\, du, \end{aligned} \tag{3.2}$$

for the choice of functions $f(\boldsymbol{w}) = \psi_1(\|\boldsymbol{w}\|^2)$ and $g(\boldsymbol{w}) = \psi_2(\|\boldsymbol{w}\|^2)$. Here the theta sum Θ_f is defined for any Schwartz function $f \in \mathcal{S}(\mathbb{R}^k)$ by

$$\Theta_f(\tau, \phi; \boldsymbol{\xi}) = v^{k/4} \sum_{\boldsymbol{m} \in \mathbb{Z}^k} f_\phi((\boldsymbol{m} - \boldsymbol{y}) v^{1/2})\, e(\tfrac{1}{2} \|\boldsymbol{m} - \boldsymbol{y}\|^2 u + \boldsymbol{m} \cdot \boldsymbol{x}), \tag{3.3}$$

where

$$\tau = u + \mathrm{i} v,\ (u \in \mathbb{R},\ v \in \mathbb{R}_{>0}), \quad \phi \in \mathbb{R}, \quad \boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}\ (\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^k). \tag{3.4}$$

The family of functions f_ϕ is an extension of $f =: f_\phi\big|_{\phi=0}$ defined by

$$f_\phi(\boldsymbol{w}) = \int_{\mathbb{R}^k} G_\phi(\boldsymbol{w}, \boldsymbol{w}') f(\boldsymbol{w}')\, dw', \tag{3.5}$$

with the integral kernel

$$G_\phi(\boldsymbol{w},\boldsymbol{w}') = e(-k\sigma_\phi/8)|\sin\phi|^{-k/2} e\left[\frac{\frac{1}{2}(\|\boldsymbol{w}\|^2+\|\boldsymbol{w}'\|^2)\cos\phi - \boldsymbol{w}\cdot\boldsymbol{w}'}{\sin\phi}\right], \tag{3.6}$$

where $\sigma_\phi = 2\nu+1$ when $\nu\pi < \phi < (\nu+1)\pi$, $\nu\in\mathbb{Z}$. The operators $U^\phi : f \mapsto f_\phi$ are unitary, and in particular $U^0 = \mathrm{id}$.

The idea behind the introduction of the extra variables ϕ and $\boldsymbol{x}$ is that the product $\Theta_f\overline{\Theta_g}$ can be identified with a function on the finite volume homogeneous space $\mathcal{M} = \Gamma\backslash G^k$ where $G^k = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^{2k}$ and Γ is a lattice in G^k. The multiplication law for G^k is $(M;\boldsymbol{\xi})(M';\boldsymbol{\xi}') = (MM';\boldsymbol{\xi}+M\boldsymbol{\xi}')$ where $M, M' \in \mathrm{SL}(2,\mathbb{R})$ and $\boldsymbol{\xi},\boldsymbol{\xi}' \in \mathbb{R}^{2k}$; the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^{2k}$ is defined by

$$M\boldsymbol{\xi} = \begin{pmatrix} a\boldsymbol{x}+b\boldsymbol{y} \\ c\boldsymbol{x}+d\boldsymbol{y}\end{pmatrix}, \quad M = \begin{pmatrix} a & b \\ c & d\end{pmatrix}, \quad \boldsymbol{\xi} = \begin{pmatrix}\boldsymbol{x} \\ \boldsymbol{y}\end{pmatrix}. \tag{3.7}$$

The connection between $M \in \mathrm{SL}(2,\mathbb{R})$ and the variables $\tau = u + \mathrm{i}v$, ϕ used above is given by the Iwasawa decomposition

$$M = \begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}\begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2}\end{pmatrix}\begin{pmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{pmatrix}. \tag{3.8}$$

The first of the two crucial ingredients in the proof of the Poisson limit of the pair correlation functions is following equidistribution theorem [24, 25] whose proof in turn uses Ratner's classification of ergodic measures invariant under a unipotent flow [33, 34]. The following theorem may be viewed (strictly speaking only in the case $\sigma = 0$) as a special case of Shah's Theorem 1.4 [40]; for a proof see [24] ($\sigma = 0$) and [25] ($\sigma > 0$).

Theorem 3 (Equidistribution of translates of unipotent orbits). *Let Γ be a subgroup of $\mathrm{SL}(2,\mathbb{Z})\ltimes\mathbb{Z}^{2k}$ of finite index, and assume the components of the vector $(\boldsymbol{y},1)\in\mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$. Let h be a continuous function $\mathbb{R}\to\mathbb{R}_{\geq 0}$ with compact support. Then, for any bounded continuous function F on $\Gamma\backslash G^k$ and any $\sigma\geq 0$, we have*

$$\lim_{v\to 0} v^\sigma \int_{\mathbb{R}} F\left(u+\mathrm{i}v, 0; \begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{y}\end{pmatrix}\right) h(v^\sigma u)\, du = \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} F\, d\mu \int_{\mathbb{R}} h(w)\, dw \tag{3.9}$$

where μ is the Haar measure of G^k.

The dynamical interpretation of the above average is the following. Let us define the flows $\Psi^u, \Phi^t : \Gamma\backslash G^k \to \Gamma\backslash G^k$ by right translation with

$$\Psi_0^u = \left(\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}; \boldsymbol{0}\right), \qquad \Phi_0^t = \left(\begin{pmatrix} \mathrm{e}^{-t/2} & 0 \\ 0 & \mathrm{e}^{t/2}\end{pmatrix}; \boldsymbol{0}\right), \tag{3.10}$$

respectively. Then

$$\Gamma\left(u + \mathrm{i}\,\mathrm{e}^{-t}, 0; \begin{pmatrix} \mathbf{0} \\ \boldsymbol{y} \end{pmatrix}\right) = \Gamma g_0 \Psi_0^u \Phi_0^t = \Phi^t \circ \Psi^u(\Gamma g_0),\ g_0 := \left(1; \begin{pmatrix} \mathbf{0} \\ \boldsymbol{y} \end{pmatrix}\right) \quad (3.11)$$

and we can thus view the integral for $t = 0$ as an integral along an orbit of the unipotent flow Ψ^u which includes (at time $u = 0$) the point g_0; for $t > 0$ we obtain a translate by Φ^t of the above orbit which, by Theorem 3, eventually becomes equidistributed in $\Gamma\backslash G^k$.

The integral on the right-hand-side of the above equidistribution theorem can be worked out explicitly for $F = \Theta_f \overline{\Theta_g}$ and yields precisely the first term in Theorem 2. The problem is that F is not a bounded function. To prove convergence in this case we need to ensure that the translated orbit stays sufficiently far away from the singularities of F; this is achieved by imposing diophantine conditions on $\boldsymbol{y}$. The only exception is a small piece of the orbit at $u = 0$ which runs into the singularity and produces an additional contribution, which in fact yields the second term in Theorem 2.

4 The Boxed Oscillator, Lattice Points in Thin Parabolic Strips, and Distribution Modulo One

The Hamiltonian of the boxed oscillator is $H = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \omega^2 y^2$, where we assume Dirichlet boundary conditions at $x = 0, \ell$. Its eigenvalues are $E_j = (\pi/\ell)^2 n^2 + (2m+1)\omega$, for $n = 1, 2, 3, \ldots$ and $m = 0, 1, 2, \ldots$. Up to overall additive and multiplicative constants these can be written as $\lambda_j = n^2\alpha + m$ with $\alpha = (\pi/\ell)^2/2\omega$. The eigenvalue number is asymptotically $\#\{j : \lambda_j \leq \lambda\} \sim c\lambda^{3/2}$ where $c = \mathrm{meas}\{x, y \geq 0, \alpha x^2 + y \leq 1\} = \frac{2}{3\sqrt{\alpha}}$.

The statistical properties of the sequence λ_j are directly related to those of $n^2\alpha \bmod 1$. For consider those $\lambda_j = n^2\alpha + m$, which fall into the interval $[\lambda, \lambda+1)$, for some fixed $\lambda > 0$. Clearly for every $n = 1, 2, \ldots$ such that $n^2\alpha < \lambda + 1$ there exists a unique $m = 0, 1, 2, \ldots$ such that $\lambda_j \in [\lambda, \lambda+1)$. The values of λ_j in this interval are thus in one-to-one correspondence with $n^2\alpha \bmod 1$, $n = 1, \ldots, N < \sqrt{(\lambda+1)/\alpha}$. The distribution of the λ_j in small random intervals can therefore be understood in terms of the distribution of $n^2\alpha \bmod 1$ in small (i.e. of size of the order of $1/N$) random intervals of the unit circle. Let $[\xi, \xi + N^{-1}\sigma] + \mathbb{Z}$ be such an interval where ξ is uniformly distributed on the unit circle; define the analogue of the counting function (1.3) by

$$\mathcal{N}(N, \xi, \sigma) = \#\{n = 1, \ldots, N : n^2\alpha \in [\xi, \xi + N^{-1}\sigma] + \mathbb{Z}\}. \quad (4.1)$$

In view of the Berry-Tabor conjecture we expect that—for generic α— this number is Poisson distributed as $N \to \infty$, i.e., for any bounded function $g : \mathbb{Z}_{\geq 0} \to \mathbb{C}$,

$$\int_0^1 g\big(\mathcal{N}(N, \xi, \sigma)\big)\, d\xi \to \sum_{k=0}^{\infty} g(k) \frac{\sigma^k}{k!}\, \mathrm{e}^{-\sigma}\,. \quad (4.2)$$

The best result we have in this direction is again for the number variance

$$\Sigma^2(N,\sigma) := \int_0^1 \{\mathcal{N}(N,\xi,\sigma)-\sigma\}^2\, d\xi, \tag{4.3}$$

which can be shown to converge to the Poisson limit for almost all α.

Theorem 4 (Poisson limit of the number variance). *There is a set $P \subset \mathbb{R}$ of full Lebsgue measure such that, for every $\alpha \in P$ and every $\sigma > 0$,*

$$\lim_{N\to\infty} \Sigma^2(N,\sigma) = \sigma. \tag{4.4}$$

As for Theorem 1 above, this theorem follows from the Poisson distribution of the more general pair correlation function

$$R_2(\psi,N) = \frac{1}{N}\sum_{j,k=1}^{N}\sum_{\nu\in\mathbb{Z}} \psi\big(N(j^2\alpha - k^2\alpha + \nu)\big) \tag{4.5}$$

where $\psi \in \mathrm{C}_0(\mathbb{R})$, continuous and with compact support. The following theorem is proved by Rudnick and Sarnak [35] by averaging $R_2(\psi,N)$ and its square over α and using a variant of the Borel-Cantelli argument.

Theorem 5 (Poisson limit of pair correlation). *There is a set $P \subset \mathbb{R}$ of full Lebsgue measure such that, for every $\alpha \in P$ and every $\psi \in \mathrm{C}_0(\mathbb{R})$, we have*

$$\lim_{N\to\infty} R_2(\psi,N) = \psi(0) + \int_{\mathbb{R}} \psi(x)\, dx. \tag{4.6}$$

The number variance is in this case in fact identical to the pair correlation function, i.e., $\Sigma^2(N,\sigma) = R_2(\psi,N) - \sigma^2$ for the choice $\psi(x) = \max\{\sigma - |x|, 0\}$, see Appendix B.

5 On $n^2\alpha$ mod 1 and the Equidistribution of Kronecker Sequences Along Closed Horocycles

In view of Theorems 1 and 2, one would like to give a more explicit characterization (in terms of diophantine conditions) for the set of α for which $n^2\alpha$ mod 1 is Poisson distributed. Would, for instance, the assertion in Theorem 5 hold for $\alpha = \sqrt{2}$? Motivated by the affirmative answer in the case of the pair correlation problem for quadratic forms discussed in the previous section, the idea is to look for an equidistribution problem involving unipotent orbits, which can be employed to understand the pair correlation densities of $n^2\alpha$ mod 1. To this end, consider a pair correlation function with smooth weighting,

$$R_2(f,h,N) = \frac{1}{N} \sum_{j,k\in\mathbb{Z}} \sum_{\nu\in\mathbb{Z}} f\left(\frac{j}{N}\right) f\left(\frac{k}{N}\right) \hat{h}\big(N(j^2\alpha - k^2\alpha + \nu)\big) \tag{5.1}$$

where $f \in \mathbb{C}_0^\infty(\mathbb{R})$, $h \in \mathrm{C}_0(\mathbb{R})$ with Fourier transform

$$\hat{h}(s) = \int_{\mathbb{R}} h(u) e(\tfrac{1}{2}us)\, du = O(|s|^{-2}) \tag{5.2}$$

for $s \to \infty$. Applying Poisson summation to the ν-sum, we obtain

$$R_2(f,h,N) = \frac{1}{N} \sum_{m\in\mathbb{Z}} h\left(\frac{m}{N}\right) \left|\Theta_f(m\alpha + \mathrm{i}N^{-2}, 0)\right|^2 \tag{5.3}$$

where $\Theta_f(\tau,\phi)$ is the theta sum (3.3) for dimension $k = 1$ at $\boldsymbol{\xi} = \mathbf{0}$, i.e.,

$$\Theta_f(\tau,\phi) = v^{1/4} \sum_{n\in\mathbb{Z}^k} f_\phi(nv^{1/2})\, e(\tfrac{1}{2}n^2 u). \tag{5.4}$$

The pair correlation function may thus be viewed as a special case of averages of the form

$$\frac{1}{M} \sum_{m=1}^{M} F\,(m\alpha + \mathrm{i}v, 0) \tag{5.5}$$

as $M \to \infty$ and $v \to 0$, where F is a continuous function on $\mathcal{M} = \Gamma\backslash \mathrm{SL}(2,\mathbb{R})$, where Γ is a lattice in $\mathrm{SL}(2,\mathbb{R})$ which contains the parabolic subgroup $\left\{\left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right) : j \in \mathbb{Z}\right\}$; we will also assume for simplicity that $-1 \in \Gamma$. In particular, for $\Gamma = \Gamma_\theta$, the invariance group of $|\Theta_f|$ (the "theta group"), one can show [28] that if for some fixed $\alpha \in \mathbb{R}$ and $F = |\Theta_f|^2$ we have

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} F\,(m\alpha + \mathrm{i}v, 0) = \frac{1}{\mu(\mathcal{M})} \int_{\mathcal{M}} F\, d\mu, \quad v = M^{-2} \to 0, \tag{5.6}$$

then the limiting pair correlation density of $n^2\alpha \bmod 1$ is Poisson.

The equidistribution theorem (5.6) we are here interested in combines two classical equidistribution problems. The first is the equidistribution of long closed horocycles [37], i.e.,

$$\lim_{y\to 0} \int_0^1 F\,(u + \mathrm{i}v, 0)\, du = \frac{1}{\mu(\mathcal{M})} \int_{\mathcal{M}} F\, d\mu, \tag{5.7}$$

for any sufficiently nice test function F, e.g., bounded continuous. The second is the distribution of the Kronecker sequence $\alpha, 2\alpha, 3\alpha \ldots, M\alpha \bmod 1$, which is well known to be equidistributed as $M \to \infty$ for all irrational α; that is

$$\lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} F\,(m\alpha + \mathrm{i}v, 0) = \int_0^1 F\,(u + \mathrm{i}v, 0)\, du \tag{5.8}$$

for fixed $v > 0$. Taking both limits $M \to \infty$, $v \to 0$ simultaneously requires a careful analysis. Of particular interest is the case when the number M of points on the horocycle grows slower than the length of the horocycle, v^{-1}. In this case the problem is that the mean distance between the points on the horocycles grows as $v \to 0$. It seems therefore difficult to show that any possible limit measure is invariant under some unipotent action, and hence Ratner's theorem cannot be applied (in the present approach, that is). The proof of the following theorem uses instead methods from spectral analysis [28].

Theorem 6 (Equidistribution of $m\alpha \bmod 1$ along closed horocycles). *Let Γ be a lattice in $\mathrm{SL}(2,\mathbb{R})$ as described above. Fix $\nu > 0$. Then there is a set $P = P(\Gamma,\nu) \subset \mathbb{R}$ of full Lebesgue measure such that for any $\alpha \in P$, any bounded continuous function $F : \mathcal{M} \to \mathrm{C}$, and any constants $0 < C_1 < C_2$, we have*

$$\frac{1}{M}\sum_{m=1}^{M} F\,(m\alpha + \mathrm{i}v, 0) \to \frac{1}{\mu(\mathcal{M})}\int_{\mathcal{M}} F\, d\mu \tag{5.9}$$

uniformly as $M \to \infty$ and $C_1 M^{-\nu} \le v \le C_2 M^{-\nu}$.

This theorem holds in fact for a larger class of test functions F which are continuous but unbounded, and which allow the choice $F = |\Theta_f|^2$. Theorem 6 thus implies Rudnick & Sarnak's result [35] that the pair correlation density of $n^2\alpha \bmod 1$ is Poisson for almost all α.

If $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ or a congruence subgroup, and we increase the number of points on the horocycles sufficiently fast (i.e., ν is chosen sufficiently small) we are able to prove equidistribution under explicit diophantine conditions. The best possible result is obtained under the assumption that the Fourier coefficients of the eigenfunctions on the Laplacian on $\Gamma\backslash\mathfrak{H}$ ($\mathfrak{H}$ denotes the complex upper half plane) are almost bounded; this hypothesis is usually referred to as the *Ramanujan conjecture for Maass wave forms.*

Theorem 7 (Equidistribution of $m\alpha \bmod 1$ along closed horocycles). *Let Γ be a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ and assume the Ramanujan conjecture for Maass waveforms on $\Gamma\backslash\mathfrak{H}$ holds. Let $\alpha \in \mathbb{R}$ be of type $\kappa \ge 2$, and fix $\nu < \min\{2, \frac{2}{\kappa-2}\}$. Then for any bounded continuous function $F : \mathcal{M} \to \mathrm{C}$, and any constant $C_1 > 0$, we have*

$$\frac{1}{M}\sum_{m=1}^{M} F\,(m\alpha + \mathrm{i}v, 0) \to \frac{1}{\mu(\mathcal{M})}\int_{\mathcal{M}} F\, d\mu \tag{5.10}$$

uniformly as $M \to \infty$, $v \to 0$ so long as $v \ge C_1 M^{-\nu}$.

This statement is proved in [28]. If $\kappa \ge 3$, then $\nu < \frac{2}{\kappa-2}$ is in fact the best possible restriction on ν, in the sense that there are otherwise counter examples for which the assertion of the theorem is wrong [28]. Thus, in contrast with the equidistribution theorem for unipotent flows (Theorem 3), we must impose diophantine conditions even in the case of bounded test functions.

It would be very interesting to extend Theorem 7 to $\nu = 2$, which, as mentioned above, is the case relevant to the pair correlation problem. Note that the theta group Γ_θ is a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$.

6 Distribution Modulo One and Almost Modular Functions

In the previous section we have presented some evidence that the distribution of $n^2\alpha$ mod 1 in intervals of size $1/N$ is described by a Poisson distribution. In the same vein (as mentioned in the introduction) it can be expected that a central limit theorem holds for slightly larger intervals. Let us here consider the case when the interval size is macroscopic. For any fixed interval $[\xi, \xi + \eta]$, $0 < \eta < 1$, we are interested in the counting function

$$\mathcal{N}_\alpha(N, \xi, \eta) = \#\{n = 1, \ldots, N : n^2\alpha \in [\xi, \xi + \eta] + \mathbb{Z}\}. \tag{6.1}$$

For irrational α, the sequence $n^2\alpha$ is equidistributed mod 1, which means that $\mathcal{N}(N, \xi, \eta) \sim N\eta$ as $N \to \infty$. The *error term* is thus

$$E_\alpha(N, \xi, \eta) = \mathcal{N}_\alpha(N, \xi, \eta) - N\eta. \tag{6.2}$$

There are two possibilities to study the fluctuations of this function. Fix the interval and take α to be uniformly distributed in $[0, 1)$, or fix α and take ξ to be uniformly distributed in $[0, 1)$ with η fixed as usual (note, however, that this time we consider large intervals compared with the mean separation $1/N$). In the first case we have the following statement.

Theorem 8 (Limit theorem for the error term). *For α uniformly distributed in $[0, 1)$, $N^{-1/2}E_\alpha(N, \xi, \eta)$ has a limit distribution as $N \to \infty$. That is, there exists a probability measure $\nu_{\xi,\eta}$ on $\mathbb{R}$ such that, for any bounded continuous function $g : \mathbb{R} \to \mathbb{R}$, we have*

$$\lim_{N\to\infty} \int_0^1 g\big(N^{-1/2}E_\alpha(N, \xi, \eta)\big) d\alpha = \int_{\mathbb{R}} g(w)\, \nu_{\xi,\eta}(dw). \tag{6.3}$$

Furthermore, $\nu_{\xi,\eta}$ is even.

This is a special case of Theorem 2.1 in [26], which also provides an explicit formula for the variance of the limit distribution. To sketch the proof of Theorem 8 let us write

$$E_\alpha(N, \xi, \eta) = \sum_{n=1}^{N} \psi(n^2\alpha) - N \int_0^1 \psi(t)\, dt, \tag{6.4}$$

where ψ is the characteristic function of $[\xi, \xi + \eta]$. ψ could in fact be a more general real- or complex-valued function; we will only require that its Fourier coefficients

$$\widehat{\psi}_k = \int_0^1 \psi(t)\, e(-kt)\, dt. \tag{6.5}$$

satisfy

$$\widehat{\psi}_0 = 0, \tag{6.6}$$

and that there are constants $\beta > 1/2$ and $C(\psi) > 0$ such that

$$|\widehat{\psi}_k| \leq \frac{C(\psi)}{|k|^\beta} \tag{6.7}$$

for all $k \neq 0$. Fourier expansion (which converges only in the L^2 sense) yields

$$E_\alpha(N, \xi, \eta) = \sum_{k \neq 0} \widehat{\psi}_k \left\{ \sum_{n=1}^{N} e(kn^2 x) \right\}. \tag{6.8}$$

It is known [19] that the theta sums inside the curly brackets individually have a limit distribution, as $N \to \infty$. This result follows from the observation (cf. previous sections) that theta sums can be identified with functions on the metaplectic cover of $\mathrm{SL}(2, \mathbb{R})$ which are invariant under certain subgroups of finite index in the metaplectic analogue of $\mathrm{SL}(2, \mathbb{Z})$. The limit theorem is then a direct consequence for the equidistribution of long closed horocycles on the metaplectic cover [20].

One can show that the truncated Fourier expansion

$$E_\alpha^{(K)}(N, \xi, \eta) = \sum_{0 < |k| \leq K} \widehat{\psi}_k \left\{ \sum_{n=1}^{N} e(kn^2 x) \right\} \tag{6.9}$$

can as well be identified with functions on the metaplectic cover of $\mathrm{SL}(2, \mathbb{R})$, where the index of the invariance subgroup is still finite but becomes large with increasing K. Following the same steps as in [19] one can therefore show that $E_\alpha^{(K)}(N, \xi, \eta)$ satisfies the limit theorem, Theorem 8.

The variance of the difference $E_\alpha(N, \xi, \eta) - E_\alpha^{(K)}(N, \xi, \eta)$ is, uniformly in $N \gg 1$, arbitrarily small for K sufficiently large; hence the distributions of $E_\alpha(N, \xi, \eta)$ and $E_\alpha^{(K)}(N, \xi, \eta)$ are arbitrarily close for K large. Theorem 8 follows now from standard probabilistic arguments.

The fact that each approximation $E_\alpha^{(K)}(N, \xi, \eta)$ is a modular function, but $E_\alpha(N, \xi, \eta)$ is not (in general), suggests the name *almost modular function*, in close analogy with almost *periodic* functions in the sense of Besicovitch. The above arguments can in fact be extended to general classes of almost modular functions, which are characterized by the approximability (with respect to a certain L^p norm) by modular functions invariant under congruence subgroups of large index [26].

Another interesting example of an almost modular function is the logarithm of

$$\prod_{n=1}^{\infty}\bigl(1-e(n^2 z)\bigr), \tag{6.10}$$

which is studied in [27]. Its limit distribution in the complex plane is in fact rotation-invariant.

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A Proof of Theorem 1

Because of (1.2) we have, for large X,

$$\varSigma^2(X,L) \sim \frac{1}{X}\int_0^\infty \mathcal{N}(T,L)^2 \rho\left(\frac{T}{X}\right) dT - L^2. \tag{A.1}$$

Expand

$$\mathcal{N}(T,L) = \sum_j \chi_1\left(\frac{X_j - T}{L}\right), \tag{A.2}$$

where χ_1 is the indicator function of the interval $[0,1]$. This yields

$$\varSigma^2(X,L) + L^2 \sim \frac{1}{X}\sum_{i,j}\int_{-\infty}^\infty \chi_1\left(\frac{X_i - T}{L}\right)\chi_1\left(\frac{X_j - T}{L}\right)\rho\left(\frac{T}{X}\right) dT. \tag{A.3}$$

We have replaced 0 in the lower limit by $-\infty$, which is permitted since ρ is supported on the positive half line. Substitute T by $T + \frac{1}{2}(X_i + X_j)$, and the right hand side becomes

$$\frac{1}{X}\sum_{i,j}\int_{-\infty}^\infty \chi_1\left(\frac{\frac{1}{2}(X_i - X_j) - T}{L}\right)\times \\ \times\chi_1\left(\frac{\frac{1}{2}(X_j - X_j) - T}{L}\right)\rho\left(\frac{\frac{1}{2}(X_i + X_j) + T}{X}\right) dT. \tag{A.4}$$

The integration in T is restricted by the inequalities

$$0 \leq \frac{1}{2}(X_i - X_j) - T \leq L, \qquad 0 \leq \frac{1}{2}(X_j - X_i) - T \leq L, \tag{A.5}$$

which imply $0 \leq -T \leq L$, so T is bounded. Therefore, by the continuity of ρ,

$$\rho\left(\frac{\frac{1}{2}(X_i + X_j) + T}{X}\right) \sim \rho\left(\frac{\frac{1}{2}(X_i + X_j)}{X}\right), \tag{A.6}$$

and it is sensible to write (A.4) as

$$\frac{1}{X}\sum_{i,j} \rho\left(\frac{\frac{1}{2}(X_i + X_j)}{X}\right) W(X_i - X_j) + \text{error term} \tag{A.7}$$

where

$$W(s) := \int_{-\infty}^{\infty} \chi_1\left(\frac{T + \frac{1}{2}s}{L}\right)\chi_1\left(\frac{T - \frac{1}{2}s}{L}\right) dT = \max\{L - |s|, 0\}. \tag{A.8}$$

Since the function $\psi(r_1, r_2, s) = \rho(\frac{1}{2}(r_1 + r_2))W(s)$ is continuous and has compact support, (2.12) yields

$$\lim_{X\to\infty} \Sigma^2(X, L) + L^2 = \int_0^\infty \psi(r, r, 0)\, dr + \int_{\mathbb{R}} \int_0^\infty \psi(r, r, s)\, dr\, ds = L + L^2, \tag{A.9}$$

which proves Theorem 1, provided the above error term is indeed small. To investigate this, note that

$$|\text{error term}| \leq \frac{1}{X}\sum_{i,j} \tilde{\rho}\left(\frac{\frac{1}{2}(X_i + X_j)}{X}\right) W(X_i - X_j) \tag{A.10}$$

where $\tilde{\rho}$ is a continuous function with compact support such that

$$\sup_{0\leq -T \leq L} \left|\rho\left(r + \frac{T}{X}\right) - \rho(r)\right| \leq \tilde{\rho}(r) \tag{A.11}$$

for all r. It is evident that for any given $\epsilon > 0$ we can find a function $\tilde{\rho}$ meeting this requirement for all X large enough and satisfying in addition $\int_0^\infty \tilde{\rho}(r)dr < \epsilon$. By (2.12) the right hand side of (A.10) converges to

$$(L^2 + L)\int_0^\infty \tilde{\rho}(r)dr < (L^2 + L)\epsilon \tag{A.12}$$

which means that the error term is smaller than any $\epsilon > 0$, hence zero. □

B Proof of Theorem 4

We have

$$
\begin{aligned}
&\Sigma^2(N,\sigma)+\sigma^2 \\
&\quad=\sum_{j,k=1}^{\infty}\sum_{\nu,\nu'\in\mathbb{Z}}\int_0^1\bigl\{\chi_{[0,\sigma]}\bigl(N(j^2\alpha+\xi+\nu)\bigr)\chi_{[0,\sigma]}\bigl(N(k^2\alpha+\xi+\nu')\bigr)\bigr\}\,d\xi \\
&\qquad=\sum_{j,k=1}^{\infty}\sum_{\nu\in\mathbb{Z}}\int_{\mathbb{R}}\bigl\{\chi_{[0,\sigma]}\bigl(N(j^2\alpha+\xi+\nu)\bigr)\chi_{[0,\sigma]}\bigl(N(k^2\alpha+\xi)\bigr)\bigr\}\,d\xi \\
&\qquad=\frac{1}{N}\sum_{j,k=1}^{\infty}\sum_{\nu\in\mathbb{Z}}\int_{\mathbb{R}}\bigl\{\chi_{[0,\sigma]}\bigl(N(j^2\alpha-k^2\alpha+\nu)+\xi\bigr)\chi_{[0,\sigma]}(\xi)\bigr\}\,d\xi \qquad \text{(B.1)}
\end{aligned}
$$

and thus $\Sigma^2(N,\sigma)+\sigma^2=R_2(\psi,N)$ for $\psi(x)=\int_{\mathbb{R}}\bigl\{\chi_{[0,\sigma]}(x+\xi)\chi_{[0,\sigma]}(\xi)\bigr\}\,d\xi=\max\{\sigma-|x|,0\}$. Since $\psi\in\mathrm{C}_0(\mathbb{R})$, and $\psi(0)+\int_{\mathbb{R}}\psi(x)\,dx=\sigma+\sigma^2$, Theorem 4 is thus indeed a special case of Theorem 5. $\square$

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