THREE LECTURES ON QUANTUM MAPS, QUANTUM ERGODICITY AND BOUNCING BALL MODES

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Introduction

Quantum maps play a central role in the theory of quantum chaos, as they provide toy models that share many features with more realistic chaotic systems. The beauty of quantum maps is that they are very simple mathematical objects: $N \times N$ unitary matrices that must satisfy certain conditions in the "semiclassical" limit $N \to \infty$, which ensure they relate to an underlying classical dynamical system. An example of a quantum chaotic map that we will discuss in detail is

$$U\, \mathcal{F}^{-1}\, \tilde{U}\, \mathcal{F} \tag{1}$$

where $\mathcal{F}$ is the N-dimensional discrete Fourier transform with matrix elements

$$\mathcal{F}_{jk} = \frac{e^{-\frac{2\pi i}{N} jk}}{\sqrt{N}}, \tag{2}$$

and $U, \tilde{U}$ are diagonal $N \times N$ matrices with coefficients

$$\tilde{U}_{jj} = e^{-\frac{2\pi i}{N} j^2}, \qquad U_{jj} = \begin{cases} 1 & \text{if } 1 \le j \le \frac{N}{2} - 1 \\ e^{\frac{2\pi i}{N} j^2} & \text{if } \frac{N}{2} \le j \le N. \end{cases} \tag{3}$$

The first lecture will explain what we mean by quantum observables and quantum maps, and illustrate the concept with the quantization of linked twist maps, an important class of maps whose ergodic properties have been studied in great detail [2, 11, 14]. This lecture follows closely the paper [9]; for more background on quantum maps see [4].

The aim of the second lecture is to prove the celebrated quantum ergodicity theorem, which states that almost all eigenstates of a quantum map are semiclassically equidistributed, if the underlying classical map is ergodic. (The original theorem due to Shnirelman, Zelditch and Colin de Verdiere is stated for eigenfunctions of the Laplacian on manifolds with ergodic geodesic flow.) We follow here the approach taken, e.g., in [15, 3, 12, 9].

The third lecture is concerned with subsequences of eigenstates that do not become equidistributed. As we shall see, these states play the analogous role of the

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famous bouncing ball modes in the stadium billiard, whose existence was recently proved in a beautiful paper by Andrew Hassell [6]. We explain the essential part of his argument in the case of quantized linked twist maps. Hassell's approach requires a quasimode construction for bouncing ball modes in the stadium billiard as in [5], cf. also [16]. We will here adapt the argument to quantum linked twist maps as in Stephen O'Keefe's PhD thesis [10], and in fact exhibit a family of maps with exceptionally accurate quasimodes, whose discrepancy is significantly smaller than the mean level spacing; see [8] for a family of billiards with a similar characteristic.

We also recommend the papers [1] and [13] for some interesting heuristics on bouncing ball modes.

Lecture I: Quantum maps

1.1. Let $\mathcal{M}$ be a d-dimensional compact smooth manifold, and μ a probability measure on $\mathcal{M}$ which is absolutely continuous with respect to Lebesgue measure. We consider bijective piecewise smooth maps $\Phi : \mathcal{M} \to \mathcal{M}$ which preserve μ. This lecture explains what we mean by a *quantization* of Φ, and discusses an important family of maps, the *linked twist maps*. By *piecewise smooth* we mean that Φ is C^∞ on $\mathcal{M} \setminus \mathcal{S}$ where $\mathcal{S}$ is a closed subset of $\mathcal{M}$ of measure zero. We will refer to $\mathcal{S} = \mathcal{S}_\Phi$ as the *singularity set* of Φ.

1.2. **Example.** *Twist maps.* Take $\mathcal{M} = \mathbb{T}^2$, where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ denotes the two-dimensional torus. A *twist map* Ψ_f is a map $\mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$\Psi_f : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p + f(q) \\ q \end{pmatrix} \mod \mathbb{Z}^2 \tag{4}$$

where $f : [0,1] \to \mathbb{R}$ is piecewise smooth and satisfies $f(1) = f(0) \bmod 1$. An example is $f(q) = mq$ with $m \in \mathbb{Z}$, or $f(q) = mq + g(q)$ where g is a smooth periodic function.

Obviously Lebesgue measure $d\mu = dp\, dq$ is invariant under Ψ_f.

1.3. **Example.** *Linked twist maps.* A *linked twist map* Φ is now obtained by combining two twist maps, Ψ_f and Ψ_g, by setting

$$\Phi = \Psi_f \circ \mathcal{R}^{-1} \circ \Psi_g \circ \mathcal{R} \tag{5}$$

with the rotation

$$\mathcal{R} : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} q \\ -p \end{pmatrix} \mod \mathbb{Z}^2. \tag{6}$$

Since Ψ_f, Ψ_g and $\mathcal{R}$ preserve μ, so does Φ. More explicitly, we have

$$\mathcal{R}^{-1} \circ \Psi_g \circ \mathcal{R} : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p \\ q + g(-p) \end{pmatrix} \mod \mathbb{Z}^2 \tag{7}$$

and thus

$$\Phi : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p + f(q + g(-p)) \\ q + g(-p) \end{pmatrix} \mod \mathbb{Z}^2. \tag{8}$$

1.4. The ergodic properties of linked twist maps are well understood [2, 11, 14]. Let $[a_i, b_i]$ $(i = 1, 2)$ be subintervals of $[0, 1]$, and choose functions $f = f_1, g = f_2 : [0, 1] \to \mathbb{R}$ with

(a) $f_i(q) = 0$ for $q \notin [a_i, b_i]$,
(b) $f_i(a_i) \in \mathbb{Z}$ and $f_i(b_i) - f_i(a_i) = k_i$ for some integer $k_i \in \mathbb{Z}$,
(c) $f_i \in C^2([a_i, b_i])$ with derivative $f_i'(q) \neq 0$ for all $q \in [a_i, b_i]$.

Let us define the constant

$$\gamma_i = \operatorname{sign}(k_i) \max_{q \in [a_i, b_i]} |f_i'(q)|. \tag{9}$$

1.5. **Theorem.** Suppose either of the following conditions is satisfied,

(i) $\gamma_1 \gamma_2 < 0$;
(ii) $|k_1|, |k_2| \geq 2$ and $\gamma_1 \gamma_2 > C_0 \approx 17.24445$.

Then the map (5) acts ergodically (with respect to Lebesgue measure μ) on the domain

$$\mathcal{D}_0 = \{(p, q) \in \mathbb{T}^2 : p \in [a_1, b_1]\} \cup \{(p, q) \in \mathbb{T}^2 : q \in [a_2, b_2]\}. \tag{10}$$

1.6. The proofs of the two parts (i) and (ii) of this statement are due to Burton and Easton [2], and Przytycki [11], respectively. Both [2] and [11] in fact establish the Bernoulli property for the action of Φ on $\mathcal{D}_0$ under conditions (i), (ii). We expect that these properties hold under weaker conditions, e.g., for smaller values of C_0. The continuity of the map at the lines $p = a_1, b_1$ and $q = a_2, b_2$, assumed in condition (b), is probably also not necessary.

We refer the reader to [14] for a detailed survey of the ergodic properties of linked twist maps.

1.7. **Definition.** Let $M_N(\mathbb{C})$ be the space of $N \times N$ matrices with complex coefficients. We say two sequences of matrices,

$$\{A_N\}_{N \in \mathbb{N}}, \qquad \{B_N\}_{N \in \mathbb{N}}, \tag{11}$$

are *semiclassically equivalent*, if

$$\|A_N - B_N\| \to 0 \tag{12}$$

as $N \in \mathbb{N}$ tends to infinity, where $\| \cdot \|$ denotes the usual operator norm

$$\|A\| := \sup_{\psi \in \mathbb{C}^N - \{0\}} \frac{\|A\psi\|}{\|\psi\|}. \tag{13}$$

We denote this equivalence relation by

$$A_N \sim B_N. \tag{14}$$

1.8. **Exercise.** Show: If $A_N \sim B_N$ then $\operatorname{Tr} A_N = \operatorname{Tr} B_N + o(N)$.

1.9. **Axiom.** *The correspondence principle for quantum observables.* Fix a measure μ as above. There is a sequence Op_N of linear maps,

$$\mathrm{Op}_N : C^\infty(\mathcal{M}) \to M_N(\mathbb{C}), \qquad a \mapsto \mathrm{Op}_N(a),$$

so that

(a) for all $a \in C^\infty(\mathcal{M})$,
$$\mathrm{Op}_N(\overline{a}) \sim \mathrm{Op}_N(a)^\dagger;$$

(b) for all $a_1, a_2 \in C^\infty(\mathcal{M})$,
$$\mathrm{Op}_N(a_1)\,\mathrm{Op}_N(a_2) \sim \mathrm{Op}_N(a_1 a_2);$$

(c) for all $a \in C^\infty(\mathcal{M})$,
$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} \mathrm{Op}_N(a) = \int_{\mathcal{M}} a \, d\mu.$$

Examples of quantum observables satisfying these conditions are given in Section 1.13. In standard quantization recipes of symplectic manifolds (such as the one discussed in Section 1.13) one in addition has the property that

$$\mathrm{Op}_N(a_1)\,\mathrm{Op}_N(a_2) - \mathrm{Op}_N(a_2)\,\mathrm{Op}_N(a_1) \sim \frac{1}{2\pi i N}\,\mathrm{Op}_N(\{a_1, a_2\}) \tag{15}$$

where $\{\ ,\ \}$ is the Poisson bracket. This assumption is however not necessary for many of the results proved in these lectures. The axioms (a)–(c) in fact apply to examples without quantum mechanical significance.

1.10. **Axiom.** *The correspondence principle for quantum maps.* There is a sequence of unitary matrices $U_N(\Phi)$ such that for any $a \in C^\infty(\mathcal{M})$ with compact support in $\mathcal{M} \setminus \Phi(\mathcal{S}_\Phi)$ (with the singularity set $\mathcal{S}_\Phi$ as defined in 1.1), we have

$$U_N(\Phi)^{-1}\,\mathrm{Op}_N(a) U_N(\Phi) \sim \mathrm{Op}_N(a \circ \Phi).$$

1.11. Note that the condition on the support of a ensures that $a \circ \Phi \in C^\infty(\mathcal{M})$.

1.12. Let us now discuss examples of semiclassical sequences of quantum maps satisfying the above Axioms 1.9 and 1.10.

We first construct a well known example of quantum observables on the two-dimensional torus $\mathcal{M} = \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ satisfying Axiom 1.9 (cf. [4]), and corresponding examples of quantum linked twist maps satisfying Axiom 1.10.

1.13. **Example.** *Quantum tori.* It is convenient to represent a vector $\psi \in \mathbb{C}^N$ as a function $\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. Let us define the translation operators

$$[t_1\psi](Q) = \psi(Q+1) \tag{16}$$

and

$$[t_2\psi](Q) = e_N(Q)\psi(Q), \tag{17}$$

where $e_N(x) := e(x/N) := \exp(2\pi i x/N)$.

1.14. **Exercise.** Show that

$$t_1^{m_1} t_2^{m_2} = t_2^{m_2} t_1^{m_1} e_N(m_1 m_2) \qquad \forall m_1, m_2 \in \mathbb{Z}. \tag{18}$$

1.15. These relations are known as the *Weyl-Heisenberg commutation relations*. For $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ put

$$T_N(\mathbf{m}) = e_N\left(\frac{m_1 m_2}{2}\right) t_2^{m_2} t_1^{m_1}. \tag{19}$$

Then

$$T_N(\mathbf{m}) T_N(\mathbf{n}) = e_N\left(\frac{\omega(\mathbf{m},\mathbf{n})}{2}\right) T_N(\mathbf{m}+\mathbf{n}) \tag{20}$$

with the symplectic form

$$\omega(\mathbf{m},\mathbf{n}) = m_1 n_2 - m_2 n_1. \tag{21}$$

For any $a \in C^\infty(\mathbb{T}^2)$, we define the quantum observable

$$\mathrm{Op}_N(a) = \sum_{\mathbf{m}\in\mathbb{Z}^2} \widehat{a}(\mathbf{m}) T_N(\mathbf{m}) \tag{22}$$

where

$$\widehat{a}(\mathbf{m}) = \int_{\mathbb{T}^2} a(\boldsymbol{\xi}) e(-\boldsymbol{\xi}\cdot\mathbf{m})\, d\xi \tag{23}$$

are the Fourier coefficients of a. The observable $\mathrm{Op}_N(a)$ is also called the *Weyl quantization of* a. Axiom 1.9 (a) is trivially satisfied. Axioms 1.9 (b) and (c) follow from the following lemmas.

1.16. **Lemma.** For all $a_1, a_2 \in C^\infty(\mathbb{T}^2)$

$$\| \mathrm{Op}_N(a_1) \mathrm{Op}_N(a_2) - \mathrm{Op}_N(a_1 a_2) \| \le \frac{\pi}{N} \Big(\sum_{\mathbf{m}\in\mathbb{Z}^2} \|\mathbf{m}\| |\widehat{a}_1(\mathbf{m})| \Big) \Big(\sum_{\mathbf{n}\in\mathbb{Z}^2} \|\mathbf{n}\| |\widehat{a}_2(\mathbf{n})| \Big). \tag{24}$$

Proof. Using the commutation relations (18) we find

$$\operatorname{Op}_N(a_1)\operatorname{Op}_N(a_2) = \sum_{m,n\in\mathbb{Z}^2} \widehat{a}_1(\mathbf{m})\widehat{a}_2(\mathbf{n})T_N(\mathbf{m})T_N(\mathbf{n}) \tag{25}$$

$$= \sum_{m,n\in\mathbb{Z}^2} e_N\left(\frac{\omega(\mathbf{m},\mathbf{n})}{2}\right)\widehat{a}_1(\mathbf{m})\widehat{a}_2(\mathbf{n})T_N(\mathbf{m}+\mathbf{n}) \tag{26}$$

$$= \sum_{m,k\in\mathbb{Z}^2} e_N\left(\frac{\omega(\mathbf{m},\mathbf{k})}{2}\right)\widehat{a}_1(\mathbf{m})\widehat{a}_2(\mathbf{k}-\mathbf{m})T_N(\mathbf{k}) \tag{27}$$

with $\mathbf{k}=\mathbf{n}+\mathbf{m}$. Furthermore

$$\operatorname{Op}_N(a_1a_2) = \sum_{m,k\in\mathbb{Z}^2} \widehat{a}_1(\mathbf{m})\widehat{a}_2(\mathbf{k}-\mathbf{m})T_N(\mathbf{k}), \tag{28}$$

and hence

$$\|\operatorname{Op}_N(a_1)\operatorname{Op}_N(a_2)-\operatorname{Op}_N(a_1a_2)\| \le \sum_{m,n\in\mathbb{Z}^2}\left| e_N\left(\frac{\omega(\mathbf{m},\mathbf{n})}{2}\right)-1\right| \left|\widehat{a}_1(\mathbf{m})\right| \left|\widehat{a}_2(\mathbf{n})\right| \tag{29}$$

The lemma now follows from

$$|e(x)-1| \le |2\pi x|, \qquad |\omega(\mathbf{m},\mathbf{n})| \le \|\mathbf{m}\|\,\|\mathbf{n}\|. \tag{30}$$

□

1.17. **Exercise.** Show that (15) holds for the above $\operatorname{Op}_N(a)$.

1.18. **Lemma.** For any $a\in C^\infty(\mathbb{T}^2)$ and $R>1$

$$\frac{1}{N}\operatorname{Tr}\operatorname{Op}_N(a) = \int_{\mathbb{T}^2} a\, d\mu + O_{a,R}(N^{-R}). \tag{31}$$

Proof. Note that

$$\operatorname{Tr} T_N(\mathbf{m}) = \begin{cases} N\, e_N\left(\frac{m_1m_2}{2}\right) & \text{if } \mathbf{m}=0 \bmod N\mathbb{Z}^2,\\ 0 & \text{otherwise.}\end{cases} \tag{32}$$

The lemma now follows from the rapid decay of the Fourier coefficients $\widehat{a}(\mathbf{m})$ for $\|\mathbf{m}\|\to\infty$. □

1.19. Note that we have the alternative representation for $\operatorname{Op}_N(a)$,

$$[\operatorname{Op}_N(a)\psi](Q) = \sum_{m\in\mathbb{Z}} \widetilde{a}\left(m, \frac{Q}{N}+\frac{m}{2N}\right)\psi(Q+m) \tag{33}$$

where

$$\widetilde{a}(m,q) = \int_{\mathbb{T}} a(p,q)\, e(-pm)\, dp, \qquad \mathbb{T}=\mathbb{R}/\mathbb{Z}, \tag{34}$$

which is sometimes useful. In fact (33) permits to quantise observables a which are piecewise smooth in the q-variable. Note that if a is a smooth function of p and, for any $\nu \geq 0$, $\frac{d^\nu}{dp^\nu} a(p,q)$ is a bounded function on $\mathbb{T}^2$, then, for any $R > 1$, there is a constant C_R such that

$$|\tilde{a}(m,q)| \leq C_R(1+|m|)^{-R} \tag{35}$$

for all m, q. This fact is proved using integration by parts. Of course (35) holds in particular for smooth observables $a \in C^\infty(\mathbb{T}^2)$.

1.20. **Example.** *Quantum twist maps*

We define the quantization of the twist map Ψ_f by the unitary operator

$$[U_N(\Psi_f)\psi](Q) = e\left[-NV\left(\frac{Q}{N}\right)\right]\psi(Q) \tag{36}$$

where V is defined by $f = -V'$ for some choice of integration constant.

1.21. **Theorem.** For any $a \in C^\infty(\mathbb{T}^2)$ with support in $\mathbb{T}^2 \setminus \Psi_f(\mathcal{S}_{\Psi_f})$,

$$\|U_N(\Psi_f)^{-1}\operatorname{Op}_N(a)U_N(\Psi_f) - \operatorname{Op}_N(a\circ\Psi_f)\| = O(N^{-2}) \tag{37}$$

where the implied constant depends on a.

Proof. We have

$$\begin{aligned}(38)\quad & [U_N(\Psi_f)^{-1}\operatorname{Op}_N(a)U_N(\Psi_f)\psi](Q) \\ &= \sum_{m\in\mathbb{Z}} \tilde{a}\left(m, \frac{Q}{N}+\frac{m}{2N}\right) e\left\{-N\left[V\left(\frac{Q+m}{N}\right) - V\left(\frac{Q}{N}\right)\right]\right\}\psi(Q+m),\end{aligned}$$

and

$$[\operatorname{Op}_N(a\circ\Psi_f)\psi](Q) = \sum_{m\in\mathbb{Z}} \tilde{a}\left(m, \frac{Q}{N}+\frac{m}{2N}\right) e\left[mf\left(\frac{Q}{N}+\frac{m}{2N}\right)\right]\psi(Q+m), \tag{39}$$

since

$$\widetilde{(a\circ\Psi_f)}(m,q) = e[mf(q)]\,\tilde{a}(m,q). \tag{40}$$

Therefore

$$\|U_N(\Psi_f)^{-1}\operatorname{Op}_N(a)U_N(\Psi_f) - \operatorname{Op}_N(a\circ\Psi_f)\| \leq \max_q \sum_{m\in\mathbb{Z}} \left|\tilde{a}\left(m, q+\frac{m}{2N}\right)c_m(q,N)\right| \tag{41}$$

with

$$c_m(q,N) = e\left\{-N\left[V\left(q+\frac{m}{N}\right) - V(q)\right]\right\} - e\left[mf\left(q+\frac{m}{2N}\right)\right]. \tag{42}$$

Since $|c_m(q,N)| \leq 2$ and $|\tilde{a}(m,q)| \leq (1+|m|)^{-5}$, we have

$$\max_q \sum_{|m|\geq N^{1/2}} \left|\tilde{a}\left(m, q+\frac{m}{2N}\right)c_m(q,N)\right| \leq N^{-2}. \tag{43}$$

Let us denote by $\mathcal{CS}$ the projection of the compact support of a onto the q axis. $\mathcal{CS}$ is a compact set that does not contain any singularity of f.

For $|m| < N^{1/2}$, Taylor expansion around $x = q + \frac{m}{2N}$ yields (the second order terms cancel)

$$V\left(x+\frac{m}{2N}\right) - V\left(x-\frac{m}{2N}\right) = V'(x)\,\frac{m}{N} + O\left(\frac{m^3}{N^3}\right) \tag{44}$$

$$= -f(x)\,\frac{m}{N} + O\left(\frac{m^3}{N^3}\right). \tag{45}$$

uniformly for all $|m| < N^{1/2}$ and all $q \in \mathcal{CS}$, provided N is sufficiently large so that $[q - N^{-1/2}, q + N^{-1/2}]$ is away from the singularities. Hence in this case

$$c_m(q, N) = O\left(\frac{m^3}{N^2}\right) \tag{46}$$

and

$$\max_q \sum_{|m|<N^{1/2}} \left|\widetilde{a}\left(m, q + \frac{m}{2N}\right) c_m(q, N)\right| \leq O(N^{-2}) \max_q \sum_{m\in\mathbb{Z}} \left|m^3 \widetilde{a}\left(m, q + \frac{m}{2N}\right)\right| \tag{47}$$

$$= O(N^{-2}). \tag{48}$$

□

1.22. **Example.** *The discrete Fourier transform* $\mathcal{F}_N$ is a unitary operator defined by

$$[\mathcal{F}_N\psi](P) = \frac{1}{\sqrt{N}} \sum_{Q=0}^{N-1} \psi(Q) e_N(-QP). \tag{49}$$

Its inverse is given by the formula

$$[\mathcal{F}_N^{-1}\psi](Q) = \frac{1}{\sqrt{N}} \sum_{P=0}^{N-1} \psi(P) e_N(PQ). \tag{50}$$

1.23. **Exercise.** Sow that, for any $a \in C^\infty(\mathbb{T}^2)$

$$\mathcal{F}_N^{-1} \operatorname{Op}_N(a) \mathcal{F}_N = \operatorname{Op}_N(a \circ \mathcal{R}) \tag{51}$$

with the rotation $\mathcal{R}$ as in (6).

Hint: Exploit the identities $\mathcal{F}_N^{-1} t_1 \mathcal{F}_N = t_2^{-1}$ and $\mathcal{F}_N^{-1} t_2 \mathcal{F}_N = t_1$.

1.24. The Fourier transform may therefore be viewed as a quantization of the rotation $\mathcal{R}$ which satisfies an *exact* correspondence principle, cf. Axiom 1.10.

1.25. **Example.** *Quantum linked twist maps*

The quantization of the linked twist map is now defined by

$$U_N(\Phi) = U_N(\Psi_f)\, \mathcal{F}_N^{-1}\, U_N(\Psi_g)\, \mathcal{F}_N. \tag{52}$$

1.26. **Theorem.** For any $a \in C^\infty(\mathbb{T}^2)$ with compact support in $\mathbb{T}^2 \setminus \Phi(\mathcal{S}_\Phi)$, we have

$$\|U_N(\Phi)^{-1}\,\mathrm{Op}_N(a)U_N(\Phi) - \mathrm{Op}_N(a \circ \Phi)\| = O(N^{-2}) \tag{53}$$

where the implied constant depends on a.

Proof. Apply Theorem 1.21 and Exercise 1.23. □

The quantum map $U_N(\Phi)$ thus satisfies Axiom 1.10.

Lecture II: Quantum ergodicity

2.27. Let us now return to the general framework of a map Φ on a general manifold $\mathcal{M}$ under the assumptions described in 1.1.

2.28. **Definition.** *Mollified characteristic functions.* Consider the characteristic function $\chi_\mathcal{D}$ of a domain $\mathcal{D} \subset \mathcal{M}$ with boundary of measure zero. An ϵ*-mollified characteristic function* $\widetilde{\chi}_\mathcal{D} \in C^\infty(\mathcal{M})$ has values in $[0, 1]$ and $\widetilde{\chi}_\mathcal{D}(x) = \chi_\mathcal{D}(x)$ on a set of x of measure $1 - \epsilon$.

2.29. **Example.** Examples of mollified characteristic functions can be constructed as follows. We first construct a continuous $\widetilde{\chi}^0_\mathcal{D}(x)$. Let us denote by $d(x, y)$ a metric on $\mathcal{M}$ and by $d(x, \mathcal{D}) = \inf\{d(x, y) : y \in \mathcal{D}\}$ the usual distance of a point x from the set $\mathcal{D}$. Let $g \in C(\mathbb{R})$ such that $g(t) = 1$ for $t \le 0$ and $g(t) = 0$ for $t \ge 1$. Then, for any $R > 0$, the function $\widetilde{\chi}^0_\mathcal{D}(x) = g(R\, d(x, \overline{\mathcal{D}}))$ is continuous and satisfies $\chi_\mathcal{D}(x) \le \widetilde{\chi}^0_\mathcal{D}(x)$. Furthermore, given $\epsilon > 0$, there is $R > 0$ (sufficiently large) such that

$$\int_\mathcal{M} [\widetilde{\chi}^0_\mathcal{D}(x) - \chi_\mathcal{D}(x)]\,d\mu < \frac{\epsilon}{2}. \tag{54}$$

Using the density of $C^\infty(\mathcal{M})$ in $C(\mathcal{M})$, we can find a function $\widetilde{\chi}_\mathcal{D} \in C^\infty(\mathcal{M})$ such that $\widetilde{\chi}^0_\mathcal{D}(x) \le \widetilde{\chi}_\mathcal{D}(x)$ and

$$\int_\mathcal{M} [\widetilde{\chi}_\mathcal{D}(x) - \widetilde{\chi}^0_\mathcal{D}(x)]\,d\mu < \frac{\epsilon}{2}, \tag{55}$$

which concludes the construction.

The support of $\widetilde{\chi}_\mathcal{D}(x)$ contains $\mathcal{D}$ in this example. Examples of mollified characteristic functions whose support is contained in $\mathcal{D}$ are obtained by taking $\widetilde{\chi}_\mathcal{D}(x) = 1 - \widetilde{\chi}_{\mathcal{D}^c}(x)$, where $\mathcal{D}^c$ is the complement of $\mathcal{D}$.

2.30. **Exercise.** Show that if $\widetilde{\chi}_\mathcal{D}$ is ϵ-mollified, so is $\widetilde{\chi}^n_\mathcal{D}$ for any $n \in \mathbb{N}$ with the same ϵ.

2.31. After mollification, we may associate with a characteristic function $\chi_{\mathcal{D}}$ a quantum observable $\mathrm{Op}_N(\widetilde{\chi}_{\mathcal{D}})$. Since $\mathrm{Op}_N(\widetilde{\chi}_{\mathcal{D}})$ is in general not hermitian, it is sometimes more convenient to consider the symmetrised version, the positive semi-definite hermitian matrix

$$\mathrm{Op}_N^{\mathrm{sym}}(\widetilde{\chi}_{\mathcal{D}}) := \mathrm{Op}_N(\widetilde{\chi}_{\mathcal{D}}^{1/2})\,\mathrm{Op}_N(\widetilde{\chi}_{\mathcal{D}}^{1/2})^{\dagger}. \tag{56}$$

Note that $\widetilde{\chi}_{\mathcal{D}}^{1/2} \in C^{\infty}(\mathcal{M})$ since $\widetilde{\chi}_{\mathcal{D}} \geq 0$. Furthermore, we have

$$\mathrm{Op}_N^{\mathrm{sym}}(\widetilde{\chi}_{\mathcal{D}}) \sim \mathrm{Op}_N(\widetilde{\chi}_{\mathcal{D}}). \tag{57}$$

2.32. The following proposition describes the distribution of eigenvalues of $\mathrm{Op}_N^{\mathrm{sym}}(\widetilde{\chi}_{\mathcal{D}})$, and suggests that the operator may be viewed as an approximate projection operator onto a subspace of dimension $\sim N \times \mu(\mathcal{D})$.

Consider a sequence $\mathbf{J} := \{J_N\}_{N\in\mathbb{N}}$ of sets $J_N \subset \{1, \ldots, N\}$. The quantity

$$\Delta(\mathbf{J}) := \lim_{N\to\infty} \frac{|J_N|}{N}, \tag{58}$$

provided the limit exists, is called the *density of* $\mathbf{J}$.

2.33. **Theorem.** Suppose $\widetilde{\chi}_{\mathcal{D}}$ is an ϵ-mollified characteristic function, and suppose $\mu_j \geq 0$ $(j = 1, \ldots, N)$ are the eigenvalues of $\mathrm{Op}_N^{\mathrm{sym}}(\widetilde{\chi}_{\mathcal{D}})$. Then there are set sequences $\mathbf{J} := \{J_N\}_{N\in\mathbb{N}}$ and $\mathbf{J}' := \{J_N'\}_{N\in\mathbb{N}}$ with densities

$$\Delta(\mathbf{J}) = \mu(\mathcal{D}) + O(\epsilon^{1/3}), \qquad \Delta(\mathbf{J}') = 1 - \mu(\mathcal{D}) + O(\epsilon^{1/3}), \tag{59}$$

such that

(i) $\mu_j = 1 + O(\epsilon^{1/3})$ for all $j \in J_N$;
(ii) $\mu_j = O(\epsilon^{1/3})$ for all $j \in J_N'$.

Proof. By Axiom 1.9, we have for every fixed integer $n \geq 1$,

$$\frac{1}{N}\,\mathrm{Tr}\left[\mathrm{Op}_N^{\mathrm{sym}}(\widetilde{\chi}_{\mathcal{D}})^n\right] = \frac{1}{N}\,\mathrm{Tr}\,\mathrm{Op}_N^{\mathrm{sym}}(\widetilde{\chi}_{\mathcal{D}}^n) + o_{\epsilon,n}(1) \tag{60}$$

$$= \int_{\mathcal{M}} \widetilde{\chi}_{\mathcal{D}}^n d\mu + o_{\epsilon,n}(1) \tag{61}$$

$$= \mu(\mathcal{D}) + O(\epsilon) + o_{\epsilon,n}(1), \tag{62}$$

where $O(\epsilon)$ does not depend on N and n. This implies for every $n \geq 1$,

$$\lim_{N\to\infty} \frac{1}{N}\,\mathrm{Tr}\left[\mathrm{Op}_N^{\mathrm{sym}}(\widetilde{\chi}_{\mathcal{D}})^n\right] = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \mu_j^n = \mu(\mathcal{D}) + O(\epsilon). \tag{63}$$

Therefore

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} (\mu_j^2 - \mu_j)^2 = O(\epsilon) \tag{64}$$

and thus

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\in H_N} (\mu_j - 1)^2 = O(\epsilon). \tag{65}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\notin H_N} \mu_j^2 = O(\epsilon), \tag{66}$$

where $H_N = \{j : \mu_j \geq 1/2\}$. By Chebyshev's inequality, (66) implies that

$$\lim_{N\to\infty} \frac{1}{N} |\{j \notin H_N : \mu_j^2 > \gamma\}| = O(\epsilon/\gamma), \tag{67}$$

for any $\gamma > 0$. This yields the bound

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\notin H_N} \mu_j = O\big(\gamma^{1/2} + \epsilon/\gamma\big), \tag{68}$$

since $0 \leq \mu_j < 1/2$. So

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\in H_N} 1 = \lim_{N\to\infty} \frac{1}{N} \sum_{j\in H_N} \big[(\mu_j - 1)^2 - \mu_j^2 + 2\mu_j\big] \tag{69}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{j\in H_N} \big[-\mu_j^2 + 2\mu_j\big] + O(\epsilon) \tag{70}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \big[-\mu_j^2 + 2\mu_j\big] + O(\epsilon) + O\big(\gamma^{1/2} + \epsilon/\gamma\big) \tag{71}$$

$$= \mu(\mathcal{D}) + O(\epsilon^{1/3}), \tag{72}$$

if we choose $\gamma = \epsilon^{2/3}$, and hence the corresponding set sequence $\mathbf{H} := \{H_N\}_{N\in\mathbb{N}}$ has density $\Delta(\mathbf{H}) = \mu(\mathcal{D}) + O(\epsilon^{1/3})$. Once more in view of Chebyshev's inequality, (65) implies

$$\lim_{N\to\infty} \frac{1}{N} |\{j \in H_N : (\mu_j - 1)^2 > \delta\}| = O(\epsilon/\delta). \tag{73}$$

Choosing $\delta = \epsilon^{2/3}$, this means that for a subsequence of $j \in H_N$ of density $\mu(\mathcal{D}) + O(\epsilon^{1/3})$ we have $\mu_j = 1 + O(\epsilon^{1/3})$. The corresponding result for $j \notin H_N$ follows by the same argument from (67). □

2.34. The following proposition is the key tool to understand the distribution of eigenvalues of $U_N(\Phi)$.

2.35. **Theorem.** *Trace asymptotics.* Suppose $\Phi : \mathcal{M} \to \mathcal{M}$ is piecewise smooth, and for every $n \neq 0$ the fixed points of Φ^n form a set of measure zero. Then for $n \neq 0$,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} U_N(\Phi)^n = 0. \tag{74}$$

Proof. Given any $\epsilon > 0$, we can find an integer R and a partition of unity on $\mathcal{M}$ by ϵ-mollified characteristic functions,

$$1 = \widetilde{\chi}_{\mathrm{bad}}(\xi) + \sum_{r=1}^{R} \widetilde{\chi}_r(\xi) \quad \forall \xi \in \mathcal{M} \tag{75}$$

with the properties

(i) the interior of the support of $\widetilde{\chi}_{\mathrm{bad}}$ contains all fixed points of Φ^n and the set $\Phi^n(\mathcal{S}_\Phi)$, and is chosen small enough so that $\int \widetilde{\chi}_{\mathrm{bad}} d\mu < \epsilon$;

(ii) the support of $\widetilde{\chi}_r$, with $r = 1, \ldots, R$, is chosen small enough, so that $\operatorname{supp} \widetilde{\chi}_r \cap \Phi^n(\operatorname{supp} \widetilde{\chi}_r) = \emptyset$.

Property (i) is possible since the fixed points form a closed set of measure zero (since Φ^n is piecewise smooth). To achieve (ii) note that the closure of $\mathcal{K} = \mathcal{M} - \operatorname{supp} \widetilde{\chi}_{\mathrm{bad}}$ does not contain any fixed points. Hence there is a sufficiently small radius $\eta = \eta(\epsilon)$ such that for all balls $\mathcal{B}_\eta \subset \mathcal{K}$ we have $\mathcal{B}_\eta \cap \Phi^n(\mathcal{B}_\eta) = \emptyset$.

By the linearity of Op_N, we have

$$\operatorname{Tr} U_N(\Phi)^n = \operatorname{Tr}[U_N(\Phi)^n \mathrm{Op}_N(\widetilde{\chi}_{\mathrm{bad}})] + \sum_{r=1}^{R} \operatorname{Tr}[U_N(\Phi)^n \mathrm{Op}_N(\widetilde{\chi}_r)] + o(N). \tag{76}$$

We begin with the first term on the right hand side:

$$\operatorname{Tr}[U_N(\Phi)^n \mathrm{Op}_N(\widetilde{\chi}_{\mathrm{bad}})] = \operatorname{Tr}[U_N(\Phi)^n \mathrm{Op}_N^{\mathrm{sym}}(\widetilde{\chi}_{\mathrm{bad}})] + o_\epsilon(N), \tag{77}$$

with the symmetrised $\mathrm{Op}_N^{\mathrm{sym}}(\widetilde{\chi}_{\mathrm{bad}})$ as defined in (56). Suppose ψ_j and $\mu_j \geq 0$ are the (normalised) eigenstates and eigenvalues of $\mathrm{Op}_N^{\mathrm{sym}}(\widetilde{\chi}_{\mathrm{bad}})$. Then

$$\begin{aligned} |\operatorname{Tr}[U_N(\Phi)^n \mathrm{Op}_N^{\mathrm{sym}}(\widetilde{\chi}_{\mathrm{bad}})]| &= \left| \sum_{j=1}^{N} \mu_j \langle \psi_j, U_N(\Phi)^n \psi_j \rangle \right| \leq \sum_{j=1}^{N} \mu_j \\ &= \operatorname{Tr} \mathrm{Op}_N^{\mathrm{sym}}(\widetilde{\chi}_{\mathrm{bad}}) = \operatorname{Tr} \mathrm{Op}_N(\widetilde{\chi}_{\mathrm{bad}}) + o_\epsilon(N) = N O(\epsilon) + o_\epsilon(N). \end{aligned} \tag{78}$$

For the last term in the sum (76) we have

$$U_N(\Phi)^n \mathrm{Op}_N(\widetilde{\chi}_r) \sim U_N(\Phi)^n \mathrm{Op}_N(\widetilde{\chi}_r^{1/2}) \mathrm{Op}_N(\widetilde{\chi}_r^{1/2}) \sim \mathrm{Op}_N(\widetilde{\chi}_r^{1/2} \circ \Phi^{-n}) U_N(\Phi)^n \mathrm{Op}_N(\widetilde{\chi}_r^{1/2}) \tag{79}$$

so

$$\operatorname{Tr}[U_N(\Phi)^n \mathrm{Op}_N(\widetilde{\chi}_r)] = \operatorname{Tr}[\mathrm{Op}_N(\widetilde{\chi}_r^{1/2} \circ \Phi^{-n}) U_N(\Phi)^n \mathrm{Op}_N(\widetilde{\chi}_r^{1/2})] + o_\epsilon(N) \tag{80}$$

$$= \operatorname{Tr}[\mathrm{Op}_N(\widetilde{\chi}_r^{1/2}) \mathrm{Op}_N(\widetilde{\chi}_r^{1/2} \circ \Phi^{-n}) U_N(\Phi)^n] + o_\epsilon(N) \tag{81}$$

$$= \operatorname{Tr}[\mathrm{Op}_N(\widetilde{\chi}_r^{1/2} \cdot \widetilde{\chi}_r^{1/2} \circ \Phi^{-n}) U_N(\Phi)^n] + o_\epsilon(N) \tag{82}$$

$$= o_\epsilon(N) \tag{83}$$

since $\widetilde{\chi}_r^{1/2} \cdot \widetilde{\chi}_r^{1/2} \circ \Phi^{-n} = 0$ in view of (ii). Therefore

$$\operatorname{Tr} U_N(\Phi)^n = O(N\epsilon) + o_\epsilon(N), \tag{84}$$

i.e.,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} U_N(\Phi)^n = O(\epsilon), \tag{85}$$

which holds for every arbitrarily small $\epsilon > 0$. This concludes the proof. $\square$

2.36. **Theorem.** *Weyl's law.* Suppose $\Phi : \mathcal{M} \to \mathcal{M}$ is piecewise smooth, and for every $n \neq 0$ the fixed points of Φ^n form a set of measure zero. Then, for every continuous function $h : \mathbb{T} \to \mathbb{C}$,

$$\lim_{N\to\infty} \sum_{j=1}^{N} h(\theta_j) = \int_{\mathbb{T}} h(\theta)\, d\theta. \tag{86}$$

Proof. Since $\operatorname{Tr} U_N(\Phi)^n = \sum_j e^{2\pi i n\theta_j}$, Theorem 2.36 follows from Theorem 2.35 by applying Weyl's criterion from the theory of distribution modulo 1; for details see any good number theory text book.

We give an explicit proof which will be useful below in generalizing this result. Let us first assume that the test function h has only finitely many non-zero Fourier coefficients, i.e.,

$$h(\theta) = \sum_{n\in\mathbb{Z}} \widehat{h}(n) e(n\theta) \tag{87}$$

is a finite sum. We then have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h(\theta_j) = \lim_{N\to\infty} \frac{1}{N} \sum_{n\in\mathbb{Z}} \widehat{h}(n) \operatorname{Tr} U_N(\Phi)^n = \widehat{h}(0) \tag{88}$$

which proves the theorem for h with finite Fourier series. We now extend this result to test functions $h \in C^1(\mathbb{T})$. Let

$$h_K(\theta) = \sum_{\substack{n\in\mathbb{Z}\\ |n|\leq K}} \widehat{h}(n) e(n\theta) \tag{89}$$

be the truncated Fourier series. Since $h \in C^1(\mathbb{T})$, its Fourier series converges absolutely and uniformly and hence, for any $\epsilon > 0$, there is a K such that $h_K(\theta) - \epsilon \leq h(\theta) \leq h_K(\theta) + \epsilon$ for all $\theta \in \mathbb{T}$. By (86), the limits of the left and right hand side of

$$\frac{1}{N} \sum_{j=1}^{N} h_K(\theta_j) - \epsilon \leq \frac{1}{N} \sum_{j=1}^{N} h(\theta_j) \leq \frac{1}{N} \sum_{j=1}^{N} h_K(\theta_j) + \epsilon \tag{90}$$

exist and differ by less than 2ϵ, hence (86) holds also for the current h. The extension of (86) to h in $C(\mathbb{T})$ is achieved by the same argument, i.e., by approximating h pointwise by functions $h_\epsilon \in C^1(\mathbb{T})$ so that $h_\epsilon(\theta) - \epsilon \leq h(\theta) \leq h_\epsilon(\theta) + \epsilon$. $\square$

2.37. Note that the last argument also allows the extension to test functions h that are characteristic functions of an interval $\subset \mathbb{T}$.

2.38. **Theorem.** *Generalised trace asymptotics.* Choose Φ and $U_N(\Phi)$ as in Theorem 2.35. Then for every $a \in C^\infty(\mathcal{M})$ and $n \neq 0$,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}[\operatorname{Op}_N(a) U_N(\Phi)^n] = 0. \tag{91}$$

Proof. By linearity of the relation (91) we may assume without loss of generality that a is real and $\min_\xi a(\xi) \geq 0$. This implies that $a^{1/2} \in C^\infty(\mathcal{M})$. Analogously to the proof of Theorem 2.35, we have

$$\begin{aligned} \operatorname{Tr}[\operatorname{Op}_N(a) U_N(\Phi)^n] &= \operatorname{Tr}[U_N(\Phi)^n \operatorname{Op}_N(\widetilde{\chi}_{\mathrm{bad}} \cdot a)] \\ &\quad + \sum_{r=1}^{R} \operatorname{Tr}[U_N(\Phi)^n \operatorname{Op}_N(\widetilde{\chi}_r \cdot a)] + o_\epsilon(N). \end{aligned} \tag{92}$$

The proof is concluded in the same way as the proof of Theorem 2.35, with all mollified characteristic functions $\widetilde{\chi}$ replaced by $\widetilde{\chi} \cdot a$. □

2.39. **Exercise.** Write out the full proof of Theorem 2.38.

2.40. **Theorem.** Generalised Weyl's law. Choose Φ and $U_N(\Phi)$ as in Theorem 2.36. Let $\varphi_j \in \mathbb{C}^N$ $(j = 1, \ldots, N)$ be an orthonormal basis of eigenstates of $U_N(\Phi)$, with corresponding eigenphases θ_j. Then, for every $a \in C^\infty(\mathcal{M})$ and for every continuous function $h : \mathbb{T} \to \mathbb{C}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h(\theta_j) \langle \operatorname{Op}_N(a) \varphi_j, \varphi_j \rangle = \int_{\mathbb{T}} h(\theta) d\theta \int_{\mathcal{M}} a \, d\mu. \tag{93}$$

Proof. We may assume again without loss of generality that a is real and $\min_\xi a(\xi) \geq 0$. In view of Theorem 2.38 and the proof of Theorem 2.36 we have for every h_K with finite Fourier expansion (as in (87))

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h_K(\theta_j) \langle \operatorname{Op}_N(a) \varphi_j, \varphi_j \rangle = \int_{\mathcal{M}} a \, d\mu \int_{\mathbb{T}} h_K(\theta) \, d\theta. \tag{94}$$

For any $h \geq 0$ we have

$$\begin{aligned} &\left| \sum_{j=1}^{N} h(\theta_j) \langle \operatorname{Op}_N(a) \varphi_j, \varphi_j \rangle - \sum_{j=1}^{N} h(\theta_j) \| \operatorname{Op}_N(a^{1/2}) \varphi_j \|^2 \right| \\ &\qquad \leq \sup h \left| \operatorname{Tr}[\operatorname{Op}_N(a) - \operatorname{Op}_N(a^{1/2}) \operatorname{Op}_N(a^{1/2})^\dagger] \right| = o(N) \sup h. \end{aligned} \tag{95}$$

Hence (94) is equivalent to

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h_K(\theta_j) \| \operatorname{Op}_N(a^{1/2}) \varphi_j \|^2 = \int_{\mathcal{M}} a \, d\mu \int_{\mathbb{T}} h_K(\theta) \, d\theta. \tag{96}$$

We now use the same approximation argument as in the proof of Theorem 2.36, for $h \in C^1(\mathbb{T})$. Given any ϵ, there is a K such that $h_K(\theta) - \epsilon \le h(\theta) \le h_K(\theta) + \epsilon$ for all $\theta \in \mathbb{T}$. The limits of the left and right hand side of

$$\frac{1}{N}\sum_{j=1}^{N}[h_K(\theta_j) - \epsilon]\| \operatorname{Op}_N(a^{1/2})\varphi_j\|^2 \le \frac{1}{N}\sum_{j=1}^{N} h(\theta_j)\| \operatorname{Op}_N(a^{1/2})\varphi_j\|^2 \tag{97}$$

$$\le \frac{1}{N}\sum_{j=1}^{N}[h_K(\theta_j) + \epsilon]\| \operatorname{Op}_N(a^{1/2})\varphi_j\|^2 \tag{98}$$

differ by less than

(99)

$$2\epsilon \sup h_K \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N}\| \operatorname{Op}_N(a^{1/2})\varphi_j\|^2 \le 2\epsilon \sup h_K \lim_{N\to\infty}\frac{1}{N}\operatorname{Tr}[\operatorname{Op}_N(a^{1/2})\operatorname{Op}_N(a^{1/2})^\dagger]$$

$$= 2\epsilon \sup h_K \int_{\mathcal{M}} a \, d\mu \tag{100}$$

which can be arbitrarily small for $\epsilon \to 0$. Thus

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N} h(\theta_j)\| \operatorname{Op}_N(a^{1/2})\varphi_j\|^2 = \int_{\mathcal{M}} a\, d\mu \int_{\mathbb{T}} h(\theta)\, d\theta. \tag{101}$$

A similar approximation argument shows that (101) holds also for all continuous h. In view of (95), the relation (101) is equivalent to (93). The assumption $h \ge 0$ can be removed by using the linearity of (93) in h. □

Let us now turn to the question of quantum ergodicity for ergodic maps. Examples of linked twist maps which are ergodic are discussed in 1.5.

2.41. **Theorem.** Let Φ and $U_N(\Phi)$ as in Theorem 2.36, and suppose in addition Φ is ergodic. Let $\varphi_1, \dots, \varphi_N \in \mathbb{C}^N$ be an orthonormal basis of eigenstates of $U_N(\phi)$. Then, for any $a \in C^\infty(\mathcal{M})$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\left| \langle \operatorname{Op}_N(a)\varphi_j, \varphi_j\rangle - \int_{\mathcal{M}} a\, d\mu \right|^2 = 0. \tag{102}$$

Proof. We may assume without loss of generality that $\int_{\mathcal{M}} a\, d\mu = 0$ and $|a| \le 1$. It is then sufficient to show

$$S_2(a, N) := \frac{1}{N}\sum_{j=1}^{N}\left|\langle \operatorname{Op}_N(a)\varphi_j, \varphi_j\rangle\right|^2 \to 0 \tag{103}$$

as $N \to \infty$. The argument presented here is inspired by the proof of quantum ergodicity for cat maps [3, 12], cf. also [15].

For any given $T \ge 1$, we may write

$$a = a_T + a_T^{\text{bad}} \tag{104}$$

where

(i) $a_T \in C^\infty$ has compact support not containing the sets $\Phi(\mathcal{S}_\Phi), \Phi^2(\mathcal{S}_\Phi), \ldots, \Phi^T(\mathcal{S}_\Phi)$, and furthermore $\int a_T \, d\mu = 0$, $|a_T| \le 1$;

(ii) $\int_{\mathcal{M}} |a_T^{\mathrm{bad}}|^2 \, d\mu < T^{-1}$.

By the triangle inequality,

$$S_2(a, N)^{1/2} \le S_2(a_T, N)^{1/2} + S_2(a_T^{\mathrm{bad}}, N)^{1/2} \tag{105}$$

Furthermore, by the Cauchy-Schwartz inequality,

$$S_2(a_T^{\mathrm{bad}}, N) \le \frac{1}{N} \sum_{j=1}^{N} \left\| \mathrm{Op}_N(a_T^{\mathrm{bad}}) \varphi_j \right\|^2 \tag{106}$$

$$= \frac{1}{N} \operatorname{Tr} \mathrm{Op}_N(|a_T^{\mathrm{bad}}|^2) + o_T(1) \tag{107}$$

and hence

$$\limsup_{N \to \infty} S_2(a_T^{\mathrm{bad}}, N) < T^{-1}. \tag{108}$$

As to the remaining term,

$$S_2(a_T, N) = \frac{1}{N} \sum_{j=1}^{N} |\langle \mathrm{Op}_N(a_T) \varphi_j, \varphi_j \rangle|^2, \tag{109}$$

it remains to be proved that the limsup of (109) can be made arbitrarily small for sufficiently large T. To this end define the ergodic average of a_T by

$$a_T^T := \frac{1}{T} \sum_{n=1}^{T} a_T \circ \Phi^n. \tag{110}$$

Since φ_j are the eigenfunctions of $U_N(\Phi)$ we have

$$S_2(a_T, N) = \frac{1}{N} \sum_{j=1}^{N} \left| \frac{1}{T} \sum_{n=1}^{T} \left\langle \mathrm{Op}_N(a_T) \varphi_j, \varphi_j \right\rangle \right|^2 \tag{111}$$

$$= \frac{1}{N} \sum_{j=1}^{N} \left| \frac{1}{T} \sum_{n=1}^{T} \left\langle \mathrm{Op}_N(a_T) e(n\theta_j) \varphi_j, e(n\theta_j) \varphi_j \right\rangle \right|^2 \tag{112}$$

$$= \frac{1}{N} \sum_{j=1}^{N} \left| \frac{1}{T} \sum_{n=1}^{T} \left\langle \mathrm{Op}_N(a_T) U_N(\Phi)^n \varphi_j, U_N(\Phi)^n \varphi_j \right\rangle \right|^2 \tag{113}$$

$$= \frac{1}{N} \sum_{j=1}^{N} \left| \frac{1}{T} \sum_{n=1}^{T} \left\langle U_N(\Phi)^{-n} \mathrm{Op}_N(a_T) U_N(\Phi)^n \varphi_j, \varphi_j \right\rangle \right|^2. \tag{114}$$

So by Cauchy-Schwartz,

$$S_2(a_T, N) \le \frac{1}{N}\sum_{j=1}^{N}\left\|\frac{1}{T}\sum_{n=1}^{T} U_N(\Phi)^{-n}\operatorname{Op}_N(a_T)U_N(\Phi)^n\varphi_j\right\|^2 \tag{115}$$

$$= \frac{1}{N}\sum_{j=1}^{N}\left\|\operatorname{Op}_N(a_T^T)\varphi_j\right\|^2 + o_T(1), \tag{116}$$

by Axiom 1.10. Now

$$\frac{1}{N}\sum_{j=1}^{N}\left\|\operatorname{Op}_N(a_T^T)\varphi_j\right\|^2 = \frac{1}{N}\sum_{j=1}^{N}\langle \operatorname{Op}_N(a_T^T)^\dagger \operatorname{Op}_N(a_T^T)\varphi_j, \varphi_j\rangle \tag{117}$$

$$= \frac{1}{N}\sum_{j=1}^{N}\langle \operatorname{Op}_N(|a_T^T|^2)\varphi_j, \varphi_j\rangle + o_T(1) \tag{118}$$

$$= \int_{\mathcal{M}} |a_T^T|^2 d\mu + o_T(1). \tag{119}$$

We have

$$\left(\int_{\mathcal{M}} |a_T^T|^2 d\mu\right)^{1/2} \le \left(\int_{\mathcal{M}} |a^T|^2 d\mu\right)^{1/2} + \left(\int_{\mathcal{M}} |a_T^{\mathrm{bad}T}|^2 d\mu\right)^{1/2} \tag{120}$$

where

$$a^T := \frac{1}{T}\sum_{n=1}^{T} a\circ\Phi^n, \qquad a_T^{\mathrm{bad}T} := \frac{1}{T}\sum_{n=1}^{T} a_T^{\mathrm{bad}}\circ\Phi^n. \tag{121}$$

We have by Cauchy-Schwartz

$$\int_{\mathcal{M}} |a_T^{\mathrm{bad}T}|^2 d\mu = \frac{1}{T^2}\sum_{j,k=1}^{T}\int_{\mathcal{M}} (a_T^{\mathrm{bad}}\circ\Phi^j)\overline{(a_T^{\mathrm{bad}}\circ\Phi^k)}\, d\mu \tag{122}$$

$$\le \frac{1}{T^2}\sum_{j,k=1}^{T}\left(\int_{\mathcal{M}} |a_T^{\mathrm{bad}}\circ\Phi^j|^2\, d\mu\right)^{1/2}\left(\int_{\mathcal{M}} |a_T^{\mathrm{bad}}\circ\Phi^k|^2\, d\mu\right)^{1/2} \tag{123}$$

$$= \int_{\mathcal{M}} |a_T^{\mathrm{bad}}|^2\, d\mu < \frac{1}{T}, \tag{124}$$

using the Φ-invariance of μ and assumption (ii). Therefore

$$\limsup_{N\to\infty} S_2(a_T, N) \le \int_{\mathcal{M}} |a^T|^2 d\mu + O(T^{-1}). \tag{125}$$

Since Φ is ergodic, we have a mean ergodic theorem for test functions $a \in L^2(\mathcal{M})$, i.e.,

$$\lim_{T\to\infty}\int_{\mathcal{M}} |a^T|^2 d\mu = 0, \tag{126}$$

and hence $\limsup_{N\to\infty} S_2(a_T, N)$ becomes arbitrarily small for T sufficiently large. □

2.42. **Exercise.** Show that there is a sequence of sets $I_N \subset \{1, \dots, N\}$ such that $|I_N|/N \to 1$ as $N \to \infty$, and

$$\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j\rangle \to \int_{\mathcal{M}} a \, d\mu \tag{127}$$

for all $j \in I_N$, as $N \to \infty$.

Hint: Apply Chebyshev's inequality with the variance given in (102).

Lecture III: Bouncing ball modes

3.43. **Definition.** A vector $\psi \in \mathbb{C}^N \setminus \{0\}$ is called a *quasimode* of the unitary matrix U_N with *quasi-eigenphase* η and *discrepancy* ϵ, if

$$\|(U_N - e^{2\pi i \eta})\psi\| \le \epsilon \|\psi\|. \tag{128}$$

3.44. By expanding ψ in an orthonormal basis of eigenfunctions, $\psi = \sum_{j=1}^N \langle \psi, \varphi_j\rangle \varphi_j$, it is easy to see that (128) implies

$$\sum_{j=1}^N |\langle \psi, \varphi_j\rangle|^2 |e^{2\pi i \theta_j} - e^{2\pi i \eta}|^2 = \|(U_N - e^{2\pi i \eta})\psi\|^2 \le \epsilon^2 \|\psi\|^2 = \epsilon^2 \sum_{j=1}^N |\langle \psi, \varphi_j\rangle|^2, \tag{129}$$

where θ_j is the eigenphase of U_N corresponding to φ_j. Hence $|e^{2\pi i \theta_j} - e^{2\pi i \eta}| \le \epsilon$ for at least one j.

3.45. For $\delta > \epsilon$, consider the set $J = \{\theta : |e^{2\pi i \theta} - e^{2\pi i \eta}| < \delta\}$. We have

$$\sum_{\theta_j \notin J} |\langle \psi, \varphi_j\rangle|^2 \le \delta^{-2} \sum_{\theta_j \notin J} |\langle \psi, \varphi_j\rangle|^2 |e^{2\pi i \theta_j} - e^{2\pi i \eta}|^2 \le (\epsilon/\delta)^2 \|\psi\|^2, \tag{130}$$

and so

$$\sum_{\theta_j \in J} |\langle \psi, \varphi_j\rangle|^2 \ge [1 - (\epsilon/\delta)^2] \|\psi\|^2. \tag{131}$$

Hence there is at least one eigenphase $\theta_j \in J$ such that

$$|\langle \psi, \varphi_j\rangle|^2 \ge \frac{1 - (\epsilon/\delta)^2}{|J|} \|\psi\|^2. \tag{132}$$

That is, the closer $\frac{1-(\epsilon/\delta)^2}{|J|}$ is to one, the closer ψ is to φ_j.

3.46. **Example.** Fix $\ell \in (0, 1)$. We consider quantum linked twist maps

$$U_N(\Phi) = U_N(\Psi_f)\, \mathcal{F}_N^{-1}\, U_N(\Psi_g)\, \mathcal{F}_N, \tag{133}$$

with

(134)

$$[U_N(\Psi_f)\psi](Q) = e\left[-NV\left(\frac{Q}{N}\right)\right] \psi(Q), \qquad [U_N(\Psi_g)\psi](P) = e\left[-NT\left(\frac{P}{N}\right)\right] \psi(P),$$

where $V(q) = 0$ when $0 \le q < \ell$ and otherwise arbitrary, and $T \in C^2(\mathbb{R})$ such that

$$NT\left(\frac{P}{N}+1\right) = NT\left(\frac{P}{N}\right) \bmod 1. \tag{135}$$

Examples are $T(p) = p^2 + \tau(p)$ or $T(p) = (p - \frac{1}{2})^2 + \tau(p)$, with a periodic function $\tau \in C^2(\mathbb{T})$.

3.47. The corresponding classical map (8) has $f = -V'$ and $g = -T'$.

3.48. Let

$$h_k(q) = \chi_\ell(q) e^{2\pi i k q}, \qquad \chi_\ell(q) = \ell^{-1/2}\chi(\ell^{-1} q), \tag{136}$$

with $k \in \mathbb{R}$ and $\chi \in C^\infty(\mathbb{R})$ a mollified characteristic function with compact support in $(0, 1)$. Define

$$\begin{aligned} \psi_k(Q) &= \frac{1}{\sqrt{N}} \sum_{m\in\mathbb{Z}} h_k\left(\frac{Q}{N}+m\right) \\ &= \frac{1}{\sqrt{N}} \sum_{m\in\mathbb{Z}} \chi_\ell\left(\frac{Q}{N}+m\right) e_N(k(Q+Nm)). \end{aligned} \tag{137}$$

By construction,

$$U_N(\Psi_f)\psi_k = \psi_k. \tag{138}$$

We will show that the ψ_k are good quasimodes of $U_N(\Phi)$, for suitable choices of k.

3.49. **Theorem.** Let $\sigma = \sigma(N) \ge 0$. Assume $T \in C^{\nu+1}(\mathbb{R})$ and $T^{(\nu+1)}$ is uniformly bounded, for some $\nu \ge 1$. Then, for every $k \in \mathbb{R}$ that satisfies

$$\left| T^{(\mu)}\left(\frac{k}{N}\right)\right| \le \left(\frac{\sigma}{N}\right)^{\nu-\mu+1} \quad \text{for all } \mu = 1, \dots, \nu, \tag{139}$$

we have

$$\|(U_N(\Phi) - e^{2\pi i \eta_k})\psi_k\| \le C_\nu \left(\frac{1+\ell\sigma}{\ell N}\right)^\nu \|\psi_k\|, \qquad \eta_k = -NT(k), \tag{140}$$

for some $C_\nu > 0$ independent of k, ℓ, σ, N.

3.50. **Example.** Let $T(p) = p^2$. Then $T'(p) = 2p$ and $T''(p) = 2$. Hence we apply the above theorem for $\nu = 1$ to obtain

$$\|(U_N(\Phi) - e^{2\pi i \eta_k})\psi_k\| \le C \left(\frac{1+\ell|k|}{\ell N}\right) \|\psi_k\|, \qquad \eta_k = -\frac{k^2}{N}, \tag{141}$$

Hence we obtain good quasimodes for k close to zero.

Furthermore, note that

$$NT\left(\frac{P}{N}\right) = N\left(\frac{P}{N}-\frac{1}{2}\right)^2 - \frac{N}{4} \bmod 1. \tag{142}$$

Thus we apply Theorem 3.49 with $\tilde{T}(p) = (p-\frac{1}{2})^2$ instead of $T(p)$. Here $\tilde{T}'(p) = 2p-1$, $\tilde{T}''(p) = 2$, so

$$\|(U_N(\Phi) - e^{2\pi i \eta_k})\psi_k\| \leq C \left(\frac{1 + \ell|k - \frac{N}{2}|}{\ell N} \right) \|\psi_k\|, \tag{143}$$

with quasi-eigenphase

$$\eta_k = -N\left(\frac{k}{N} - \frac{1}{2}\right)^2 + \frac{N}{4} = -\frac{k^2}{N} + k. \tag{144}$$

So (143) says that good quasimodes will have k close to $\frac{N}{2}$.

Proof of Theorem 3.49. We first calculate the L^2 norm:

$$\begin{aligned} \|\psi_k\|^2 &= \frac{1}{N} \sum_{Q=0}^{N-1} \sum_{m,m' \in \mathbb{Z}} \chi_\ell\left(\frac{Q}{N} + m\right) \chi_\ell\left(\frac{Q}{N} + m'\right) \\ &= \frac{1}{N} \sum_{Q=0}^{N-1} \chi_\ell\left(\frac{Q}{N}\right)^2 \\ &= \int_{\mathbb{R}} \chi_\ell(q)^2 dq + O(N^{-R}) \\ &= \int_{\mathbb{R}} \chi(q)^2 dq + O(N^{-R}) \end{aligned} \tag{145}$$

for any $R > 0$, since χ is C^∞ and of compact support in $(0,1)$.

By (138), it suffices to show that $\widehat{\psi}_k := \mathcal{F}\psi_k$ is a good quasimode of $U_N(\Psi_g)$. Let us first derive an explicit formula for $\widehat{\psi}_k$. By the Poisson summation formula

$$\begin{aligned} \psi_k(Q) &= \frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} h_k\left(\frac{Q}{N} + q\right) e^{-2\pi i n q} dq \\ &= \frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}} \widehat{h}_k(n) e_N(nQ) \\ &= \frac{1}{\sqrt{N}} \sum_{P=0}^{N-1} \sum_{n \in \mathbb{Z}} \widehat{h}_k(P + Nn) e_N(PQ), \end{aligned} \tag{146}$$

with the Fourier transform

$$\widehat{h}_k(p) = \int_{\mathbb{R}} h_k(q) e^{-2\pi i q p} dq. \tag{147}$$

The above allows us to read off the Fourier transform of $\psi_k(Q)$:

$$\widehat{\psi}_k(P) := [\mathcal{F}_N \psi_k](P) = \frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}} \widehat{h}_k(P + Nn). \tag{148}$$

With the above choice of h_k,

$$\widehat{h}_k(p) = \sqrt{\ell}\,\widehat{\chi}(\ell(p-k)). \tag{149}$$

Since χ is C^∞ and has compact support, for any $R > 0$ there is a constant $C_R > 0$ (independent of p, k, ℓ) such that

$$\widehat{h}_k(p) \leq C_R \frac{\sqrt{\ell}}{(1+\ell|p-k|)^R}. \tag{150}$$

Thus

$$\widehat{\psi}_k(P) = \sqrt{\frac{\ell}{N}} \sum_{n\in\mathbb{Z}} \widehat{\chi}\big(\ell(P-k+Nn)\big), \tag{151}$$

which converges rapidly.

Now,

$$[U_N(\Psi_g)\widehat{\psi}_k](P) = e\left[-NT\left(\frac{P}{N}\right)\right]\widehat{\psi}_k(P) \tag{152}$$

Suppose $T \in C^{\nu+1}(\mathbb{R})$; then

$$\begin{aligned} &T\left(\frac{P}{N}+n\right) \\ &= T\left(\frac{k}{N}\right) + \sum_{\mu=1}^{\nu} \frac{1}{\mu!} T^{(\mu)}\left(\frac{k}{N}\right)\left(\frac{P-k}{N}+n\right)^{\mu} + \frac{1}{(\nu+1)!}T^{(\nu+1)}(\xi)\left(\frac{P-k}{N}+n\right)^{\nu+1} \end{aligned} \tag{153}$$

for some ξ between $\frac{P}{N}+n$ and $\frac{k}{N}$.

If $\left|T^{(\mu)}\left(\frac{k}{N}\right)\right| \leq \left(\frac{\sigma}{N}\right)^{\nu-\mu+1}$ for $1 \leq \mu \leq \nu$, and $|T^{(\mu)}|$ uniformly bounded, then

$$T\left(\frac{P}{N}+n\right) = T\left(\frac{k}{N}\right) + O\left(\left(\frac{P-k}{N}+n\right)\max\left\{\left(\frac{\sigma}{N}\right)^{\nu}, \left|\frac{P-k}{N}+n\right|^{\nu}\right\}\right). \tag{154}$$

So

$$\begin{aligned} &[U_N(\Psi_g)\widehat{\psi}_k](P) = e\left[-NT\left(\frac{k}{N}\right)\right]\widehat{\psi}_k(P) \\ &+ \sqrt{\frac{\ell}{N}}\sum_{n\in\mathbb{Z}} O\left(N\left(\frac{P-k}{N}+n\right)\max\left\{\left(\frac{\sigma}{N}\right)^{\nu}, \left|\frac{P-k}{N}+n\right|^{\nu}\right\}\right)\widehat{\chi}\big(\ell(P-k+Nn)\big) \end{aligned} \tag{155}$$

and so

$$\left|\left[\left(U_N(\Psi_g) - e\left[-NT(k)\right]\right)\widehat{\psi}_k\right](P)\right| \leq \frac{1}{(\ell N)^{\nu+\frac{1}{2}}}\sum_{n\in\mathbb{Z}} M\big(\ell(P-k+Nn)\big) \tag{156}$$

where

$$M(\xi) = O\big(\xi \max\left\{\ell^{\nu}\sigma^{\nu}, |\xi|^{\nu}\right\}\big)\widehat{\chi}(\xi) \tag{157}$$

is a rapidly decaying function on $\mathbb{R}$. The calculation of the L^2 is similar to that of ψ_k above, and we conclude

$$\left\|\left(U_N(\Psi_g) - e\left[-NT(k)\right]\right)\widehat{\psi}_k\right\| \leq C_\nu\left(\frac{1+\ell\sigma}{\ell N}\right)^{\nu}\|\widehat{\psi}_k\| \tag{158}$$

for some constant $C_\nu > 0$. This completes the proof of Theorem 3.49. □

3.51. Note that the inner product of two quasimodes is

$$
\begin{aligned}
\langle \psi_k, \psi_{k'} \rangle &= \sum_{Q=0}^{N-1} \psi_k(Q) \overline{\psi}_{k'}(Q) \\
&= \frac{1}{N} \sum_{Q=0}^{N-1} \chi_\ell \left(\frac{Q}{N} \right)^2 e_N((k-k')Q) \\
&= \int_{\mathbb{R}} \chi_\ell(q)^2 e((k-k')q) dq + O(N^{-R}) \\
&= O\big((1+|k-k'|)^{-R}\big) + O\big(N^{-R}\big)
\end{aligned} \tag{159}
$$

for any $R > 0$, since χ is C^∞ and of compact support.

3.52. Let $U_N = U_N(\Phi)$ be the quantum linked twist maps considered above. For $\kappa \in (-1, 1)$ and $a \in C^\infty(\mathbb{T}^2)$ let us consider the perturbation

$$U_N^{(\kappa)} = e\big(\kappa \operatorname{Op}_N(a)\big)\, U_N. \tag{160}$$

We denote the corresponding eigenfunctions by $\varphi_j^{(\kappa)}$, and eigenphases by $\theta_j^{(\kappa)}$, so that

$$U_N^{(\kappa)} \varphi_j^{(\kappa)} = e\big(\theta_j^{(\kappa)}\big) \varphi_j^{(\kappa)}. \tag{161}$$

Clearly $U_N^{(\kappa)}$ defines a holomorphic family of matrices, and one can apply standard perturbation theory to show that the eigenprojections and eigenvalues depend analytically on κ; see e.g. Chapter II of [7]. By expanding the eigenfunctions and eigenvalues in power series in κ, one finds (except for at most finitely many κ),

$$\frac{d\theta_j^{(\kappa)}}{d\kappa} = \langle \varphi_j^{(\kappa)}, \operatorname{Op}_N(a)\ \varphi_j^{(\kappa)} \rangle. \tag{162}$$

3.53. **Exercise.** Prove (162).

3.54. **Example.** A natural choice is $a(p, q) = a(q)$. In this case $U_N^{(\kappa)}$ is of the same form as in the previous section, with $V(q)$ replaced by

$$V_\kappa(q) = V(q) + \frac{\kappa}{N}\ a(q). \tag{163}$$

This is a second order perturbation, and thus has not affect on the underlying classical dynamics. Here (162) becomes

$$
\begin{aligned}
\frac{d\theta_j^{(\kappa)}}{d\kappa} &= \langle \varphi_j^{(\kappa)}, a\ \varphi_j^{(\kappa)} \rangle \\
&= \sum_{Q=0}^{N-1} a\left(\frac{Q}{N}\right) \big|\varphi_j^{(\kappa)}(Q)\big|^2.
\end{aligned} \tag{164}
$$

If a is supported in $[\ell, 1]$ mod 1 (as is V) then the quasimodes ψ_k, their discrepancy and quasi-eigenphase remain the same, for all κ.

3.55. In order to show that our quasimodes have non-vanishing (as $N \to \infty$) overlap with some sequence of eigenfunctions $\varphi^{(\kappa)}$, we need to show that (recall 3.45)

$$\mathcal{N}_N^{(\kappa)}(\lambda) = \#\{j = 1, \dots, N : \theta_j^{(\kappa)} \in [-\lambda, \lambda] + \mathbb{Z}\} \tag{165}$$

is bounded, when λ is of the order of the quasimode discrepancy $(\frac{\sigma}{N})^\nu$. This is difficult to show for fixed κ, but (following [6]) better estimates are possible on average over κ.

3.56. Let us fix $a \in C^\infty(\mathbb{T}^2)$ as in 3.54 with $a \geq 0$ and $\int_{\mathbb{T}^2} a \, d\mu = 1$. Given $b \in (0, 1]$, take

$$Z_N(b) = \left\{ \kappa \in (-1, 1) : \langle \varphi_j^{(\kappa)}, \mathrm{Op}_N(a) \ \varphi_j^{(\kappa)} \rangle \geq b \text{ for all } j \right\}. \tag{166}$$

Then, in view of (162),

$$\begin{aligned} \frac{1}{|Z_N(b)|} \int_{Z_N(b)} \mathcal{N}_N^{(\kappa)}(\lambda) d\kappa &= \frac{1}{|Z_N(b)|} \sum_{j=0}^{N-1} \int_{Z_N(b)} \chi(\theta_j^{(\kappa)} \in [-\lambda, \lambda] + \mathbb{Z}) d\kappa \\ &\leq \frac{1}{b|Z_N(b)|} \sum_{j=N_0}^{N-1} \int_{Z_N(b)} \chi(\theta_j^{(\kappa)} \in [-\lambda, \lambda] + \mathbb{Z}) \frac{d\theta_j^{(\kappa)}}{d\kappa} d\kappa \\ &\leq \frac{1}{b|Z_N(b)|} \sum_{j=0}^{N-1} \int_{-1}^{1} \chi(\theta_j^{(\kappa)} \in [-\lambda, \lambda] + \mathbb{Z}) \frac{d\theta_j^{(\kappa)}}{d\kappa} d\kappa \\ &= \frac{2N\lambda}{b|Z_N(b)|}. \end{aligned} \tag{167}$$

Thus, if $0 < \epsilon < 1$, then there is a set $W_\epsilon \subset Z_N(b)$ of measure

$$|W_\epsilon| > |Z_N(b)|(1 - \epsilon), \tag{168}$$

such that

$$\mathcal{N}_N^{(\kappa)}(\lambda) \leq \frac{2N\lambda}{\epsilon b |Z_N(b)|} \tag{169}$$

for all $\kappa \in W_\epsilon$. Hence for quasimodes with discrepancy of order $1/N$, we can choose $\lambda \asymp 1/N$ and the above bound is finite, provided $|Z_N(b)| \geq c$ for some constant $c > 0$. Hence in this case our bouncing ball quasimodes have indeed finite overlap with at most finitely many eigenstates for each N.

3.57. Let us now discuss the case when $|Z_N(b)|$ is small. Then the the complement of $Z_N(b)$ is

$$(-1, 1) \setminus Z_N(b) = \left\{ \kappa \in (-1, 1) : \langle \varphi_j^{(\kappa)}, \mathrm{Op}_N(a) \ \varphi_j^{(\kappa)} \rangle < b \text{ for some } j \right\}. \tag{170}$$

This set has measure $2 - |Z_N(b)|$, i.e., close to full measure. In this case there is at least one eigenstate which does not become equidistributed with respect to μ, since its mass in the support of a is less than $b < 1$. That is, quantum unique ergodicity fails for all $\kappa \notin Z_N(b)$. Note that the corresponding eigenstates might not necessarily correspond to bouncing ball modes, since they could in principle be localized on other invariant measures which are singular with respect to μ, and have their support outside the support of a. (In the case of the stadium [6] the analogue of a is supported in a sufficiently big set so that indeed the bouncing ball modes are the only possibility.)

Gallery of bouncing ball eigenstates

The following pictures and tables are taken from Stephen O'Keefe's PhD Thesis [10]. They show eigenstates of the quantum linked twist map

$$\mathcal{F}_N^{-1}\, U_N(\Psi_g)\, \mathcal{F}_N\, U_N(\Psi_f) \tag{171}$$

which is clearly conjugate to (133), with the particular choice $T(p) = p^2$ and

$$V(q) = \begin{cases} 0 & (0 \le q < \frac{1}{2}) \\ -q^2 & (\frac{1}{2} \le q < 1). \end{cases} \tag{172}$$

Note that $g(p) = -T'(p) = -2p$ and

$$f(q) = -V'(q) = \begin{cases} 0 & (0 \le q < \frac{1}{2}) \\ 2q & (\frac{1}{2} \le q < 1), \end{cases} \tag{173}$$

hence the corresponding classical map is ergodic; recall Theorem 1.5 (this corresponds to case (i)).

The quantum states $\psi \in \mathbb{C}^N$ are represented as density plots of the Husimi functions

$$H_\psi(p,q) = \left| (2N)^{1/4} \sum_{Q=0}^{N-1} \sum_{m\in\mathbb{Z}} e^{-\pi N[(\frac{Q}{N}+m-q)^2 + 2ip(\frac{Q}{N}+m-q)]} \psi(Q) \right|^2 . \tag{174}$$

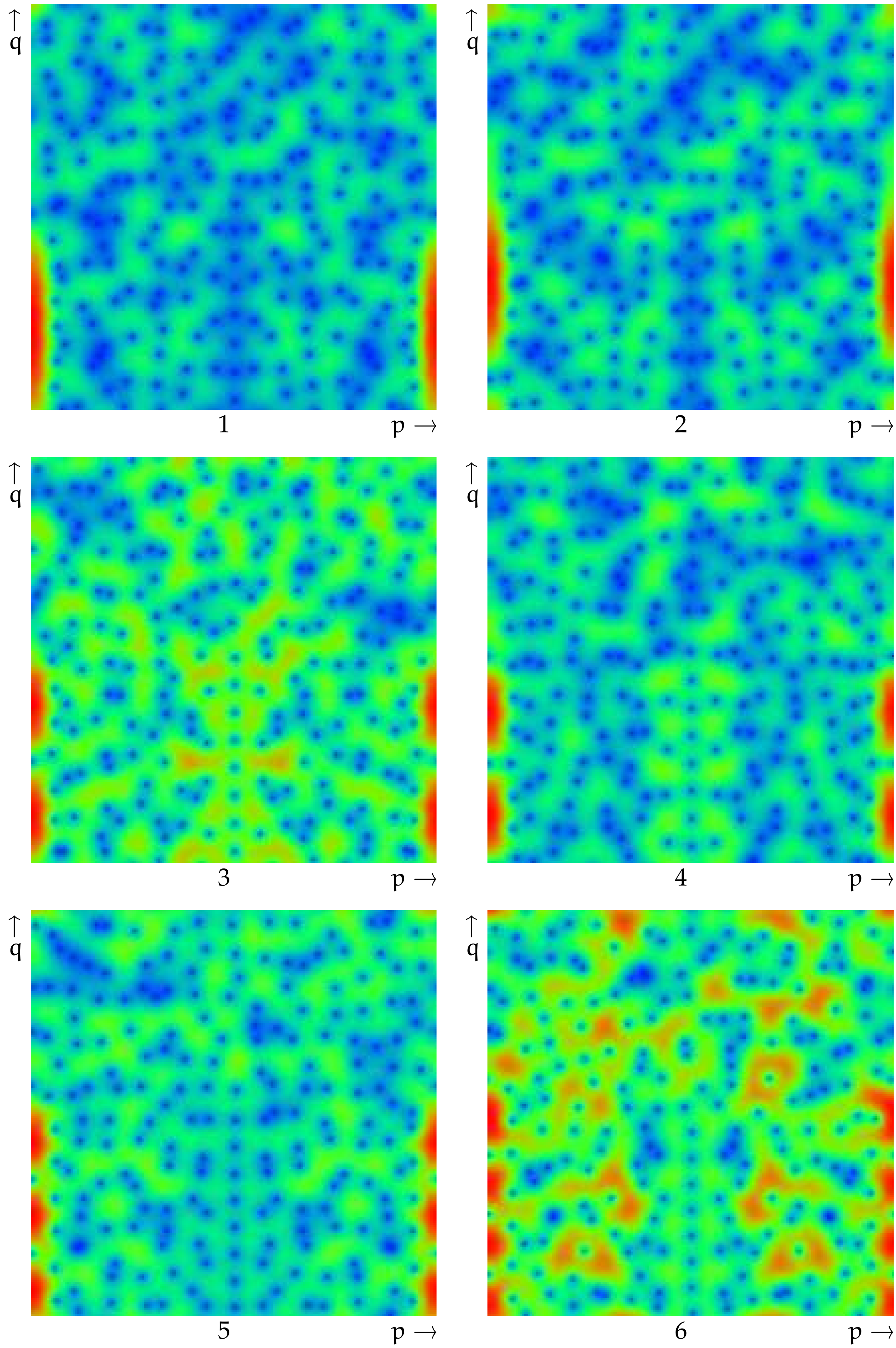
In the following tables we list data corresponding to 17 bouncing ball quantum eigenstates. We list the eigenphases of the bouncing ball mode, θ_j, together with the eigenphases of the previous and following state, θ_{j-1} and θ_{j+1}. Our above analysis shows that a large gap between eigenphases increases the accuracy of the approximation by quasimodes. We also list the parameter k corresponding to the quasimode ψ_k (assuming θ_j equals the quasi-eigenphase, which need not be the optimal choice). In the first set of examples (1-8) we have $\theta_j = -\frac{k^2}{N}$, in the second (9-17) we have $\theta_j = -\frac{k^2}{N} + k$. We also list the quantities

$$L = \sum_{0\le Q<1/2} |\varphi_j(Q)|^2, \qquad R = \sum_{1/2\le Q<N} |\varphi_j(Q)|^2, \tag{175}$$

to compare the mass distribution of each eigenstate.

The calculations were performed in dimension $N = 201$.

Ref.	$N\theta_j$	$N\theta_{j-1}$	$N\theta_{j+1}$	$\|k\|$	L	R
1	-0.77644	-1.11428	0.28588	0.8812	0.782	0.218
2	-1.11428	-2.48920	-0.77644	1.056	0.714	0.286
3	-3.30602	-3.73872	-2.48920	1.818	0.667	0.333
4	-3.73872	-4.57805	-3.30602	1.934	0.744	0.266
5	-7.36481	-7.79200	-6.42704	2.714	0.669	0.331
6	-11.70866	-11.99325	-10.80775	3.422	0.544	0.456
7	-15.91510	-17.53742	-13.97987	3.989	0.584	0.416
8	-30.03456	-30.35960	-28.09362	5.480	0.494	0.506
Ref.	$N\theta_j$	$N\theta_{j-1}$	$N\theta_{j+1}$	$\|k-\frac{N}{2}\|$	L	R
9	49.44264	48.57090	50.36306	0.899	0.963	0.037
10	47.26942	46.44128	48.32636	1.726	0.777	0.223
11	46.44128	46.25964	47.26942	1.952	0.617	0.383
12	43.66588	43.40560	45.24036	2.566	0.657	0.343
13	41.98385	41.06977	43.40560	2.875	0.640	0.360
14	41.06977	39.80613	41.98385	3.030	0.524	0.476
15	37.91422	37.33891	39.80613	3.512	0.555	0.445
16	35.22250	33.37426	36.38226	3.877	0.612	0.388
17	30.19735	29.63694	32.46626	4.478	0.473	0.527


↑ q
1
p →
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2
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5
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6
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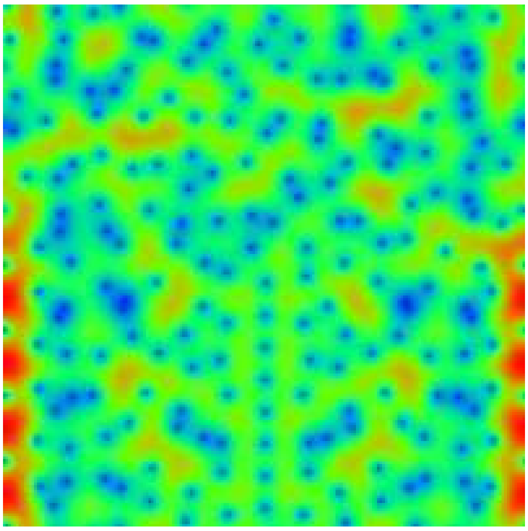
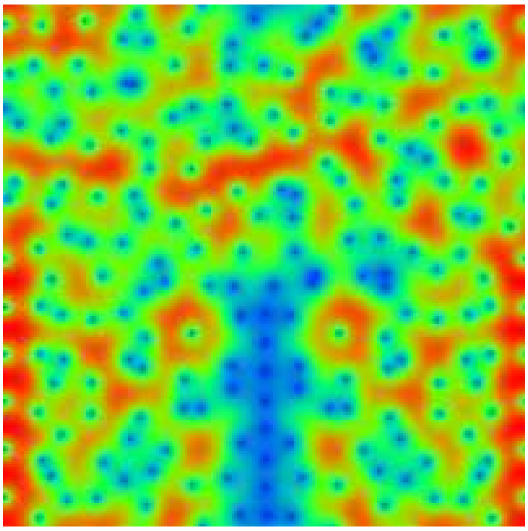
↑ q



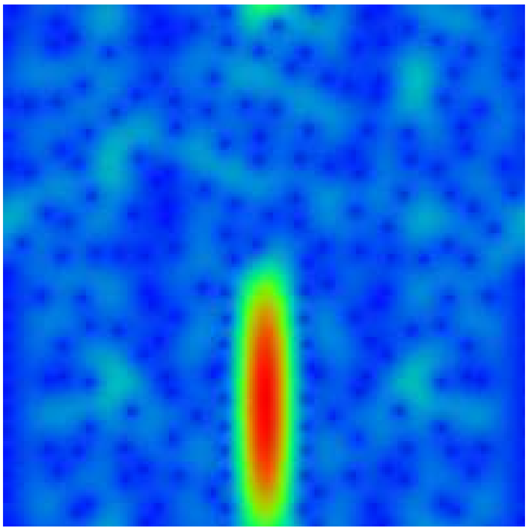
7 p →

↑ q



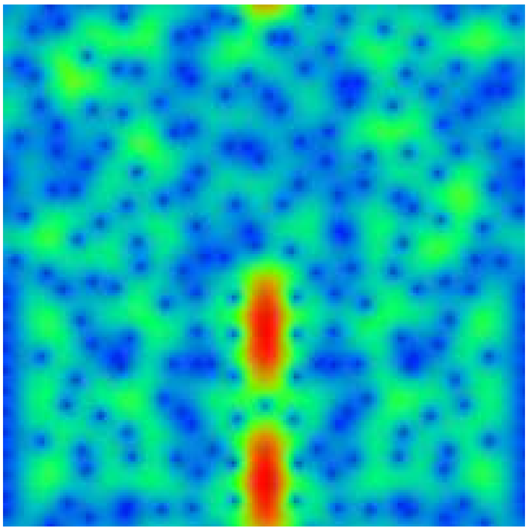
8 p →

↑ q



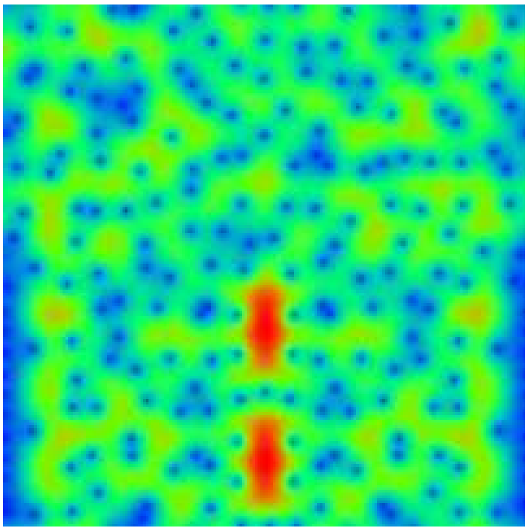
9 p →

↑ q



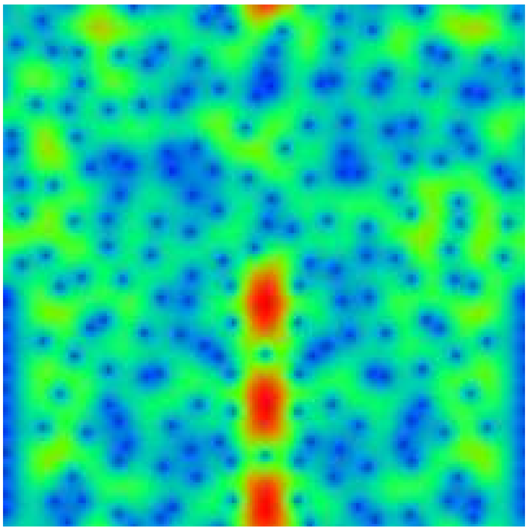
10 p →

↑ q



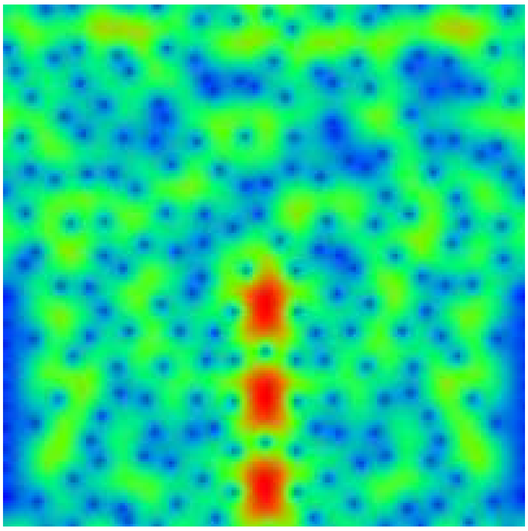
11 p →

↑ q



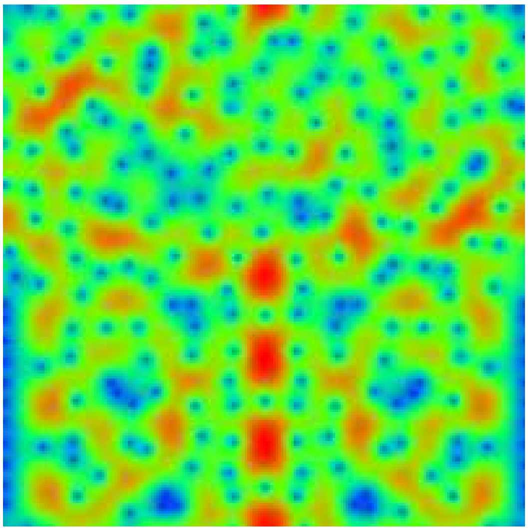
12 p →

↑ q



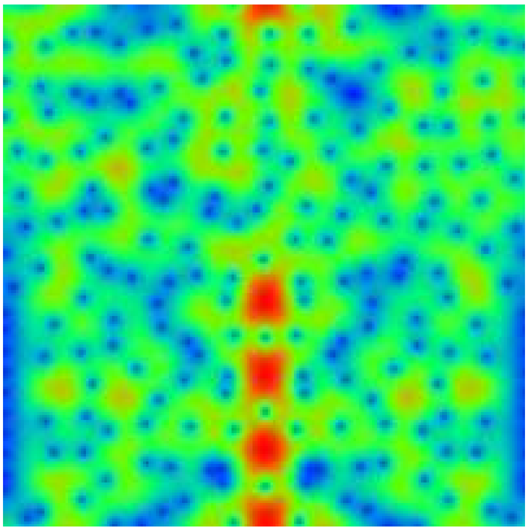
13 p →

↑ q



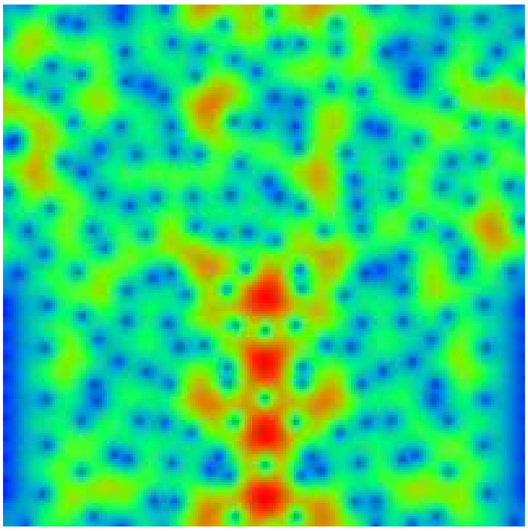
14 p →

↑ q



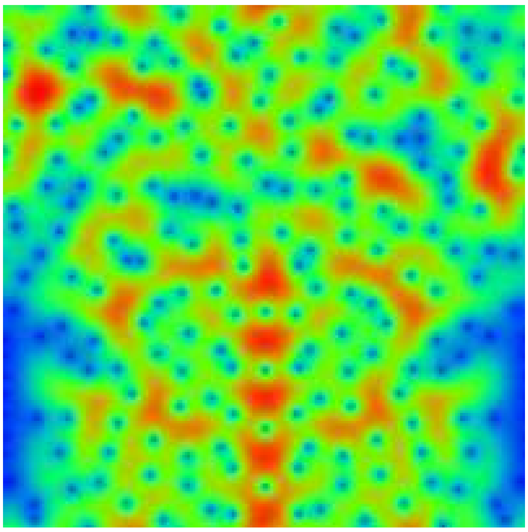
15 p →

↑ q



16 p →

↑ q



17 p →

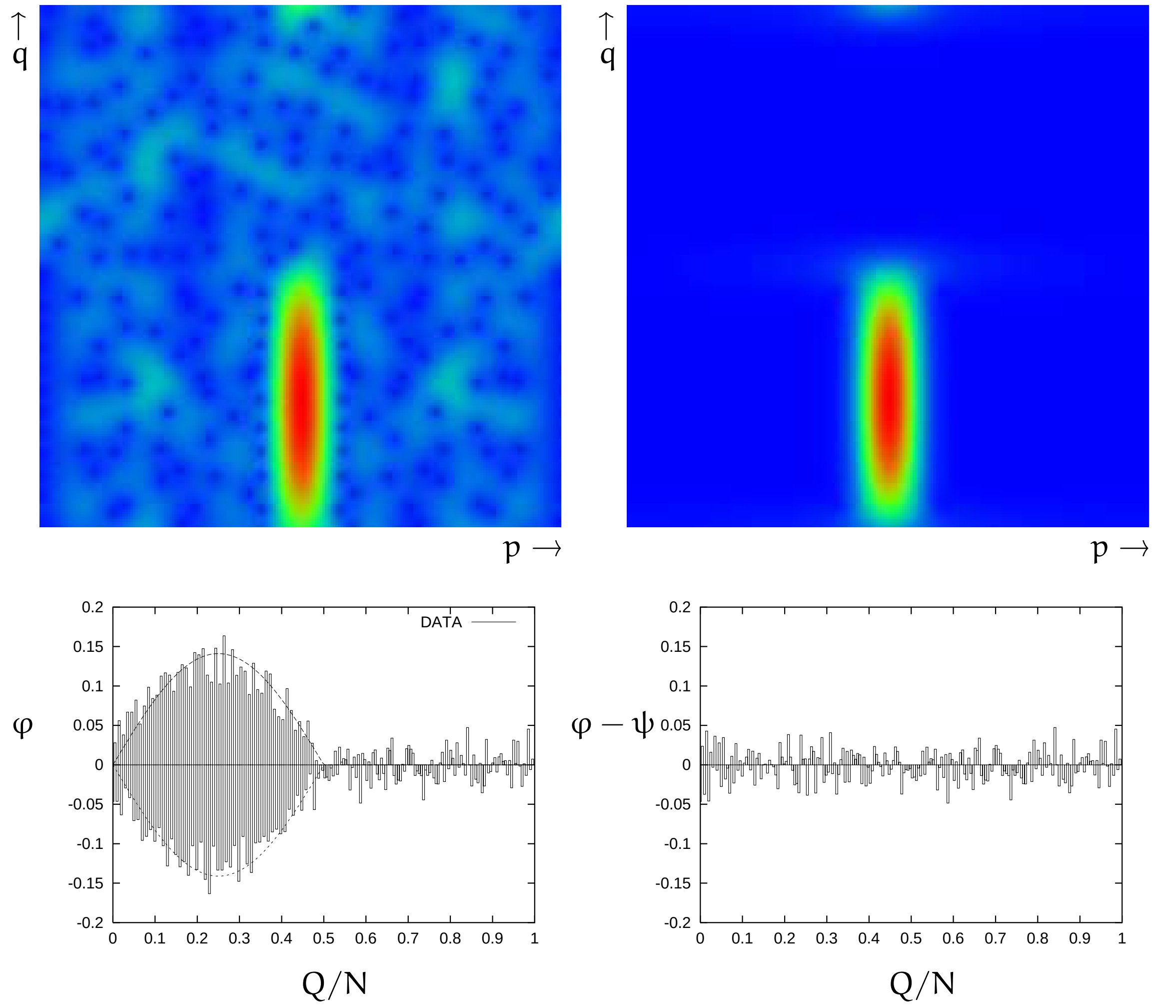
The numerical computation suggests that eigenstate 9 is approximated exceptionally well by the quasimode

$$\psi(Q) = \frac{1}{i\sqrt{2}}\{\psi_{N/2+1}(Q) - \psi_{N/2-1}(Q)\} \tag{176}$$

where ψ_k as in (137), with $\ell = 1/2$ and χ the (*unsmoothed*) characteristic function of $[0, 1]$. Explicitly:

$$\psi(Q) = \begin{cases} N^{-1/2}(-1)^Q \sin(2\pi Q/N) & (0 \leq Q \leq N/2) \\ 0 & (N/2 \leq Q \leq N). \end{cases} \tag{177}$$

The following shows the comparison of eigenstate φ and quasimode ψ: both as a Husimi plot and as a function of Q.



3.58. **Project.**

(1) Define your own favourite ergodic linked twist map Φ on the torus.

(2) Plot the iterates under the map of initial data which is (a) generic and (b) close to a fixed point.
(3) Find the quantization $U_N(\Phi)$.
(4) Diagonalize $U_N(\Phi)$ (using Maple, Mathematica, Matlab, etc.).
(5) Plot its eigenstates as Husimi functions.
(6) Identify bouncing ball modes or other localized eigenstates (“scars”), and compare them with the corresponding quasimodes.
(7) Plot the distribution of spacings between consecutive eigenphases.

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