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PERFECT RETROREFLECTORS AND BILLIARD DYNAMICS

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ABSTRACT. We construct semi-infinite billiard domains which reverse the direction of most incoming particles. We prove that almost all particles will leave the open billiard domain after a finite number of reflections. Moreover, with high probability the exit velocity is exactly opposite to the entrance velocity, and the particle's exit point is arbitrarily close to its initial position. The method is based on asymptotic analysis of statistics of entrance times to a small interval for irrational circle rotations. The rescaled entrance times have a limiting distribution in the limit when the length of the interval vanishes. The proof of the main results follows from the study of related limiting distributions and their regularity properties.

1. INTRODUCTION

The present paper is motivated by the problem of constructing open billiard domains with exact velocity reversal (EVR), which means that the velocity of every incoming particle is reversed when the particle eventually leaves the domain. This problem arises in the construction of perfect retroreflectors—optical devices that exactly reverse the direction of an incident beam of light and preserve the original image. A well-known example of a perfect retroreflector is the Eaton lens [4, 14] which is a spherically symmetric lens that, unlike our model, also reverses the original image. A second application lies in the search for domains that maximize the pressure of a flow of particles [12]: for a particle of mass $m > 0$, which moves towards a wall with velocity $\bar{v}$, the impulse transmitted to the wall at the moment of reflection is equal to $2m|\bar{v}_n|$, where $\bar{v}_n$ is the normal component of $\bar{v}$. It is maximized when $\bar{v} = \bar{v}_n$, *i.e.,* when the direction of the particle is reversed.

We construct a family of domains $D_{\epsilon,\sigma}$ for which EVR holds up to a set of initial condition whose measure tends to zero in the limit $\epsilon \to 0$. The domain

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$D_{\epsilon,\sigma}$ is the semi-infinite tube $[0,\infty)\times[0,1]$ with vertical barriers of height $\epsilon/2$ at the points $(\sigma n,0)$ and $(\sigma n,1)$, with $n\in\mathbb{N}$, as illustrated in Figure 1, where $\sigma>0$ denotes the spacing between the barriers. Inside the domain the particle moves with constant speed and elastic reflections from the boundary. Since the kinetic energy of the particle is preserved, we can assume without loss of generality that the speed of the particle is equal to one. The particle enters the tube at $x_{\text{in}}=0$, $y_{\text{in}}\in[0,1]$ with initial velocity $v_{\text{in}}=(\cos(\pi\varphi),\sin(\pi\varphi))$, where $\varphi\in[-1/2,1/2]$. Define $\Omega=[0,1]\times[-1/2,1/2]$.

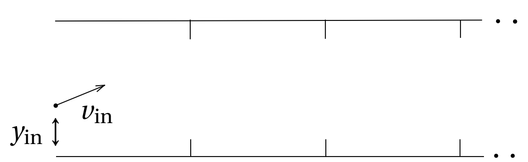

FIGURE 1. The billiard domain $D_{\epsilon,\sigma}$: a semi-infinite tube with regularly spaced vertical barriers of height $\epsilon/2$

THEOREM 1.1. *For every $\epsilon\in(0,1)$ there exists a set $\Omega(\epsilon)\subset\Omega$ of full Lebesgue measure, such that for every $(y_{\text{in}},\varphi)\in\Omega(\epsilon)$, the particle eventually leaves the tube.*

We now consider random initial conditions $(y_{\text{in}},\varphi)\in\Omega$ with respect to a fixed Borel probability measure $\mathbb{P}$. We assume throughout this paper that $\mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure on Ω, but otherwise arbitrary. The position and the velocity with which it leaves the tube are denoted by $(y_{\text{out}},v_{\text{out}})$. By Theorem 1.1, for every $\epsilon\in(0,1)$ the functions $y_{\text{out}}=y_{\text{out}}(y_{\text{in}},\varphi)$ and $v_{\text{out}}=v_{\text{out}}(y_{\text{in}},\varphi)$ are defined $\mathbb{P}$-almost everywhere.

THEOREM 1.2. *For any $\delta>0$,*

$$\mathbb{P}\{(y_{\text{in}},\varphi)\colon v_{\text{out}}=-v_{\text{in}},\,|y_{\text{out}}-y_{\text{in}}|<\delta\}\to 1 \qquad \textit{as } \epsilon\to 0. \tag{1.1}$$

Theorem 1.2 follows from the results on the existence of certain limiting distributions for the exit statistics of the billiard particle as $\epsilon\to 0$. Below we formulate these results as Theorem 1.3(ii) and Theorem 1.4(ii). In the last section of the paper, we show how they imply Theorem 1.2. The relevant statistics are $\mathcal{Q}_{\epsilon,\sigma}=\mathcal{Q}_{\epsilon,\sigma}(y_{\text{in}},\varphi)$, the number of reflections from the vertical walls before the particle leaves the tube, and $T_{\epsilon,\sigma}=T_{\epsilon,\sigma}(y_{\text{in}},\varphi)$, the time that particle spends inside the tube. By Theorem 1.1, both $\mathcal{Q}_{\epsilon,\sigma}$ and $T_{\epsilon,\sigma}$ are finite $\mathbb{P}$-a.e.

To prove our results it is natural to consider the bi-infinite tubular domain obtained by extending the semi-infinite tube described above. It consists of a strip of width one and a σ-periodic configuration of vertical walls of height $\epsilon/2$ at $(\sigma n,0)$, $(\sigma n,1)$ with $n\in\mathbb{Z}$. Let x be the horizontal coordinate and assume that the particle starts at $x=0$ with random $(y_{\text{in}},\varphi)\in\Omega$ as above. We denote by $\xi^k_{\epsilon,\sigma}\in\sigma\mathbb{Z}$ the x-coordinate of the particle at the moment of kth reflection from a vertical wall. Since the tube is now bi-infinite, $\{\xi^k_{\epsilon,\sigma}\}$ is a discrete time process on $\sigma\mathbb{Z}$, defined for any $k\in\mathbb{N}$. We also define a continuous version of this process: $\{\xi_{\epsilon,\sigma}(t)\}$ is the projection of the trajectory of a billiard particle in the bi-infinite tube to the x-axis. We rescale the velocity of the original particle in such a way that the horizontal coordinate $\xi_{\epsilon,\sigma}(t)$ has speed σ/ϵ.

THEOREM 1.3. *Fix $\sigma > 0$.*

(i) *The process $\{\frac{\epsilon}{\sigma}\xi^k_{\epsilon,\sigma}\}$ converges, as $\epsilon \to 0$, in distribution (with respect to $\mathbb{P}$) to a stochastic process $\{\xi^k\}$.*
(ii) *There exists a limiting probability distribution function $G\colon \mathbb{N} \to [0,1]$ such that for every $k \in \mathbb{N}$,*

$$\lim_{\epsilon\to 0}\mathbb{P}\{\mathcal{Q}_{\epsilon,\sigma}(y_{\text{in}},\varphi) = k\} = G(k).$$

The limits $\{\xi^k\}$ and G do not depend on the choice of σ and $\mathbb{P}$.

The second part of Theorem 1.3 says that for the limiting stochastic process $\{\xi^k\}$, with probability one there exists $k \in \mathbb{N}$, such that $\xi^k < 0$. Analogous results are true for the continuous process $\{\xi_{\epsilon,\sigma}(t)\}$.

THEOREM 1.4. *Fix $\sigma > 0$.*

(i) *The process $\{\frac{\epsilon}{\sigma}\cdot\xi_{\epsilon,\sigma}(t)\}$ converges, as $\epsilon \to 0$, in distribution with respect to $\mathbb{P}$ to a stochastic process $\xi(t)$.*
(ii) *There exists a limiting probability distribution function $H\colon \mathbb{R}_{\geq 0} \to [0,1]$, such that for every $t \geq 0$,*

$$\lim_{\epsilon\to 0}\mathbb{P}\{\epsilon\cos(\pi\varphi)T_{\epsilon,\sigma}(y_{\text{in}},\varphi) < \sigma t\} = H(t).$$

The limits $\{\xi(t)\}$ and H do not depend on the choice of σ and $\mathbb{P}$.

Note that the above rescaling of time by a factor of $\cos(\pi\varphi)$ in (ii) corresponds in (i) to normalizing the horizontal component of the particle velocity to σ/ϵ.

Our model is of course a special example of a general class of infinite periodic polygonal billiard tables. We refer the interested reader in particular to the recent study of the Ehrenfest model [5], which appeared shortly after the first version of this paper on the arXiv.

2. REDUCTION TO CIRCLE ROTATIONS AND POINTWISE EXITS

We will begin by reformulating the problem in terms of circle rotations. Let us identify $[0,1)$ with $S^1 = \mathbb{R}/\mathbb{Z}$. For $\alpha \in \mathbb{R}$, let $R_\alpha\colon S^1 \to S^1$ be the circle rotation by angle α:

$$R_\alpha x = x + \alpha \mod 1.$$

We assume in the following that $\alpha \in \mathbb{R}\smallsetminus\mathbb{Q}$. Let $I_\epsilon = [-\epsilon/2, \epsilon/2] \subset S^1$. We define several sequences measuring the return times to the interval I_ϵ, which will be used throughout the proofs. The hitting times $m^k_\epsilon = m^k_\epsilon(x,\alpha)$, $k = 0,1,2,\ldots$, are defined for $x \in S^1$ by $m^0_\epsilon = 0$ and for $k \geq 1$ inductively as

$$m^k_\epsilon = \min\{l > m^{k-1}_\epsilon : R^l_\alpha x \in I_\epsilon\}.$$

The sequence $n_\epsilon^k = n_\epsilon^k(x,\alpha)$, $k = 1,2,\dots$, of *relative* return times to the interval I_ϵ is defined for $x \in \mathrm{S}^1$ by

$$n_\epsilon^k = m_\epsilon^k - m_\epsilon^{k-1}.$$

Recall the sequence $\{\xi_{\epsilon,\sigma}^k\}$ defined in the introduction as the sequence of the horizontal coordinates of points of the reflection from the vertical walls. Note that if $x = y_{\rm in}$, and $\alpha = \sigma\tan(\pi\varphi) \bmod 1$, then $\sigma n_\epsilon^i(x,\alpha)$ is the distance between horizontal coordinate of the place of the $(i-1)$st and the ith reflections of the particle from vertical walls. Therefore,

$$\xi_{\epsilon,\sigma}^k = \sigma\big[n_\epsilon^1 - n_\epsilon^2 + \cdots + (-1)^{k+1} n_\epsilon^k\big],$$

and

$$\mathcal{Q}_{\epsilon,\sigma} = \min\big\{j \in \mathbb{N}\colon n_\epsilon^1 - n_\epsilon^2 + \cdots + (-1)^{j+1} n_\epsilon^j \le 0\big\} - 1.$$

Let $\bar n_\epsilon^k = (n_\epsilon^1,\dots,n_\epsilon^k)^\top$, $\bar m_\epsilon^k = (m_\epsilon^1,\dots,m_\epsilon^k)^\top$ and $\bar\xi_\epsilon^k = (\xi_{\epsilon,\sigma}^1,\dots,\xi_{\epsilon,\sigma}^k)^\top$. Then,

$$\bar\xi_\epsilon^k = \sigma\mathbf{A}\bar n_\epsilon^k, \qquad \bar m_\epsilon^k = \mathbf{B}\bar n_\epsilon^k, \tag{2.1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are two $k\times k$ matrices with

$$\mathbf{A}_{i,j} = \begin{cases} 0 & \text{if } i<j, \\ (-1)^{j+1} & \text{if } i \ge j, \end{cases} \qquad \mathbf{B}_{i,j} = \begin{cases} 0 & \text{if } i<j, \\ 1 & \text{if } i \ge j. \end{cases}$$

The probability measure $\mathbb{P}$ on the initial conditions $(y_{\rm in},\varphi) \in [0,1]\times[-1/2,1/2]$ for the billiard particle induces a probability measure on the coordinates $(x,\alpha) \in [0,1)\times[0,1) \simeq \mathbb{T}^2$ which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^2$ and which will also be denoted by $\mathbb{P}$.

We now prove Theorem 1.1. Let $\hat T_{\alpha,\epsilon}\colon I_\epsilon \to I_\epsilon$ be the map induced on I_ϵ by the circle rotation R_α,

$$\hat T_{\alpha,\epsilon}(x) = R_\alpha^{m_\epsilon^1}(x).$$

We first show that this map is weakly mixing.

PROPOSITION 2.1. *For every $\epsilon \in (0,1)$ there exists a set of full Lebesgue measure $\Lambda(\epsilon) \subset \mathrm{S}^1$, such that for every $\alpha \in \Lambda$, the map $\hat T_{\alpha,\epsilon}$ is weakly mixing.*

Proof. The proof of the proposition will follow from a combination of results of Boshernitzan [1] and Boshernitzan and Nogueira [3]. We start by providing some well-known background.

The map $\hat T_{\alpha,\epsilon}$ is an interval-exchange transformation of at most 3 intervals. In particular, for every $\epsilon > 0$ there exists a full Lebesgue measure set $\Lambda'(\epsilon) \subset \mathrm{S}^1$, such that for every $\alpha \in \Lambda'(\epsilon)$, the corresponding map $\hat T_{\alpha,\epsilon}$ has combinatorial type (3 2 1) and satisfies the infinite distinct orbit condition. The first fact is elementary, and the second can be deduced by considering orbits of the underlying rotation.

We recall properties P and P' introduced by Boshernitzan in [1] for a general interval-exchange transformation T. A set $\mathscr{A} \subset \mathbb{N}$ is said to be *essential*, if for any integer $l \ge 2$ there exists $\lambda > 1$, such that the system

$$\begin{cases} n_{i+1} > 2n_i & \text{for } 1 \le i \le l-1, \\ n_l < \lambda n_1 \\ n_i \in \mathscr{A} & \text{for } 1 \le i \le l. \end{cases}$$

has an infinite number of solutions $(n_1, n_2, \dots, n_l)$. Let $m_n(T)$ be the length of the smallest interval of continuity of T^n.

An interval-exchange transformation T has *property P*, if for some $\delta > 0$ the set $\mathscr{A}(\delta) = \{n \in \mathbb{N} \mid m_n(T) > \delta/n\}$ is essential; T has *property P′*, if for some $\delta > 0$ the set $\mathscr{A}(\delta)$ is infinite.[1]

Theorem 9.4(a) in [1] shows that, for every $\epsilon \in (0,1)$ there exist a full Lebesgue measure set $\Lambda(\epsilon) \subset \mathrm{S}^1$, such that for every $\alpha \in \Lambda$, the map $\hat{T}_{\alpha,\epsilon}$ has property P. This implies $\hat{T}_{\alpha,\epsilon}$ has property P', and Proposition 2.1 is now a direct corollary of Theorem 5.3 in [3]. □

The next two statements are well-known. We include their proofs to keep the exposition self-contained.

LEMMA 2.2. *Let T be a weakly mixing transformation on (X,μ). Then T^2 is weakly mixing.*

Proof. The map T is weakly mixing if and only if for every $f, g \in L^2(X,\mu)$ with $\int f\,d\mu = 0$, there is a subsequence $\{n_j\}$ of full density in $\mathbb{N}$ such that

$$\lim_{j\to\infty} \int f(T^{n_j}x)g(x)\,d\mu(x) = 0.$$

Because $\{n_j\}$ has full density, the sequence obtained from $\{n_j\}$ by removing all odd integers has full density in $2\mathbb{N}$. Therefore, weak mixing of T implies weak mixing of T^2. □

PROPOSITION 2.3. *Let T be an ergodic transformation on (X,μ), $\mu(X) = 1$, and let $f \in L^1(X,\mu)$, $\int f\,d\mu = 0$ and $S_n(f,x) = f(x) + f(Tx) + \cdots + f(T^{n-1}x)$ be its Birkhoff sums. Then either $S_n(f,x)$ is unbounded from below for almost every $x \in X$, or f is a coboundary,* i.e., *there exists a measurable $g(x)$, such that $f(x) = g(x) - g(Tx)$.*

Proof. Since T is ergodic, the set of points x for which $S_n(f,x)$ is bounded from below has measure either equal to zero or one. In the first case, the proposition is proved, so assume that it has measure one. Then the function $g(x) = \liminf_{n\ge 1} S_n(f,x)$ is the desired coboundary. □

Proof of Theorem 1.1. In view of Proposition 2.1 and Lemma 2.2, for every $\alpha \in \Lambda(\epsilon)$ the map $\hat{T}^2_{\alpha,\epsilon}$ is ergodic. Let $x \in I_\epsilon$ and $f(x) = n^1_\epsilon(x) - n^1_\epsilon(\hat{T}_{\alpha,\epsilon}x)$. Then the

[1] Boshernitzan showed in [1] that for minimal T, property P implies unique ergodicity. He conjectured in the same study that in fact the weaker property P' should suffice. This was subsequently proved by Veech [15], with a simpler proof supplied in [2].

Birkhoff sums for $\hat{T}^2_{\alpha,\epsilon}$ and $f(x)$ are

$$S_m(f,x) = f(x) + f(\hat{T}^2_{\alpha,\epsilon}x) + f(\hat{T}^4_{\alpha,\epsilon}x) + \cdots + f(\hat{T}^{2m}_{\alpha,\epsilon}x)$$
$$= n^1_\epsilon(x) - n^2_\epsilon(x) + \cdots - n^{2m}_\epsilon(x).$$

By Proposition 2.3, for Lebesgue almost every $x \in I_\epsilon$, either there exists $m_0 \in \mathbb{N}$, such that $S_{m_0}(f,x) \le 0$ (and therefore $\mathcal{Q}_{\epsilon,\sigma}(x,\varphi) < \infty$) or $f(x)$ is a coboundary. But in the second case, $S_m(f,x) = g(x) - g(\hat{T}^{2m+2}_{\alpha,\epsilon}x)$ for a measurable $g(x)$. Either $g(x) < \operatorname{ess\,sup} g(x)$, or $g(x) = \operatorname{ess\,sup} g(x)$ on a positive Lebesgue measure set. In either case, the ergodicity of $\hat{T}^2_{\alpha,\epsilon}$, implies that for Lebesgue-a.e. x, there exists $m_0 \in \mathbb{N}$, such that $S_{m_0}(f,x) \le 0$ and so $\mathcal{Q}_{\epsilon,\sigma}(x,\varphi) < \infty$.

Now let $x \in \mathrm{S}^1 \setminus I_\epsilon$. Since $\alpha \notin \mathbb{Q}$, there exists $n_0 > 0$, such that $R_\alpha^{-n_0}x \in I_\epsilon$. Then,

$$\mathcal{Q}_{\epsilon,\sigma}(x,\varphi) \le \mathcal{Q}_{\epsilon,\sigma}(\hat{T}_\alpha^{-n_0}x,\varphi) < \infty$$

for Lebesgue almost every $x \in \mathrm{S}^1$. □

3. LIMITING DISTRIBUTIONS

Let us now turn to the proof of Theorems 1.3 and 1.4.

3.1. **Notations and the formulation of the main limiting distribution result.** In the following, we denote by $\mathbb{P}$ an arbitrary Borel probability measure on $\mathbb{T}^2$, absolutely continuous with respect to Lebesgue measure. Let

$$F^{(n)}_\epsilon(t_1,\ldots,t_n) = \mathbb{P}\{\epsilon m^1_\epsilon > t_1, \epsilon m^2_\epsilon > t_2, \ldots, \epsilon m^n_\epsilon > t_n\}$$

be the joint distribution function of the vector $\epsilon \bar{m}^n_\epsilon = (\epsilon m^1_\epsilon, \epsilon m^2_\epsilon, \ldots, \epsilon m^n_\epsilon)^\top$. It is also convenient to introduce

$$\mathcal{N}_\epsilon(x,\alpha,T) = \#\{k \in \mathbb{Z} \cap (0,\epsilon^{-1}T] : k\alpha + x \subset I_\epsilon + \mathbb{Z}\},$$

the number of times the orbit of a rotation hits the interval I_ϵ during the time $\epsilon^{-1}T$. Note that

$$F^{(n)}_\epsilon(t_1,\ldots,t_n) = \mathbb{P}\{\mathcal{N}_\epsilon(x,\alpha,t_k) \le k-1,\ k = 1,\ldots,n\}. \tag{3.1}$$

Let χ_I denote the characteristic function of the interval $I \subset \mathbb{R}$ and $\psi_T(x,y) = \chi_{(0,1]}(x)\chi_{[-T/2,T/2]}(y)$ be the characteristic function of a corresponding rectangle. Then

$$\begin{aligned}
\mathcal{N}_\epsilon(x,\alpha,T) &= \sum_{m=1}^{[\epsilon^{-1}T]} \sum_{n\in\mathbb{Z}} \chi_{I_\epsilon}(\alpha m + n + x) \\
&= \sum_{(m,n)\in\mathbb{Z}^2} \chi_{(0,1]}\left(\frac{m}{\epsilon^{-1}T}\right) \chi_{[-T/2,T/2]}\big(\epsilon^{-1}T(\alpha m + n + x)\big) \\
&= \sum_{(m,n)\in\mathbb{Z}^2} \psi_T\left((m,n+x)\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}\begin{pmatrix}\epsilon T^{-1} & 0\\ 0 & \epsilon^{-1}T\end{pmatrix}\right).
\end{aligned}$$

Therefore,

$$\mathcal{N}_\epsilon(x,\alpha,T) = \#\left\{(m,n) \in \mathbb{Z}^2 : (m,n+x)\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon^{-1}\end{pmatrix} \in \mathcal{R}(T)\right\}, \tag{3.2}$$

where $\mathscr{R}(T) = (0, T] \times [-1/2, 1/2]$.

Let $\mathrm{ASL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ be the semidirect product group with multiplication law

$$(M, \mathbf{v})(M', \mathbf{v}') = (MM', \mathbf{v}M' + \mathbf{v}').$$

The action of an element $(M, \mathbf{v})$ of this group on $\mathbb{R}^2$ is defined by

$$\mathbf{w} \mapsto \mathbf{w}M + \mathbf{v}. \tag{3.3}$$

Each affine lattice of covolume one in $\mathbb{R}^2$ can then be represented as $\mathbb{Z}^2 g$ for some suitable $g \in \mathrm{ASL}(2,\mathbb{R})$, and the space of affine lattices is represented by $X = \mathrm{ASL}(2,\mathbb{Z}) \backslash \mathrm{ASL}(2,\mathbb{R})$, where $\mathrm{ASL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$. Denote by ν the Haar probability measure on X.

THEOREM 3.1. *As $\epsilon \to 0$, the limit of* (3.1) *exists and is equal to*

$$F^{(n)}(t_1, \ldots, t_n) = \nu\big(\big\{g \in X : \#\{\mathbb{Z}^2 g \cap \mathscr{R}(t_k)\} \le k-1\,(k=1,\ldots,n)\big\}\big), \tag{3.4}$$

which is a C^1 function $\mathbb{R}^n_{\ge 0} \to [0,1]$ and independent of the choice of $\mathbb{P}$.

We define the associated limiting probability density $\phi^{(n)}(t_1, \ldots, t_n)$ by

$$F^{(n)}(t_1, \ldots, t_n) = \int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} \phi^{(n)}(t_1, \ldots, t_n)\, dt_1 \cdots dt_n.$$

We postpone the proof of Theorem 3.1 to the end of this section.

3.2. **The reduction of Theorem 1.3 to Theorem 3.1.** Because of relation (2.1), Theorem 3.1 implies the convergence in distribution for the sequences $\{\epsilon n_\epsilon^k\}$ and $\{\frac{\epsilon}{\sigma}\xi^k_{\epsilon,\sigma}\}$ (part (i) of Theorem 1.3).

Indeed, let $k \in \mathbb{N}$ and $I_1, \ldots, I_k$ be a collection of k intervals on the real line. Let $I = I_1 \times \cdots \times I_k \subset \mathbb{R}^k$. Then,

$$\begin{aligned}\lim_{\epsilon\to 0} \mathbb{P}\big\{\epsilon n_\epsilon^1 \in I_1, \ldots, \epsilon n_\epsilon^k \in I_k\big\} &= \lim_{\epsilon\to 0} \mathbb{P}\big\{\epsilon \bar{m}_\epsilon^k \in \mathbf{B} I\big\} \\ &= \int_{\mathbf{B} I} \phi^{(k)}(t_1, \ldots, t_k)\, dt_1 \cdots dt_k\end{aligned}$$

and

$$\begin{aligned}\lim_{\epsilon\to 0} \mathbb{P}\big\{\tfrac{\epsilon}{\sigma}\xi^1_{\epsilon,\sigma} \in I_1, \ldots, \tfrac{\epsilon}{\sigma}\xi^k_{\epsilon,\sigma} \in I_k\big\} &= \lim_{\epsilon\to 0} \mathbb{P}\big\{\epsilon \bar{m}_\epsilon^k \in \mathbf{B}\mathbf{A}^{-1} I\big\} \\ &= \int_{\mathbf{B}\mathbf{A}^{-1} I} \phi^{(k)}(t_1, \ldots, t_k)\, dt_1 \cdots dt_k.\end{aligned}$$

As for part (ii) of Theorem 1.3, the convergence for the random variable $\mathscr{Q}_{\epsilon,\sigma}$ also follows from Theorem 3.1. Indeed, for any $k \ge 1$ let χ_{A_k} be the characteristic function of the set

$$\Delta_k = \big\{(y_1, \ldots, y_k) \in \mathbb{R}^k : \; y_1 > 0, \ldots, y_{k-1} > 0, y_k < 0\big\}.$$

Then for every $\epsilon > 0$ we have

$$\mathscr{Q}_{\epsilon,\sigma} = \min\big\{j \in \mathbb{Z}_+ : \; \xi^j_{\epsilon,\sigma} \le 0\big\} - 1.$$

Therefore,

$$\begin{aligned}\mathbb{P}\{\mathcal{Q}_{\epsilon,\sigma}=k\} &= \mathbb{P}\big\{\tfrac{\epsilon}{\sigma}\xi^1_{\epsilon,\sigma}>0,\dots,\tfrac{\epsilon}{\sigma}\xi^{k-1}_{\epsilon,\sigma}>0,\tfrac{\epsilon}{\sigma}\xi^k_{\epsilon,\sigma}\le 0\big\}\\ &= \mathbb{P}\big\{\epsilon\bar{m}^k_\epsilon\in\mathbf{BA}^{-1}\Delta_k\big\}\\ &= \int_{\mathbf{BA}^{-1}\Delta_k} dF^{(k)}_\epsilon,\end{aligned}$$

and by Theorem 3.1 and the Helly–Bray Theorem [7, p.183], the following limit exists

$$G(k)=\lim_{\epsilon\to 0}\mathbb{P}\{\mathcal{Q}_{\epsilon,\sigma}=k\}=\int_{\mathbf{BA}^{-1}\Delta_k}\phi^{(k)}(t_1,\dots,t_k)\,dt_1\cdots dt_k.$$

Notice that this representation implies that $\sum_{k=1}^\infty G(k)\le 1$.

PROPOSITION 3.2.

$$\sum_{k=1}^\infty G(k)=1. \tag{3.5}$$

Proof. Let $\{\eta^k\}$ be the limiting process for the sequence $\{\epsilon n^k_\epsilon\}$. From the explicit description (3.4) of the limiting distribution in Theorem 3.1, we have the following description of the process $\{\eta^k\}$.

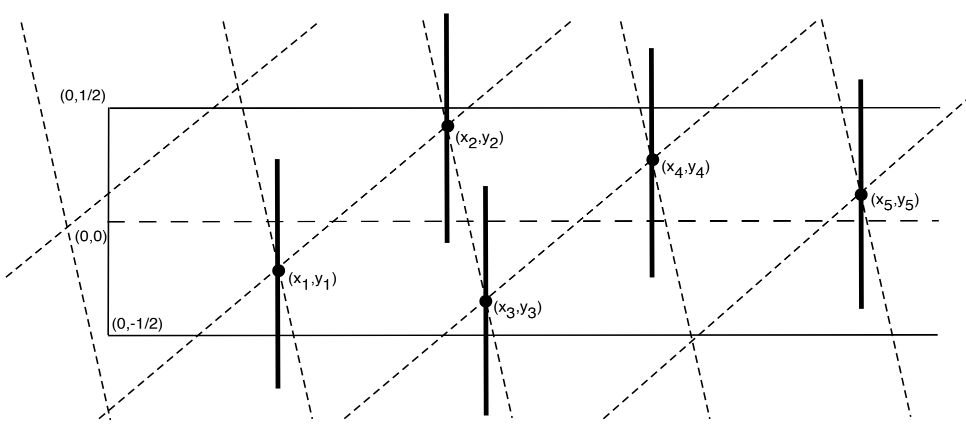

FIGURE 2. The horizontal ray through $(0,0)$ generates the sequence $\{-y_k\}$ as an orbit of an interval-exchange map

Let $g\in X$ be an affine lattice which has no points either with the same horizontal coordinates, or on the boundary of the semi-infinite tube $\mathcal{R}_\infty=[0,+\infty)\times[-1/2,1/2]$. The set of such lattices has full Haar measure in X. Let us enumerate points of g which lie in $\mathcal{R}_\infty$ according to their horizontal coordinates: if the coordinates of the kth lattice point of g in $\mathcal{R}_\infty$ are $(x_k,y_k)=(x_k(g),y_k(g))$, $k\in\mathbb{N}$, then $x_k<x_{k+1}$ for any $k\in\mathbb{N}$. Notice that ν-almost every lattice g has infinitely many points in $\mathcal{R}_\infty$.

The sequence of random variables $\epsilon n^k_\epsilon=\epsilon n^k_\epsilon(x,\alpha)$ with respect to the probability measure $\mathbb{P}$ on $\mathbb{T}^2$ converges in distribution to the sequence $\eta^1=\eta^1(g)=$

$x^1(g)$, and $\eta^k = \eta^k(g) = x_k(g) - x_{k-1}(g)$ for $k \geq 2$ with respect to Haar measure ν on X.

Therefore, in order to prove (3.5), it is enough to show that for ν-almost every affine lattice $g \in X$, there exists an even $k > 0$, such that

$$\eta^1 - \eta^2 + \eta^3 - \cdots - \eta^k = x_1 - (x_2 - x_1) + (x_3 - x_2) - \cdots - (x_k - x_{k-1}) \leq 0. \tag{3.6}$$

We will now show that the sequence $y_k(g)$ is an orbit of a certain map of an interval into itself, reduce (3.6) to a Birkhoff sum over this map and treat it in the way as in Section 2.

First, we describe the map. Consider set $\mathscr{I} \subset \mathbb{R}^2$ of vertical segments of unit length centered at every lattice point of g. We identify each segment in $\mathscr{I}$ with the base $I = [-1/2, 1/2]$ of the tube $\mathscr{R}_\infty$ by parallel translation. Let $\pi\colon \mathscr{I} \to I$ be the projection, which sends a point on some interval through a lattice point to the corresponding point in I.

Consider a unit speed flow in the positive horizontal direction on $\mathbb{R}^2$. Its first-return map to $\mathscr{I}$ is a well-defined map $\hat{T} = \hat{T}(g)$ of $\mathscr{I}$ into itself. We define the corresponding invertible map $T\colon I \to I$ in such a way that $\pi \circ \hat{T} = T \circ \pi$. It is easy to see that the map T is an exchange of three intervals. For ν-almost every lattice g it has combinatorial type (3 2 1).

For every $y \in I$, we let $\psi(y)$ to be the Euclidean distance between $\hat{y} \in \pi^{-1}(y)$ and its image under $\hat{T}$. Clearly, this does not depend on the choice of $\hat{y} \in \pi^{-1}(y)$.

Notice that the sequence $\{y_k\}$ of the vertical coordinates of the lattice points of g in $\mathscr{R}_\infty$ is related to the map T described above: for $k \in \mathbb{N}$, $y_k = -T^{k-1}(-y_1)$ (see Figure 2). Also for $k \in \mathbb{N}$, we have $\psi(-y_k) = x_{k+1} - x_k$. Let $-y_0 = T^{-1}(-y_1)$. Then the sum in (3.6) has the form (recall, k is even)

$$\begin{aligned} &x_1 - \psi(-y_1) + \psi(-y_2) - \cdots - \psi(-y_{k-1}) \\ &\qquad \leq \psi(-y_0) - \psi\big(-T(-y_0)\big) + \psi\big(T^2(-y_0)\big) - \cdots - \psi\big(T^{k-1}(-y_0)\big). \end{aligned}$$

Therefore, similar to Section 2, the alternating sum (3.6) is reduced to a Birkhoff sum for the function $f(y) = \psi(-y) - \psi(-T(-y))$ and the map T^2.

Let the lengths of the interval-exchange map T be equal to $(\lambda_1, \lambda_2, 1-\lambda_1-\lambda_2)$. Denote the simplex of possible λ_is by

$$\Lambda = \{(\lambda_1, \lambda_2) \mid \lambda_1 > 0, \lambda_2 > 0, \lambda_1 + \lambda_2 < 1\} \subset \mathbb{R}^2,$$

and the corresponding interval-exchange map of combinatorial type (3 2 1) by T_{λ_1,λ_2}. The following theorem was first proved by Katok and Stepin in [6].

THEOREM 3.3. *For Lebesgue almost every pair $(\lambda_1, \lambda_2) \in \Lambda$, the map T_{λ_1,λ_2} of the interval I onto itself is weakly mixing.*

Similar to the proof of Theorem 1.1, Theorem 3.3 and Proposition 2.3 imply that there exists a full Lebesgue measure subset $\Lambda_1 \subset \Lambda$, such that for every $(\lambda_1, \lambda_2) \in \Lambda_1$, there exists a full Lebesgue measure subset $I' = I'(\lambda_1, \lambda_2) \subset I$, such that for every $y \in I'$ there exists $k > 0$, such that

$$\psi(-y_0) - \psi\big(-T_{\lambda_1,\lambda_2}(-y_0)\big) + \psi\big(T^2_{\lambda_1,\lambda_2}(-y_0)\big) - \cdots - \psi\big(T^{k-1}_{\lambda_1,\lambda_2}(-y_0)\big) \le 0.$$

Let $\tilde{X} \subset X$ be the set of lattices, for which the construction above gives an interval-exchange transformation of combinatorial type (3 2 1). Then $\tilde{X}$ is open and $\nu(\tilde{X}) = 1$. Notice that for any $g \in \tilde{X}$, the map $\mathscr{X}\colon g \mapsto (\lambda_1, \lambda_2, y_0)$ is differentiable and its differential is surjective. Therefore, the preimage of any Lebesgue measure zero set under $\mathscr{X}$ has Haar measure zero in X. Therefore, the set of lattices $g \in X$, such that $\mathscr{X}(g) \in \{(\lambda_1, \lambda_2, y_0) \mid (\lambda_1, \lambda_2) \in \Lambda_1,\ y_0 \in I'(\lambda_1, \lambda_2)\}$ has full Haar measure in X and so (3.5) is proved. □

REMARK 3.4. The condition (3.5) is equivalent to the tightness of the family of distributions $\{\mathcal{Q}_{\epsilon,\sigma}\}$ as $\epsilon \to 0$. Namely, for any $\delta > 0$ there exists $N = N(\delta)$ and $\epsilon_1 = \epsilon_1(\delta)$ such that

$$1 - \delta \le \sum_{k=1}^{N} \mathbb{P}\{\mathcal{Q}_{\epsilon,\sigma} = k\} \le 1 \tag{3.7}$$

for $\epsilon < \epsilon_1$.

3.3. **Continuous case.**

PROPOSITION 3.5. *For any $s > 0$ and $\delta > 0$, there exists $\epsilon_0 > 0$ and $k \in \mathbb{N}$, such that*

$$\mathbb{P}\big\{\epsilon m_\epsilon^k \le s\big\} < \delta$$

for all $\epsilon < \epsilon_0$.

Proof. We have

$$\mathbb{P}\big\{\epsilon m_\epsilon^k \le s\big\} = \mathbb{P}\{\mathcal{N}_\epsilon(x, \alpha, s) \ge k\}.$$

The limit as $\epsilon \to 0$ exists and, in view of [9, p.1131, first equation], is bounded above by $C_s k^{-3}$ for some constant C_s. □

Proof of Theorem 1.4. We begin with the proof of part (i). For any integer $N \in \mathbb{N}$ and intervals $I_1, \ldots, I_N \subset \mathbb{R}$,

$$\begin{aligned}\mathbb{P}\big\{&\tfrac{\epsilon}{\sigma}\xi_{\epsilon,\sigma}(s_1) \in I_1, \ldots, \tfrac{\epsilon}{\sigma}\xi_{\epsilon,\sigma}(s_N) \in I_N\big\} \\ &= \sum_{\bar{k} \in \mathbb{Z}^N_{\ge 0}}^{\infty} \mathbb{P}\big\{\tfrac{\epsilon}{\sigma}\xi_{\epsilon,\sigma}(s_j) \in I_j, \epsilon m_\epsilon^{k_j} \le s_j < \epsilon m_\epsilon^{k_{j+1}}\ (j = 1, \ldots, N)\big\}.\end{aligned}$$

Notice that

$$\tfrac{\epsilon}{\sigma}\xi_{\epsilon,\sigma}(s) = \begin{cases} s & \text{if } 0 \le s < \epsilon m_\epsilon^1, \\ \tfrac{\epsilon}{\sigma}\xi^k_{\epsilon,\sigma} + (-1)^k (s - \epsilon m_\epsilon^k) & \text{if } \epsilon m_\epsilon^k \le s < \epsilon m_\epsilon^{k+1}, \end{cases}$$

and

$$\xi^k_{\epsilon,\sigma} = \sigma \sum_{i=1}^{k} (-1)^{i-1}(k - i + 1) m_\epsilon^i.$$

Therefore, by Theorem 3.1, for every *fixed* $\mathbf{k} \in \mathbb{Z}^N_{\geq 0}$ we have

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\tfrac{\epsilon}{\sigma}\xi_{\epsilon,\sigma}(s_j) \in I_j, \epsilon m_\epsilon^{k_j} \leq s_j < \epsilon m_\epsilon^{k_{j+1}}\,(j=1,\dots,N)\right\} = \int_{B_{\mathbf{k}}} \phi^{(k+1)}(t_1,\dots,t_{k+1})\,dt_1\cdots dt_{k+1},$$

with $k = \max(\mathbf{k})$, and the range of integration restricted to the set

$$B_{\mathbf{k}} = \left\{(t_1,\dots,t_{k+1})\colon\, t_{k_j} \leq s_j < t_{k_j+1},\ \sum_{i=1}^{k}(-1)^{i-1}(k-i+1)\,t_i + (-1)^{k_j}(s_j - t_{k_j}) \in A_j\right\}.$$

Furthermore,

$$\begin{aligned}\sum_{\substack{\mathbf{k}\in\mathbb{Z}^N_{\geq 0}\\ \max(\mathbf{k})\geq R}}^{\infty} & \mathbb{P}\left\{\tfrac{\epsilon}{\sigma}\xi_{\epsilon,\sigma}(s_j) \in I_j, \epsilon m_\epsilon^{k_j} \leq s_j < \epsilon m_\epsilon^{k_{j+1}}\,(j=1,\dots,N)\right\}\\ &\leq \sum_{\substack{\mathbf{k}\in\mathbb{Z}^N_{\geq 0}\\ k_1\geq R}}^{\infty} \mathbb{P}\left\{\tfrac{\epsilon}{\sigma}\xi_{\epsilon,\sigma}(s_j) \in I_j, \epsilon m_\epsilon^{k_j} \leq s_j < \epsilon m_\epsilon^{k_{j+1}}\,(j=1,\dots,N)\right\}\\ &\leq \mathbb{P}\left\{\epsilon m_\epsilon^R \leq s_1\right\}.\end{aligned}$$

Part (i) of Theorem 1.4 now follows from Proposition 3.5.

For part (ii) of Theorem 1.4, we have

$$\mathbb{P}\{T_{\epsilon,\sigma} \leq t\} = \sum_{k\in\mathbb{N}} \mathbb{P}\{T_{\epsilon,\sigma} \leq t, \mathcal{Q}_{\epsilon,\sigma} = k\}.$$

Notice that if $\mathcal{Q}_{\epsilon,\sigma} = k$, then k is necessarily odd the time which a particle moving with unit speed spends in the tube is equal to

$$T_{\epsilon,\sigma} = \frac{2\sigma}{\cos(\pi\varphi)}\,(n_\epsilon^1 + n_\epsilon^3 + \cdots + n_\epsilon^k).$$

Thus,

$$\mathbb{P}\left\{\epsilon\cos(\pi\varphi)\,T_{\epsilon,\sigma} \leq \sigma s, \mathcal{Q}_{\epsilon,\sigma} = k\right\} = \mathbb{P}\left\{2\epsilon(n_\epsilon^1 + n_\epsilon^3 + \cdots + n_\epsilon^k) < s, \mathcal{Q}_{\epsilon,\sigma} = k\right\}. \tag{3.8}$$

By Theorem 3.1, for any $s > 0$ there exists joint limiting distribution of

$$\mathbb{P}\left\{\epsilon m_\epsilon^k > t_k\,(k=1,\dots,n)\right\},$$

as $\epsilon \to 0$, and therefore, the limit of (3.8) exists as well. On the other hand,

$$\mathbb{P}\{\epsilon\cos(\pi\varphi)\,T_{\epsilon,\sigma} \leq \sigma s, \mathcal{Q}_{\epsilon,\sigma} = k\} \leq \mathbb{P}\{\epsilon m_\epsilon^k \leq s\},$$

and so, Proposition 3.5 and the convergence of (3.8) imply the existence of the limit

$$H(s) = \lim_{\epsilon\to 0}\mathbb{P}\{\epsilon\cos(\pi\varphi)\,T_{\epsilon,\sigma} \leq \sigma s\}.$$

Finally, since

$$\mathbb{P}\{\epsilon\cos(\pi\varphi)T_{\epsilon,\sigma}\le\sigma s\}\ge\sum_{k=1}^{N}\mathbb{P}\{\epsilon\cos(\pi\varphi)T_{\epsilon,\sigma}\le\sigma s,\,\mathcal{Q}_{\epsilon,\sigma}=k\},$$

the tightness (3.7) implies that $H(s)\to 1$ as $s\to\infty$. This completes the proof of part (ii) of Theorem 1.4. □

3.4. **Proof of Theorem 3.1.** By (3.1), it is enough to show that for any $n\in\mathbb{N}$ and any n-tuples $(t_1,\dots,t_n)\in\mathbb{R}^n_{>0}$, $\mathbf{k}=(k_1,\dots,k_n)\in\mathbb{Z}^n_{\ge 0}$ there exists the limit

$$\begin{aligned}G^{(n)}(t_1,\dots,t_n)&=\lim_{\epsilon\to 0}\mathbb{P}\{\mathcal{N}_\epsilon(x,\alpha,t_j)=k_j,\,(j=1,\dots,n)\}\\&=\nu\big(\{g\in X\colon \#\{\mathbb{Z}^2g\cap\mathcal{R}(t_j)\}=k_j\,(j=1,\dots,n)\}\big),\end{aligned}\tag{3.9}$$

and that $G^{(n)}(t_1,\dots,t_n)$ is a C^1-function of $(t_1,\dots,t_n)$.

For $n=1$, the convergence in (3.9) was first proved by Mazel and Sinai [11]. It was later reproved and generalized by the third author [8, 9] using different methods. The proof of the convergence in (3.9) follows the one in [8]. The proof of the regularity of the limiting function is similar to the one in [10].

We reduce the convergence in (3.9) to an equidistribution result for the geodesic flow on X. The action of the geodesic flow on X is given by right action of a one-parameter subgroup of X with

$$\Phi^t=\left(\begin{pmatrix}e^{-t/2}&0\\0&e^{t/2}\end{pmatrix},(0,0)\right).$$

The unstable horocycle of the flow Φ^t on X is then parametrized by the subgroup $H=\{n_-(x,\alpha)\}_{(x,\alpha)\in\mathbb{T}^2}$ with

$$n_-(x,\alpha)=\left(\begin{pmatrix}1&\alpha\\0&1\end{pmatrix},(0,x)\right).$$

For $g\in X$ let $F_T(g)$ be equal to the number of lattice points of $\mathbb{Z}^2g$ in the rectangle $\mathcal{R}(T)$. Then by (3.2)

$$\mathcal{N}_\epsilon(x,\alpha,T)=F_T(n_-(x,\alpha)\Phi^t)$$

with $t=-2\ln(\epsilon)$.

THEOREM 3.6 ([8])**.** *For any bounded $f\colon \mathrm{ASL}(2,\mathbb{Z})\backslash\mathrm{ASL}(2,\mathbb{R})\to\mathbb{R}$ such that the discontinuities of f are contained in a set of ν-measure zero and any Borel probability measure $\mathbb{P}$ that is absolutely continuous with respect to Lebesgue measure on $[0,1)\times[0,1)$,*

$$\lim_{t\to\infty}\int_0^1\int_0^1 f(n_-(x,\alpha)\Phi^t)\,d\mathbb{P}(x,\alpha)=\int_{\mathrm{ASL}(2,\mathbb{Z})\backslash\mathrm{ASL}(2,\mathbb{R})}f\,d\nu.$$

Let

$$D(g)=\begin{cases}1&\text{if }F_{t_j}(g)=k_j,\,(j=1,\dots,n),\\0&\text{otherwise.}\end{cases}$$

Then $D(g)$ satisfies the conditions of Theorem 3.6. The convergence in (3.9) now follows from Theorem 3.6 applied to the function $D(g)$. We now prove C^1

regularity of the limiting function $G^{(n)}(t_1,\dots,t_n)$. It is enough to consider the case when all t_j are different. We also assume that all $k_j>0$. The case when some $k_j=0$ is similar.

Let $X_1=\mathrm{SL}(2,\mathbb{Z})\setminus\mathrm{SL}(2,\mathbb{R})$ be the homogeneous space of lattices of covolume one and let ν_1 be the probability Haar measure on X_1. For a given $\mathbf{y}\in\mathbb{R}^2$, let

$$X(\mathbf{y})=\{g\in X\colon \mathbf{y}\in\mathbb{Z}^2g\},$$

where the action of X on $\mathbb{R}^2$ is given by the formula (3.3). There is a natural identification of the sets $X(\mathbf{y})$ and X_1 through

$$X(\mathbf{y})=\{(M,\mathbf{y})\colon M\in X_1\}.$$

Under this identification the probability Haar measure ν_1 on X_1 induces a Borel probability measure $\nu_{\mathbf{y}}$ on $X(\mathbf{y})$.

We will need the following two results.

PROPOSITION 3.7 (Siegel's formula, [13])**.** *Let $f\in L^1(\mathbb{R}^2)$. Then,*

$$\int_{X_1}\sum_{\mathbf{k}\in\mathbb{Z}^2\smallsetminus\mathbf{0}} f(\mathbf{k}M)\,d\nu_1(M)=\int_{\mathbb{R}^2} f(x)\,dx.$$

PROPOSITION 3.8 ([10])**.** *Let $\mathscr{E}\subset X$ be any Borel set, then $\mathbf{y}\mapsto\nu_{\mathbf{y}}(\mathscr{E}\cap X(\mathbf{y}))$ is a measurable function from $\mathbb{R}^2$ to $\mathbb{R}$. If $U\subset\mathbb{R}^2$ is any Borel set such that $\mathscr{E}\subset\bigcup_{\mathbf{y}\in U}X(\mathbf{y})$, then*

$$\nu(\mathscr{E})\le\int_U\nu_{\mathbf{y}}(\mathscr{E}\cap X(\mathbf{y}))\,d\mathbf{y}.\tag{3.10}$$

Furthermore, if $X(\mathbf{y}_1)\cap X(\mathbf{y}_2)\cap\mathscr{E}=\emptyset$ for all $\mathbf{y}_1\ne\mathbf{y}_2\in U$, then equality holds in (3.10).

Notice that Propositions 3.7 and 3.8 imply that if there are two different indices $1\le i,j\le n$, such that

$$\begin{aligned}\Delta_{ij}(h_i,h_j)=\big\{g\in X\colon \big|\mathbb{Z}^2g\cap\big(\mathscr{R}(t_i)\triangle\mathscr{R}(t_i+h_i)\big)\big|>0\big\}&\\ \cap\big\{g\in X\colon \big|\mathbb{Z}^2g\cap\big(\mathscr{R}(t_j)\triangle\mathscr{R}(t_j+h_j)\big)\big|>0\big\}&\ne\emptyset,\end{aligned}$$

then, $\nu\{\Delta_{ij}(h_i,h_j)\}=\bar{o}(\|\mathbf{h}\|)$ as $\|h\|\to0$. Therefore,

$$\begin{aligned}&G^{(n)}(t_1+h_1,\dots,t_n+h_n)-G^{(n)}(t_1,\dots,t_n)\\ &=\sum_{j=1}^{n}G^{(n)}(t_1,\dots,t_{j-1},t_j+h_j,t_{j+1},\dots,t_n)-G^{(n)}(t_1,\dots,t_n)+\bar{o}(\|\mathbf{h}\|)\\ &=\sum_{j=1}^{n}\Big[\nu\big\{g\in X\colon |\mathbb{Z}^2g\cap\mathscr{R}(t_j)|=k_j-1,\,|\mathbb{Z}^2g\cap\mathscr{R}(t_j+h_j)|=k_j,\\ &\qquad\qquad |\mathbb{Z}^2g\cap\mathscr{R}(t_i)|=k_i,\,i\ne j\big\}\\ &\qquad-\nu\big\{g\in X\colon |\mathbb{Z}^2g\cap\mathscr{R}(t_j)|=k_j,\,|\mathbb{Z}^2g\cap\mathscr{R}(t_j+h_j)|=k_j+1,\\ &\qquad\qquad |\mathbb{Z}^2g\cap\mathscr{R}(t_i)|=k_i,\,i\ne j\big\}\Big]+\bar{o}(\|\mathbf{h}\|).\end{aligned}\tag{3.11}$$

Consider a single term in the expression above. Let

$$\mathscr{E}_j = \mathscr{E}_j(h_j) = \Big\{ g \in X : |\mathbb{Z}^2 g \cap \mathscr{R}(t_j)| = k_j, \\ |\mathbb{Z}^2 g \cap \mathscr{R}(t_j + h_j)| = k_j + 1, |\mathbb{Z}^2 g \cap \mathscr{R}(t_i)| = k_i, i \neq j \Big\},$$

and let $U = \mathscr{R}(t_j + h_j) \smallsetminus \mathscr{R}(t_j)$. Then by Proposition 3.8,

$$\nu(\mathscr{E}_j) = \int_U \nu_{\mathbf{y}}(\mathscr{E}_j \cap X(\mathbf{y}))\, d\mathbf{y} = \int_{t_j}^{t_j+h_j} \int_{-1/2}^{1/2} \nu_{(x,y)}(\mathscr{E}_j \cap X(x,y))\, dx\, dy.$$

Therefore, by Proposition 3.7,

$$\lim_{h_j \to 0} \frac{1}{h_j} \nu\big(\mathscr{E}_j(h_j)\big) = \int_{-1/2}^{1/2} \nu_1\big(\{ g \in X_1 : \\ \big|\mathbb{Z}^2 g \cap \big(\mathscr{R}(t_i) - (t_j, y)\big)\big| = k_i, i = 1, \ldots, n\}\big)\, dy.$$

For every fixed $y \in [-1/2, 1/2]$ continuity of the expression under the integral sign with respect to $(t_1, \ldots, t_n)$ again follows from Proposition 3.7. It is clearly uniform in y and therefore the integral is continuous with respect to $(t_1, \ldots, t_n)$. Each term in (3.11) can be treated in a similar way and this proves C^1 regularity of the function $G^{(n)}(t_1, \ldots, t_n)$ and finishes the proof of Theorem 3.1.

4. PROOF OF THEOREM 1.2

We now deduce (1.1) from part (ii) of Theorem 1.3. Consider the unfolding of the tube to $\mathbb{R}^2$ obtained by the reflections from the horizontal boundary of the tube. Let $\mathbf{p_k} = (\xi^k_{\epsilon,\sigma}, \zeta^k_{\epsilon,\sigma})$ be the position of the particle at the moment of kth reflection from the wall in this unfolding. Then,

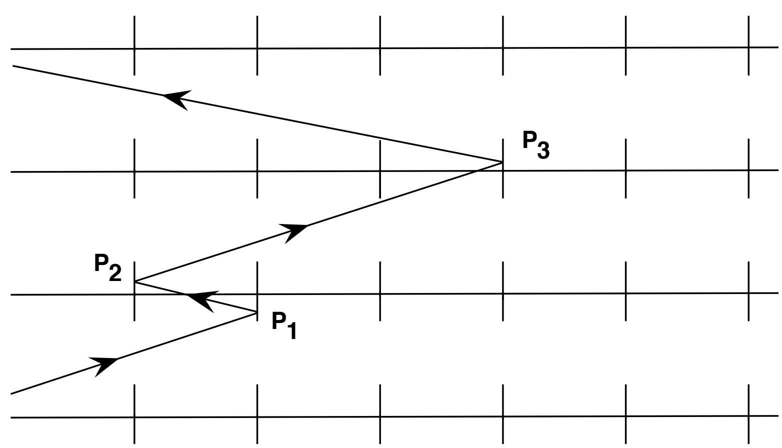

FIGURE 3. An unfolded trajectory. In this example, $\mathscr{Q}_{\epsilon,\sigma} = 3$, $\lfloor\bar{\zeta}\rfloor = 2$, and $n^1_\epsilon = 2$, $n^2_\epsilon = 1$, $n^3_\epsilon = 3$, $n^4_\epsilon > 4$

$$\xi^k_{\epsilon,\sigma} = \sigma[n^1_\epsilon - n^2_\epsilon + \cdots + (-1)^{k+1} n^k_\epsilon]$$

and

$$\zeta^k_{\epsilon,\sigma} = y_{\rm in} + \alpha(n^1_\epsilon + n^2_\epsilon + \cdots + n^k_\epsilon)$$

where we re-define $\alpha := \sigma\tan(\pi\varphi) \in \mathbb{R}$ (and not mod 1 as earlier). At the moment of the exit from the tube, the vertical coordinate of the particle is

$$\bar\zeta = 2\big(y_{\rm in} + \alpha n^1_\epsilon + \alpha n^3_\epsilon + \cdots + \alpha n^{\mathcal{Q}_{\epsilon,\sigma}}_\epsilon\big) - y_{\rm in}. \tag{4.1}$$

Let

$$z = y_{\rm in} + \alpha n^1_\epsilon + \alpha n^3_\epsilon + \cdots + \alpha n^{\mathcal{Q}_{\epsilon,\sigma}}_\epsilon$$

and let $\|\cdot\|$ denote the distance to the nearest integer. Then

$$\|y_{\rm in} + \alpha n^1_\epsilon\| \le \epsilon/2 \qquad \text{and} \qquad \|\alpha n^i_\epsilon\| \le \epsilon \quad \text{for } i > 1.$$

Therefore,

$$\|z\| \le \frac{\epsilon\mathcal{Q}_{\epsilon,\sigma}}{2}. \tag{4.2}$$

Notice that $v_{\rm out} = -v_{\rm in}$, if both the number of reflections from vertical walls and from horizontal walls is odd. The former is obviously odd at the moment of exit. The number of reflections from the horizontal walls is equal to the integer part $\lfloor\bar\zeta\rfloor$.

If $z - \lfloor z\rfloor \le 1/2$, then by (4.1), $\lfloor\bar\zeta\rfloor$ is odd provided that $2\|z\| < y_{\rm in}$, and if $z - \lfloor z\rfloor > 1/2$, then $\lfloor\bar\zeta\rfloor$ is odd provided that $1 - 2\|z\| > y_{\rm in}$. By (4.2) this is the case, when

$$\epsilon\mathcal{Q}_{\epsilon,\sigma} < \min\{y_{\rm in}, 1 - y_{\rm in}\}.$$

By assumption, the probability measure $\mathbb{P}$ on the initial conditions $(y_{\rm in}, \alpha)$ is absolutely continuous with respect to the Lebesgue measure. Therefore, for any $k \in \mathbb{N}$,

$$\begin{aligned}&\mathbb{P}\big\{\mathcal{Q}_{\epsilon,\sigma} = k, \epsilon k < \min\{y_{\rm in}, 1 - y_{\rm in}\}\big\}\\ &\qquad = \mathbb{P}\{\mathcal{Q}_{\epsilon,\sigma} = k\} - \mathbb{P}\big\{\mathcal{Q}_{\epsilon,\sigma} = k, \min\{y_{\rm in}, 1 - y_{\rm in}\} \le \epsilon k\big\} \to G(k) \quad \text{as } \epsilon \to 0.\end{aligned}$$

Together with the tightness condition (3.7), this implies

$$\lim_{\epsilon\to 0} \mathbb{P}\big\{\epsilon\mathcal{Q}_{\epsilon,\sigma} < \min\{y_{\rm in}, 1 - y_{\rm in}\}\big\} = 1$$

and so,

$$\lim_{\epsilon\to 0} \mathbb{P}\{v_{\rm out} = -v_{\rm in}\} = 1.$$

Note that the existence of the limiting probability distribution for $\{\mathcal{Q}_{\epsilon,\sigma}\}$ as $\epsilon \to 0$ also implies that for any $\delta > 0$,

$$\lim_{\epsilon\to 0} \mathbb{P}\{|y_{\rm out} - y_{\rm in}| > \delta\} = 0.$$

Indeed, after each reflection from a vertical wall, the particle backtracks itself with an error at most ϵ, so at the moment of exit it backtracks the incoming trajectory with total error of at most $\epsilon\mathcal{Q}_{\epsilon,\sigma}$. This completes the proof of Theorem 1.2.

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