Nodal Domain Statistics for Quantum Maps, Percolation, and Stochastic Loewner Evolution

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We develop a percolation model for nodal domains in the eigenvectors of quantum chaotic torus maps. Our model follows directly from the assumption that the quantum maps are described by random matrix theory. Its accuracy in predicting statistical properties of the nodal domains is demonstrated for perturbed cat maps and supports the use of percolation theory to describe the wave functions of general Hamiltonian systems. We also demonstrate that the nodal domains of the perturbed cat maps obey the Cardy crossing formula and find evidence that the boundaries of the nodal domains are described by stochastic Loewner evolution with diffusion constant κ close to the expected value of 6, suggesting that quantum chaotic wave functions may exhibit conformal invariance in the semiclassical limit.

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One of the central problems in the field of quantum chaos is to understand the morphology of quantum eigenfunctions in classically chaotic systems. In time-reversal-symmetric systems one can always find a basis in which these eigenfunctions are real. They can thus be divided into nodal domains—connected regions of the same sign, separated by nodal lines on which the eigenfunctions vanish. The statistical properties of these nodal domains then constitute a natural way to characterize the morphology of the eigenfunctions.

Nodal domain statistics were studied for separable billiards in [1], where it was shown that if ν_n is the number of nodal domains in the nth energy eigenstate then $\chi_n = \nu_n/n$ has a limiting distribution as $n \to \infty$ with a square-root singularity at a system-dependent maximum value $\chi_{\max} < 1$.

In chaotic systems the eigenfunctions may be modeled statistically, far from boundaries and turning points, by random superpositions of plane waves [2]:

$$u(\mathbf{x}) = \sqrt{\frac{2}{J}} \sum_{j=1}^{J} \cos(kx\cos\theta_j + ky\sin\theta_j + \phi_j), \quad (1)$$

where θ_j and ϕ_j are random phases. This is known as the random wave model. Since plane waves are solutions of the Schrödinger equation for a free particle, $\nabla^2\psi = -k^2\psi$, the maxima of any superposition are positive and the minima are negative. Hence the nodal domains correspond to groups of either maxima or minima. A given pair of adjacent maxima (minima) lie in the same nodal domain if the saddle point between them is positive (negative). The density of saddles in the random wave model is asymptotically twice the density maxima or minima. This would be exactly the case, for example, if the maxima and minima lay on alternate sites of a square lattice and the saddles on the corresponding dual lattice, e.g., midway between diagonally adjacent maxima (or minima) [although it is important to note that typical realizations of $u(\mathbf{x})$ are in fact highly irregular]. The saddles may be thought of as lying at the midpoints of bonds of the dual lattice connecting the maxima, for example. If the saddle height is positive, then the corresponding maxima are connected and the bond may be thought of as "open"; if it is negative, the maxima are not directly connected, and the bond may be thought of as "closed." This was the basis of the very interesting suggestion put forward by Bogomolny and Schmit [3] that statistical properties of nodal domains in the random wave model, and hence in quantum chaotic eigenfunctions, correspond to those in critical percolation—percolation at the critical probability where there is a phase transition and an infinite spanning cluster emerges. Specifically, Bogomolny and Schmit assumed that the heights of the saddles are uncorrelated and have equal probability of being positive or negative, and proposed bond percolation on a square lattice as a model for nodal domain statistics. This implies that χ_n is Gaussian distributed as n varies in the semiclassical limit. Moreover, it leads to the conclusion that the scaling exponents associated with critical percolation also characterize properties of the nodal domains in quantum chaotic eigenfunctions, for example, their area distribution and fractal dimension.

The predictions of the percolation model are consistent with numerical computations [3] and experimental measurements [4] of the fluctuation statistics for the nodal domains of quantum billiards, but the data do not provide conclusive verification. This is important, because the model has been the subject of considerable debate. Foltin has shown that the heights of the saddles in the random wave model exhibit long range correlations [5], contradicting one of the key assumptions of the percolation model. Bogomolny has argued that oscillations in the two-point correlation function are sufficient to ensure the applicability of the Harris criterion [6] and so guarantee that the scaling exponents are unaffected [7], but the issue awaits a more systematic investigation. Moreover, Foltin, Gnutzmann, and Smilansky have devised a particular statistical measure for which the percolation model fails [8]. The range of validity of the model and the precise assumptions upon which it relies thus remain to be determined.

Our purpose here is to establish a percolation model for quantum torus maps. These are some of the most important examples of quantum chaotic systems, because one can find maps that are fully chaotic and quantum mechanically they are finite dimensional and so easily tractable. We will show that for these systems there is a critical percolation model that follows directly from the Bohigas-Giannoni-Schmit (BGS) conjecture, which asserts that local quantum fluctuation statistics in classically chaotic systems are modeled by random matrix theory [9]. This model corresponds to site percolation on a triangular lattice, which falls into the same universality class as bond percolation on a square lattice and so has the same critical exponents. The advantages of investigating the percolation model for maps are, first, that the assumptions underlying it are very much more straightforward—one only has to assume the BGS conjecture, and there are no problems analogous to those relating to the slow decay of correlations in billiard eigenfunctions—and, second, that one can perform more extensive and controlled numerical computations, leading to significantly more precise tests of the predictions.

We find that the percolation model for maps is extremely accurate in that the critical scaling exponents associated with the nodal domains are very close to those predicted by percolation theory. Moreover, the agreement goes beyond scaling laws: the nodal domains of the quantum maps we study also obey the Cardy crossing formula, which, in percolation theory, gives the probability of there being a cluster spanning the system between specified sections of the boundary [10]. We verify that Cardy's formula is satisfied within the random wave model as well. This suggests that both linear superpositions of random waves and quantum chaotic eigenfunctions may exhibit conformal invariance in the semiclassical limit. Finally, the link between processes governed by stochastic Loewner evolution (SLE) and statistical models has recently been the focus of considerable attention. SLE constitutes a method for analyzing random self-avoiding curves whose continuum limit is conformally invariant. Essentially, these curves are generated by conformal transformations which satisfy a stochastic differential equation depending upon a driving function that is proportional to a Brownian motion [see, e.g., [11]]. Critical percolation is believed to relate to SLE with diffusion constant $\kappa = 6$. For percolation on a triangular lattice this has been established rigorously [12]. On the basis of the percolation model one would expect nodal lines to behave like processes governed by SLE with $\kappa = 6$. We find evidence that this is the case for quantum maps.

The systems we study correspond to chaotic sympletic maps acting on the unit $2L$-dimensional torus, which is viewed as their phase space. Such maps can be quantized using an approach introduced by Hannay and Berry [13]. The Hilbert space has finite dimension N^L, where N plays the role of the inverse of Planck's constant. Quantum maps correspond to unitary matrices U acting on wave functions in this Hilbert space so as to generate their (discrete) time evolution. In the position representation these wave functions take values on an L-dimensional lattice. For example, when $L = 1$ the wave functions take values $(c_1, c_2, \ldots, c_N)$ at positions $q = Q/N$, $0 \leq Q < N$; and when $L = 2$ they take values $(c_1, c_2, \ldots, c_{N^2})$ at positions on the square lattice $\mathbf{q} = (Q_1/N, Q_2/N)$, $0 \leq Q_1, Q_2 < N$. We shall be concerned with the quantum map eigenvectors. If the map is time-reversal symmetric (and so U is symmetric), the components of the eigenvectors are real. For a given eigenvector, we can thus split the quantum lattice into regions such that the components associated with neighboring sites have the same sign. These regions then correspond to nodal domains.

When $L = 1$ this can be done straightforwardly: if sites lying next to each other on the one-dimensional lattice have eigenvector components c_j with the same sign then they constitute part of the same nodal domain. When $L = 2$ the situation requires more careful consideration, because one needs a convention for deciding whether lattice points that are diagonal neighbors and have eigenvector components with the same sign lie in the same nodal domain or not. Consider, for example, when the eigenvector components associated with a group of four lattice points which form a square have signs in a checkerboard arrangement, e.g., on the top row $+ -$, and underneath $- +$. Are the pluses automatically part of the same nodal domain, or the minuses? We take as our convention that lattice points are connected along diagonals running from the top left to the bottom right; so in the example just given it is the pluses that are connected. This takes us from the original square lattice to the triangular lattice. Nodal domains then correspond to regions on this triangular lattice in which connected points have the same sign. Our convention is, of course, one of many possibilities. However, we note that it is necessary to incorporate diagonal neighbors for the definition of nodal domains to be consistent with that in billiards, and that all of the conventions we have tested which do this lead to the same results.

In order to develop a statistical model for the nodal domains we now need to introduce a statistical ansatz for the signs. According to the BGS conjecture, for generic, classically chaotic, time-reversal-symmetric systems statistical properties of the matrix U should coincide with those of random matrices taken from the circular orthogonal ensemble of random matrix theory. The probability density for the eigenvectors $\mathbf{c} = (c_1, c_2, \ldots, c_{N^L})$ is then uniform on the hypersphere $\mathbf{c} \cdot \mathbf{c} = 1$. Crucially for us, it follows immediately that the sign of a given component is equally likely to be positive or negative and that these signs are independent of each other at different sites, i.e., they are uncorrelated.

When $L = 1$ this model was explored in [14]. When $L = 2$ it corresponds directly to site percolation on a triangular lattice, which falls into the same universality class as the Bogomolny-Schmit model. This means that the critical exponents associated with the nodal domain statis-

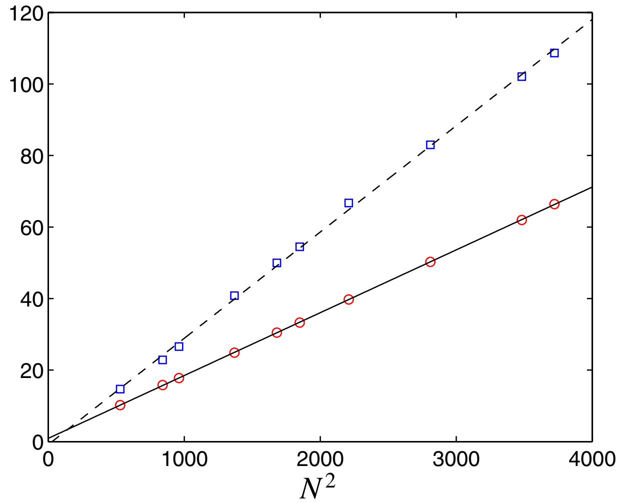

FIG. 1 (color online). The mean (red circles) and variance (blue squares) of the number of nodal domains for the quantum map with $k_1 = 0.04$ and $k_2 = 0.01$. The linear fit for the mean (solid line) gives $n_c = 0.0176$ and $b = 0.902$, while the fit for the variance (dashed line) gives $c = 0.0297$.

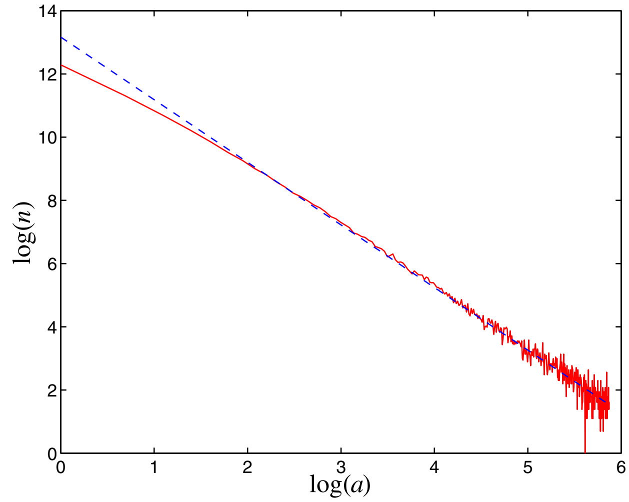

FIG. 2 (color online). Frequency of nodal domains as a function of area. The solid (red) line shows the data and the dashed (blue) line the theoretical power law with exponent $\tau = 187/91$. Here $N = 61$, $k_1 = 0.04$, and $k_2 = 0.01$.

tics will be the same. We note that in our case these have been established rigorously for percolation [12].

We now test the percolation model for a particular family of quantum torus maps. In essence, we are seeing whether this family is described sufficiently accurately by random matrix theory (RMT) for the model to apply. Linear maps are not sufficient for our purpose: because of nongeneric arithmetical symmetries they are not described by RMT [15]. Instead, we take a linear map composed with a nonlinear perturbation. Specifically, we use $M = \rho \circ A \circ \rho$ with

$$A: \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -2 & -2 & -1 \\ -2 & 6 & -1 & 0 \\ 16 & -39 & 2 & -2 \\ -39 & 94 & -2 & 6 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \mathrm{mod} 1 \qquad (2)$$

and ρ a nonlinear periodic shear in momentum: $p_1 \rightarrow p_1 + \frac{k_1}{4\pi}\cos(2\pi q_1)$, $p_2 \rightarrow p_2 + \frac{k_2}{4\pi}\cos(2\pi q_2)$. The map M is time-reversal symmetric and, for sufficiently small values of the perturbation parameters, completely chaotic. The corresponding quantum map, a unitary matrix of dimension N^2, can be written down easily using the prescriptions in [13,16] [for the explicit formula, see [17]]. We now compare statistical properties of the nodal domains of this map with those of percolation clusters.

Consider first the distribution of the number n of nodal domains. For percolation on N^2 sites this should be a Gaussian with mean $n_c N^2 + b + o(1)$ and variance cN^2, where Monte Carlo simulations give $n_c = 0.0176\ldots$, $b = 0.878\ldots$, and $c = 0.0309\ldots$ [18]. For the quantum map we find a Gaussian distribution with a mean and variance consistent with the percolation formulas. The data are shown in Fig. 1. For the distribution of areas a of the nodal domains, the percolation model predicts a scaling law $a^{-\tau}$, with $\tau = 187/91$. A log-log plot of the data for the eigenvectors of the quantum map is shown in Fig. 2.

The percolation model also implies that the nodal domains should have a fractal dimension $D = 91/48 = 1.89\ldots$. Data for the quantum map, obtained using a box-counting algorithm and shown as a log-log plot in Fig. 3, are consistent with this.

One of the key features of critical percolation is that it has a conformally invariant limit. This underlies the use of conformal field theory in deriving the Cardy crossing formula, for example, and the link with SLE. Given the success of the percolation model in describing scaling exponents associated with their nodal domains, it is natural to ask whether random waves and quantum eigenfunctions are also conformally invariant in the semiclassical limit.

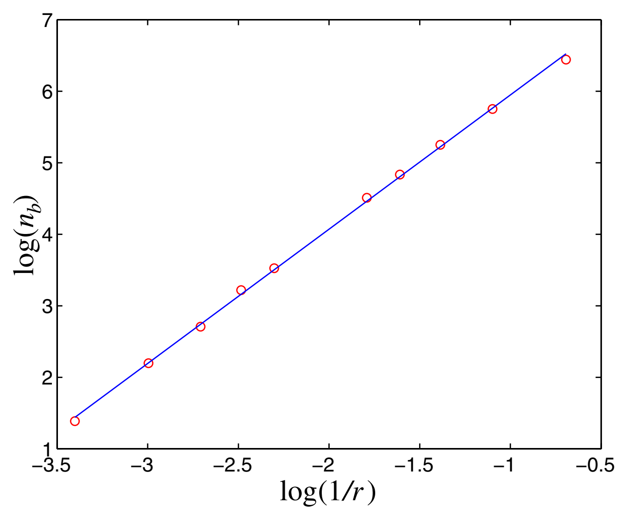

FIG. 3 (color online). A box count for the largest nodal domain of an eigenvector of the map with $N = 61$, $k_1 = 0.01$, and $k_2 = 0.02$. The linear fit corresponds to a fractal dimension of 1.8774.

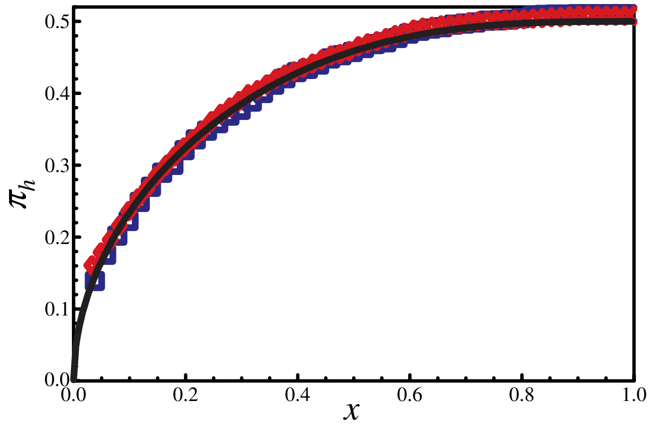

FIG. 4 (color online). Numerically computed crossing probabilities for map eigenvectors when $N = 61$, $k_1 = 0.01$, and $k_2 = 0.02$ (red diamonds) and 2000 realizations of the random wave model (blue squares), compared with Cardy's formula (solid line).

For the eigenvectors of the quantum map, we can determine the probability $\pi_h(x)$, defined with respect to an average over different eigenstates, that there exists a nodal domain that spans the triangular lattice defined above, connecting a specified section of length x on the left-hand boundary of the unit square representing the torus to any part of the right-hand boundary. We also test the nodal domains of realizations of the random wave model using the same geometry. The results are shown in Fig. 4 together with Cardy's formula for $\pi_h(x)$ [10].

To explore the connection with SLE we use an idea introduced in [19] to test conformal invariance in 2D turbulence. SLE generates curves from a stochastic differential equation, the Loewner equation, dependent on a driving function $\xi(t)$, which is proportional to a Brownian motion. We consider curves constructed from the nodal lines of our quantum map and invert the Loewner equation to find the corresponding driving function $\xi(t)$. For each eigenvector, we select curves by following a nodal line, keeping lattice points at which the eigenvector is positive to the right. Upon hitting the left boundary of the lattice, the curve continues along the boundary, in such a way that it can always be connected to the right-hand boundary without crossing itself, until it reaches another nodal line, which it then follows. The process is stopped when the curve reaches any of the other boundaries. The empirical driving is then computed by inverting the Loewner equation. The results are shown in Fig. 5. The fact that $\xi(t)$ generated from the curves has variance κt is consistent with Brownian motion, and so with the SLE interpretation. This is further supported by the observation that $\xi(t)$ has a Gaussian value distribution (see the inset). The best fit to the diffusion constant is $\kappa \approx 5.3$, which is close to the theoretical value, $\kappa = 6$, for the boundaries of percolation clusters.

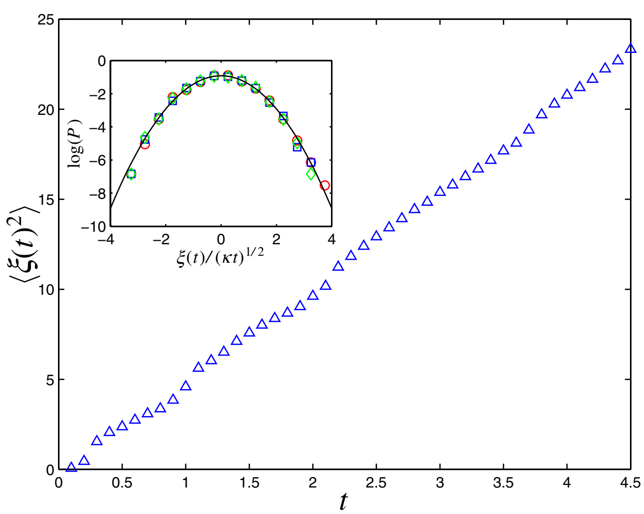

FIG. 5 (color online). Variance of $\xi(t)$ as a function of t. The inset shows the distribution of the rescaled driving function for $t = 0.3$ (red circles), $t = 0.4$ (blue squares), and $t = 0.45$ (green diamonds) with the expected Gaussian shown as a solid line. Here $N = 61$, $k_1 = 0.02$, and $k_2 = 0.01$.

The fact that the crossing formula applies and the link with SLE holds is evidence of conformal invariance. It is natural to conjecture that this will extend to generic quantum chaotic eigenfunctions in the semiclassical limit.

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