polynomial type estimate for the size of the smallest nontrivial integral solution of the inequality $\mid Q(x) \mid < \varepsilon$ (for the case $n \geq 5$).

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Horospheres, Farey fractions and Frobenius numbers

Jens Marklof

Frobenius numbers. Let $\widehat{\mathbb{Z}}^d = \{\boldsymbol{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d : \gcd(a_1, \ldots, a_d) = 1\}$ be the set of primitive lattice points, and $\widehat{\mathbb{Z}}^d_{\geq 2}$ the subset with coefficients $a_j \geq 2$. Given $\boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2}$, the Frobenius number $F(\boldsymbol{a})$ is defined as the largest integer that does not have a representation of the form $\boldsymbol{m} \cdot \boldsymbol{a}$ with $\boldsymbol{m} \in \mathbb{Z}^d_{\geq 0}$. In the case of two variables ($d = 2$) Sylvester showed that $F(\boldsymbol{a}) = a_1 a_2 - a_1 - a_2$. No such explicit formulas are known in higher dimensions [10]. In his studies of "arithmetic turbulence", Arnold [2] conjectured that $F(\boldsymbol{a})$ should fluctuate wildly as a function of $\boldsymbol{a}$. The following theorem establishes the existence of a limit distribution for these fluctuations. As we shall see, the key in the proof of this statement uses a novel interpretation of the Frobenius number in terms of the dynamics of a certain flow Φ^t on the space of lattices $\Gamma \backslash G$, with $G := \mathrm{SL}(d, \mathbb{R})$, $\Gamma := \mathrm{SL}(d, \mathbb{Z})$.

Theorem 1 ([7])**.** *Let $d \geq 3$. There exists a continuous non-increasing function $\Psi_d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\Psi_d(0) = 1$, such that for any bounded set $\mathcal{D} \subset \mathbb{R}^d_{\geq 0}$ with boundary of Lebesgue measure zero, and any $R \geq 0$,*

$$\lim_{T \to \infty} \frac{1}{T^d} \# \left\{ \boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2} \cap T\mathcal{D} : \frac{F(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} > R \right\} = \frac{\mathrm{vol}(\mathcal{D})}{\zeta(d)} \, \Psi_d(R). \tag{1}$$

Variants of Theorem 1 were previously known only in dimension $d = 3$ in the work of Bourgain and Sinai [4], and Shur, Sinai and Ustinov [13]. For $d = 3$

Ustinov [14] derived an explicit formula for the limit density,

(2)
$$-\Psi_3'(t) = \begin{cases} 0 & (0 \le t \le \sqrt{3}) \\ \frac{12}{\pi}\left(\frac{t}{\sqrt{3}} - \sqrt{4-t^2}\right) & (\sqrt{3} \le t \le 2) \\ \frac{12}{\pi^2}\left(t\sqrt{3}\arccos\left(\frac{t+3\sqrt{t^2-4}}{4\sqrt{t^2-3}}\right) + \frac{3}{2}\sqrt{t^2-4}\log\left(\frac{t^2-4}{t^2-3}\right)\right) & (2 \le t). \end{cases}$$

For arbitrary $d \ge 3$, the limit distribution $\Psi_d(R)$ is given by the distribution of the covering radius of the simplex $\Delta = \{\boldsymbol{x} \in \mathbb{R}^{d-1}_{\ge 0} : \boldsymbol{x}\cdot\boldsymbol{e} \le 1\}$, $\boldsymbol{e} := (1,1,\dots,1)$, with respect to a random lattice in $\mathbb{R}^{d-1}$ [7]. Here, the *covering radius* (sometimes also called *inhomogeneous minimum*) of a set $K \subset \mathbb{R}^{d-1}$ with respect to a lattice $\mathcal{L} \subset \mathbb{R}^{d-1}$ is defined as the infimum of all $\rho > 0$ with the property that $\mathcal{L} + \rho K = \mathbb{R}^{d-1}$. To state this result precisely, let $\mathbb{Z}^{d-1}A$ be a lattice in $\mathbb{R}^{d-1}$ with $A \in G_0 := \mathrm{SL}(d-1,\mathbb{R})$. The *space of lattices* (of unit covolume) is $\Gamma_0\backslash G_0$ with $\Gamma_0 := \mathrm{SL}(d-1,\mathbb{Z})$. We denote by μ_0 the unique G_0-right invariant probability measure on $\Gamma_0\backslash G_0$.

Theorem 2 ([7]). *Let $\rho(A)$ be the covering radius of the simplex Δ with respect to the lattice $\mathbb{Z}^{d-1}A$. Then $\Psi_d(R) = \mu_0\big(\{A \in \Gamma_0\backslash G_0 : \rho(A) > R\}\big)$.*

The connection between Frobenius numbers and lattice free simplices is well understood [6], [12]. In particular, Theorem 2 connects nicely to the sharp lower bound of [1] (see also [11]): $F(\boldsymbol{a}) + \boldsymbol{e}\cdot\boldsymbol{a} \ge \rho_*(a_1\cdots a_d)^{1/(d-1)}$, with $\rho_* := \inf_{A\in\Gamma_0\backslash G_0}\rho(A)$. It is proved in [1] that $\rho_* > ((d-1)!)^{1/(d-1)} > 0$, and so in particular $\Psi_d(R) = 1$ for $0 \le R < \rho_*$.

Horospheres. Let $G := \mathrm{SL}(d,\mathbb{R})$ and $\Gamma := \mathrm{SL}(d,\mathbb{Z})$, and define

(3)
$$n_+(\boldsymbol{x}) = \begin{pmatrix} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{x} & 1 \end{pmatrix}, \qquad n_-(\boldsymbol{x}) = \begin{pmatrix} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{x} \\ \boldsymbol{0} & 1 \end{pmatrix}, \qquad \Phi^t = \begin{pmatrix} \mathrm{e}^{-t}1_{d-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & \mathrm{e}^{(d-1)t} \end{pmatrix}.$$

The right action $\Gamma\backslash G \to \Gamma\backslash G$, $\Gamma M \mapsto \Gamma M \Phi^t$, defines a flow on the space of lattices $\Gamma\backslash G$. The horospherical subgroups generated by $n_+(\boldsymbol{x})$ and $n_-(\boldsymbol{x})$ parametrize the stable and unstable directions of the flow Φ^t as $t \to \infty$. Let us now identify a function W_δ on $\Gamma\backslash G$ that, when evaluated along a specific orbit of the flow Φ^t, produces the Frobenius number. Brauer and Shockley [5] proved that $F(\boldsymbol{a}) = \max_{r \bmod a_d} N_r(\boldsymbol{a}) - a_d$, where N_r is the smallest positive integer that has a representation in $r \bmod a_d$. A short calculation shows that

$$(4)\quad N_r(\boldsymbol{a}) = \begin{cases} a_d & (r \equiv 0 \bmod a_d) \\ \min\{\boldsymbol{m}'\cdot\boldsymbol{a}' : \boldsymbol{m}' \in \mathbb{Z}^{d-1}_{\ge 0},\ \boldsymbol{m}'\cdot\boldsymbol{a}' \equiv r \bmod a_d\} & (r \not\equiv 0 \bmod a_d) \end{cases}$$

with $\boldsymbol{a}' = (a_1,\dots,a_{d-1})$. This formula is the starting point in [7] of the construction of the function $W_\delta : \mathbb{R}^{d-1}_{\ge 0} \times G \to \mathbb{R}$, $(\boldsymbol{\alpha}, M) \mapsto W_\delta(\boldsymbol{\alpha}, M)$, given by

$$(5)\qquad W_\delta(\boldsymbol{\alpha}, M) = \sup_{\boldsymbol{\xi}\in\mathbb{T}^d} \min\nolimits_+ \big\{(\boldsymbol{m}+\boldsymbol{\xi})M\cdot(\boldsymbol{\alpha},0) : \boldsymbol{m}\in\mathbb{Z}^d,\ (\boldsymbol{m}+\boldsymbol{\xi})M \in \mathcal{R}_\delta\big\}$$

where $\mathcal{R}_\delta = \mathbb{R}_{\geq 0}^{d-1} \times (-\delta, \delta)$. Note that for every $\gamma \in \Gamma$, we have $W_\delta(\boldsymbol{\alpha}, \gamma M) = W_\delta(\boldsymbol{\alpha}, M)$, and thus W_δ can be viewed as a function on $\mathbb{R}_{\geq 0}^{d-1} \times \Gamma\backslash G$. The relation with the Frobenius number is as follows:

Theorem 3. *Let* $\boldsymbol{a} = (a_1, \ldots, a_d) \in \widehat{\mathbb{Z}}_{\geq 2}^d$ *with* $a_1, \ldots, a_{d-1} \leq a_d \leq \mathrm{e}^{(d-1)t}$, *and* $0 < \delta \leq \frac{1}{2}$. *Then* $F(\boldsymbol{a}) = \mathrm{e}^t W_\delta\big(\boldsymbol{a}', n_-(\widehat{\boldsymbol{a}})\Phi^t\big) - \boldsymbol{e} \cdot \boldsymbol{a}$, *where* $\widehat{\boldsymbol{a}} := \frac{\boldsymbol{a}'}{a_d} = \big(\frac{a_1}{a_d}, \ldots, \frac{a_{d-1}}{a_d}\big)$.

By exploiting standard probabilistic arguments [7], Theorem 1 now follows from Theorem 3 and the below equidistribution theorem for Farey fractions on a certain embedded submanifold of the space of lattices $\Gamma\backslash G$.

Farey fractions. Denote by $\mu = \mu_G$ the Haar measure on $G = \mathrm{SL}(d, \mathbb{R})$, normalized so that it represents the unique right G-invariant probability measure on the homogeneous space $\Gamma\backslash G$, where $\Gamma = \mathrm{SL}(d, \mathbb{Z})$. We will use the notation μ_0 for the right G_0-invariant probability measure on $\Gamma_0\backslash G_0$, with $G_0 = \mathrm{SL}(d-1, \mathbb{R})$ and $\Gamma_0 = \mathrm{SL}(d-1, \mathbb{Z})$ Consider the subgroups $H = \left\{ \begin{pmatrix} A & {}^{\mathrm{t}}\boldsymbol{b} \\ \boldsymbol{0} & 1 \end{pmatrix} : A \in G_0,\ \boldsymbol{b} \in \mathbb{R}^{d-1} \right\}$ and $\Gamma_H = \Gamma \cap H$. We normalize the Haar measure μ_H of H so that it becomes a probability measure on $\Gamma_H\backslash H$; explicitly: $d\mu_H(M) = d\mu_0(A)\, d\boldsymbol{b}$.

Let us denote the Farey sequence of level Q by

$$\mathcal{F}_Q = \left\{ \frac{\boldsymbol{p}}{q} \in [0,1)^{d-1} : (\boldsymbol{p}, q) \in \widehat{\mathbb{Z}}^d,\ 0 < q \leq Q \right\}. \tag{6}$$

Note that $|\mathcal{F}_Q| \sim \frac{Q^d}{d\,\zeta(d)}$ as $Q \to \infty$.

Theorem 4 ([7])**.** *Let* $f : \mathbb{T}^{d-1} \times \Gamma\backslash G \to \mathbb{R}$ *be bounded continuous. Then, for* $Q = \mathrm{e}^{(d-1)t}$,

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{|\mathcal{F}_Q|} \sum_{\boldsymbol{r} \in \mathcal{F}_Q} f\big(\boldsymbol{r}, n_-(\boldsymbol{r})\Phi^t\big) \\ = d(d-1) \int_0^\infty \int_{\mathbb{T}^{d-1}\times\Gamma_H\backslash H} \widetilde{f}(\boldsymbol{x}, M\Phi^{-s})\, d\boldsymbol{x}\, d\mu_H(M)\, \mathrm{e}^{-d(d-1)s} ds \end{aligned} \tag{7}$$

with $\widetilde{f}(\boldsymbol{x}, M) := f(\boldsymbol{x}, {}^{\mathrm{t}}M^{-1})$.

This statement can be established as a consequence of the mixing property of the flow Φ^t on $\Gamma\backslash G$, see [7] for details. It is interesting to note that, if one replaces $\Gamma = \mathrm{SL}(d, \mathbb{Z})$ with a lattice Γ not commensurable with $\mathrm{SL}(d, \mathbb{Z})$, the Farey sequence becomes uniformly distributed in all of $\Gamma\backslash G$ with respect to Haar measure [8].

Open problems. In the case $d = 2$ the proof of Theorem 4 is very simple. In fact one can prove a stronger statement on the equidistribution of rationals with denominator $= q$. For every bounded continuous $f : \mathbb{T} \times \mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}$ ($\mathbb{H}$ is the upper half plane, and $\mathrm{SL}(2,\mathbb{R})$ acts by fractional linear transformations)

$$\lim_{q\to\infty}\frac{1}{\varphi(q)}\sum_{\substack{p=1\\ \gcd(p,q)=1}}^{q-1} f\left(\frac{p}{q},\frac{p}{q}+\mathrm{i}\frac{\sigma}{q^2}\right)=\int_0^1\int_0^1 f(\xi,x+\mathrm{i}\sigma^{-1})\,d\xi\,dx, \tag{8}$$

where $\varphi(q)$ is Euler's totient function. To prove this notice that $\frac{p}{q}+\mathrm{i}\frac{\sigma}{q^2}$ is mapped by a suitable element from $\mathrm{SL}(2,\mathbb{Z})$ to the point $-\frac{\overline{p}}{q}+\mathrm{i}\frac{1}{\sigma}$, where $\overline{p}$ denotes the inverse of p mod q. Eq. (8) then follows from Fourier expanding f and applying standard bounds on Kloosterman sums. In analogy with the Corollary of Theorem 2 in [8], I conjecture that for every $\alpha\notin\mathbb{Q}$ and f as above,

$$\lim_{q\to\infty}\frac{1}{\varphi(q)}\sum_{\substack{p=1\\ \gcd(p,q)=1}}^{q-1} f\left(\frac{p}{q},\alpha\frac{p}{q}+\mathrm{i}\frac{\sigma}{q^2}\right)=\frac{3}{\pi}\int_0^1\int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}} f(\xi,x+\mathrm{i}y)\,d\xi\,\frac{dx\,dy}{y^2}. \tag{9}$$

(Here $\pi/3$ is the area of the modular surface $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$.) It is not hard to see that for bounded continuous f, eq. (9) implies

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f\left(\frac{n}{N},\alpha\frac{n}{N}+\mathrm{i}\frac{\sigma}{N^2}\right)=\frac{3}{\pi}\int_0^1\int_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}} f(\xi,x+\mathrm{i}y)\,d\xi\,\frac{dx\,dy}{y^2}. \tag{10}$$

If (10) could be shown also for unbounded continuous functions with $|f(\xi,x+\mathrm{i}y)|\leq Cy^{1/2}$ for all $y\geq 1$ (presumably under some additional diophantine condition on α), then (10) would imply that the pair correlation function of the fractional parts of $n^2\alpha/N$ converges to that of independent random variables (see [9] for details of the analogous argument for the fractional parts of $n^2\alpha$). This in turn would prove a special instance of the Berry-Tabor conjecture in quantum chaos for the eigenvalues of the "boxed oscillator" [3, 15]!

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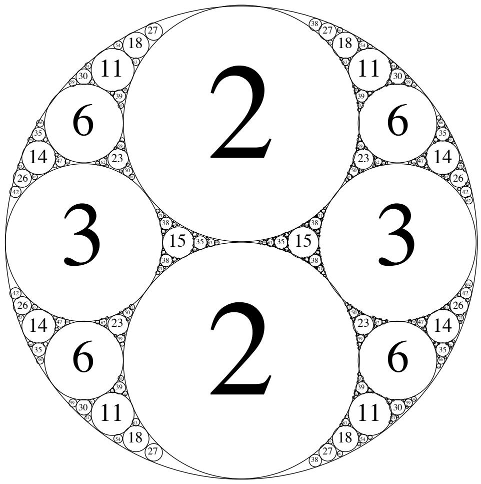

FIGURE 1. An Apollonian circle packing labeled by curvatures.

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Distribution of circles in Apollonian circle packings and beyond

HEE OH

Given a set of four mutually tangent circles in the plane $\mathbb{C}$ with distinct points of tangency, one can construct four new circles, each of which is tangent to three of the given ones. Continuing to repeatedly fill the interstices between mutually tangent circles with further tangent circles, we obtain an infinite circle packing, called an *Apollonian circle packing*, after the great geometer Apollonius of Perga (262-190 BC).

Let $\mathcal{P}$ be an Apollonian circle packing. For $\mathcal{P}$ bounded and $T > 0$, denote by $N_T(\mathcal{P})$ the number of circles in $\mathcal{P}$ whose curvature (=the reciprocal of its radius) is at most T. Note that $N_T(\mathcal{P}) = \infty$ for a general unbounded packing. However in the special case of unbounded packing $\mathcal{P}$ which lies between two parallel lines, the altered definition of $N_T(\mathcal{P})$ to count circles in a fixed period is a well-defined finite number for any $T > 0$.