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Weyl's law and quantum ergodicity for maps with divided phase space

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With an appendix

Converse quantum ergodicity

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Abstract

For a general class of unitary quantum maps, whose underlying classical phase space is divided into several invariant domains of positive measure, we establish analogues of Weyl's law for the distribution of eigenphases. If the map has one ergodic component, and is periodic on the remaining domains, we prove the Schnirelman–Zelditch–Colin de Verdière theorem on the equidistribution of eigenfunctions with respect to the ergodic component of the classical map (quantum ergodicity). We apply our main theorems to quantized linked twist maps on the torus. In the appendix, Zelditch connects these studies to some earlier results on 'pimpled spheres' in the setting of Riemannian manifolds. The common feature is a divided phase space with a periodic component.

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1. Introduction

The quantization of an invertible map Φ on a d-dimensional compact manifold $\mathcal{M}$ produces a unitary $N \times N$ matrix, the *quantum map* $U_N(\Phi)$. If there is a quantization recipe that works for an infinite sequence of integers N, one natural question is whether dynamical properties of the *classical map* Φ are recovered in the *semiclassical limit* $N \to \infty$. In this paper, we prove that this is indeed possible for maps whose phase space $\mathcal{M}$ is divided into several Φ-invariant sets of positive measure, provided the quantization recipe satisfies a *correspondence principle*: quantum and classical evolution of observables must be equivalent in the semiclassical limit. The precise formulation of the necessary axioms is given in section 2. These are kept fairly general and allow applications outside of quantum mechanics. In particular, $\mathcal{M}$ is not required to be symplectic.

Let $\varphi_j \in \mathbb{C}^N$ $(j = 1, \ldots, N)$ be an orthonormal basis of eigenvectors of $U_N(\Phi)$, and $\theta_j \in \mathbb{R}$ the corresponding eigenphases defined by the relation

$$U_N(\Phi)\,\varphi_j = \mathrm{e}^{2\pi\mathrm{i}\theta_j}\,\varphi_j. \tag{1.1}$$

In section 5, we prove an analogue of Weyl's law [2, 22, 43, 44] for the limiting distribution of the eigenphases as $N \to \infty$. Let us denote by μ an invariant probability measure of Φ on $\mathcal{M}$ which is absolutely continuous with respect to Lebesgue measure. We furthermore assume that the map is periodic on a collection of sets $\mathcal{D}_1, \mathcal{D}_2, \ldots$ with positive measure $\mu(\mathcal{D}_\nu) > 0$ and boundary whose fractal d-dimensional Minkowski content is zero[1], and that on the remaining set $\mathcal{D}_0$ the periodic orbits of Φ of any given period form a set with d-dimensional Minkowski content equal to zero. The number of eigenphases in some generic[2] interval $[a, b]$ with $b - a \leqslant 1$ is then

$$\lim_{N\to\infty} \frac{1}{N}\#\{j = 1, \ldots, N : \theta_j \in [a, b] \bmod 1\} = \sum_{\nu=0}^{\infty} \mu(\mathcal{D}_\nu) \int_a^b \rho_\nu(\theta)\,\mathrm{d}\theta, \tag{1.2}$$

where $\rho_0(\theta) = 1$ is the uniform probability density mod 1 and, for $\nu \geqslant 1$, $\rho_\nu(\theta)$ is the uniform probability density supported on the points $(k/n_\nu) + \alpha_{\mathcal{D}_\nu}$; here, n_ν denotes the period of Φ on $\mathcal{D}_\nu$ and the constant $\alpha_{\mathcal{D}_\nu} \in \mathbb{R}$ depends on the chosen quantization recipe. This means in particular that if $\mathcal{M} = \mathcal{D}_0$, the spectrum of the quantum map is uniformly distributed on the unit circle. A formula analogous to (1.2) has been obtained by Zelditch [46], theorem 3.20, in the case of the wave group for a compact Riemannian manifold. A special case of formula (1.2) is proved in [49] for quantized contact transformations, and in [9] for perturbed cat maps; in both cases the set of periodic orbits has measure zero and thus $\mathcal{M} = \mathcal{D}_0$.

The main result of this paper describes the semiclassical distribution of the quantum map's eigenstates on the classical manifold. If the quantization constants $\alpha_{\mathcal{D}_1}, \alpha_{\mathcal{D}_2}, \ldots$ are linearly independent over the rationals, then, for N large, approximately $N \times \mu(\mathcal{D}_\nu)$ of the N eigenstates localize on the set $\mathcal{D}_\nu$ (section 7). If in addition the classical map acts ergodically on $\mathcal{D}_0$, then almost all of the eigenstates localized on $\mathcal{D}_0$ are in fact equidistributed on $\mathcal{D}_0$ (section 8). This latter result may be viewed as an extension of the *Schnirelman–Zelditch–Colin de Verdière theorem*, originally formulated for completely ergodic Hamiltonian flows [13, 21, 23, 40, 45, 50] and maps [8, 15, 48, 49], to maps with partially ergodic phase space. The possibility that eigenfunctions localize exclusively on the ergodic or on the integrable component had been conjectured by Percival in the 1970s [36], and is known as the *semiclassical wave function hypothesis*.

Two typical and two untypical examples of eigenstates of a quantized linked twist map on a two-dimensional torus (see section 3.2), with $N = 201$, are displayed as a Husimi density

[1] Fractal Minkowski contents and dimensions also play an important role in the Weyl–Berry conjecture for the distribution of eigenvalues of the Laplacian of domains in $\mathbb{R}^d$ with fractal boundary, see [4, 5, 27, 33] and references therein.

[2] *Generic* means here that a, b are not in the singular support of the limit density (which in the present case is a countable set).

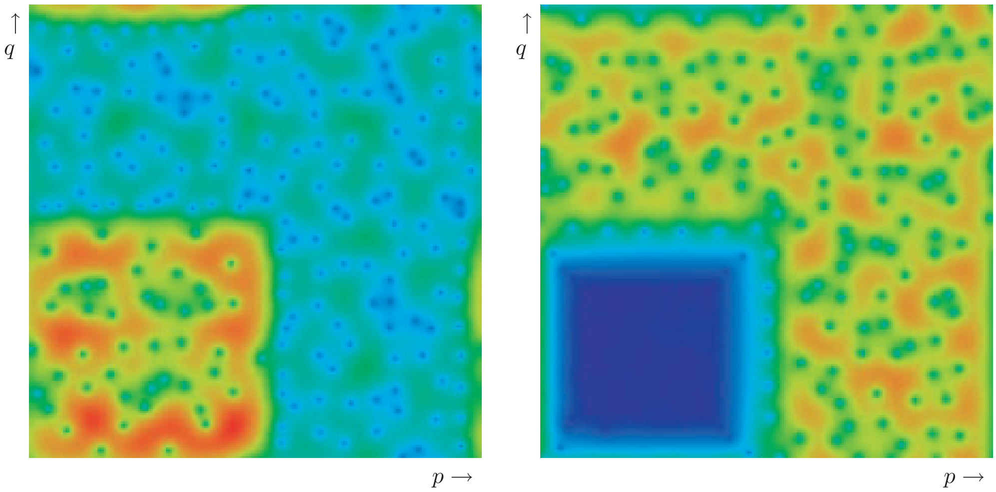

Figure 1. Two typical quantum eigenstates of a linked twist map on the torus for $N = 201$, one localized in the lower left quadrant $\mathcal{D}_1$ (left image) and the other in the complement $\mathcal{D}_0$ (right image). According to theorem 7.1, approximately 25% respectively 75% of all eigenstates behave in this way—the latter states are in fact equidistributed on $\mathcal{D}_0$, cf theorem 8.1.

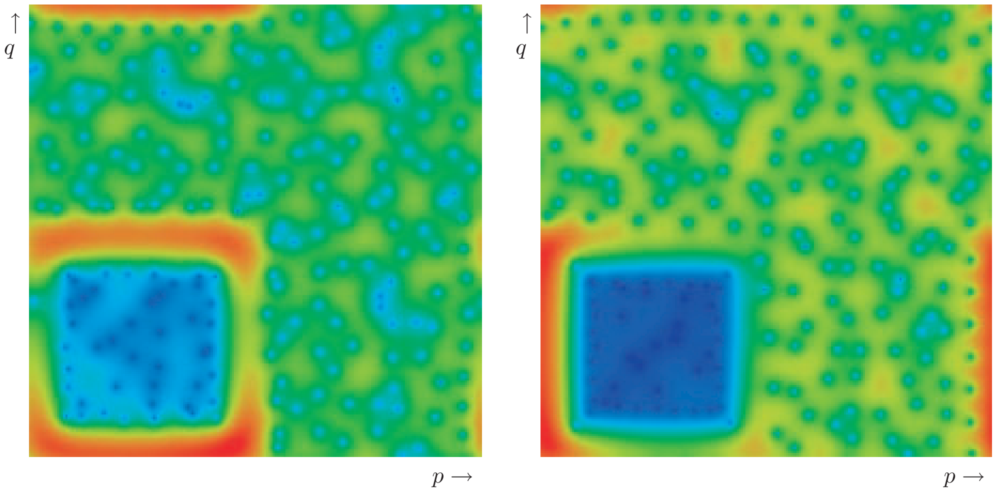

Figure 2. Two untypical quantum eigenstates of the same linked twist map as in figure 1, with $N = 201$. These eigenstates are localized near the boundary between the domains $\mathcal{D}_0$ and $\mathcal{D}_1$; the eigenstate on the right has comparable mass in both domains. Eigenstates of this type form a sequence of density zero, cf theorem 7.1.

plot[3] in figures 1 and 2. The classical map acts as the identity on the lower left quadrant and ergodically on the L-shaped complement (section 3.3). Figure 3 shows the integrated density

$$\mathcal{I}_N(\theta) := \frac{1}{N}\#\left\{j = 1, \ldots, N : \theta_j \in \left[-\frac{1}{2}, \theta\right] \bmod 1\right\} \tag{1.3}$$

[3] See, e.g. [1, 34] for a detailed discussion of Husimi functions.

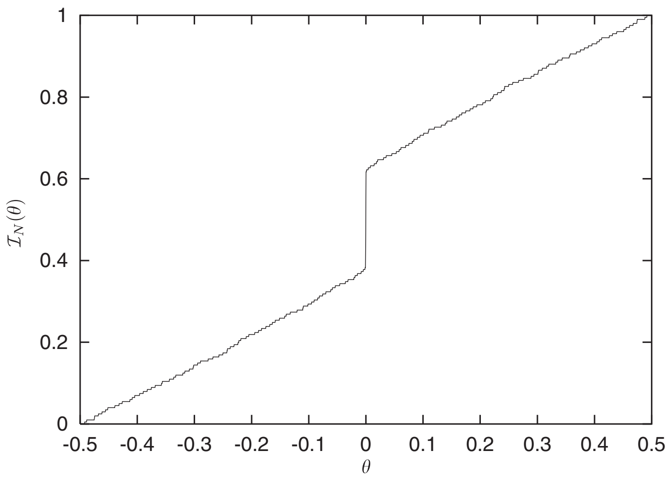

Figure 3. The integrated density of quantum eigenphases $\mathcal{I}_N(\theta)$ of the linked twist map in figure 1, with $N = 201$. The jump at $\theta = 0$ by $\frac{1}{4}$ reflects the fact that 25% of eigenphases are asymptotically zero. The remaining 75% of eigenphases are uniformly distributed (modulo one) in the interval $[-\frac{1}{2}, \frac{1}{2}]$, in accordance with Weyl's law (1.4), cf theorem 5.2.

of quantum eigenphases of the map with θ ranging from $-\frac{1}{2}$ to $\frac{1}{2}$. In this case, the limit (1.2) evaluates to

$$\lim_{N\to\infty} \mathcal{I}_N(\theta) = \begin{cases} \frac{3}{4}\theta + \frac{3}{8} & \text{for } \theta \in \left[-\frac{1}{2}, 0\right), \\ \frac{3}{4}\theta + \frac{5}{8} & \text{for } \theta \in \left(0, \frac{1}{2}\right]. \end{cases} \tag{1.4}$$

Very similar quantum maps have been investigated numerically in [30]; our theorems apply to those cases with sharply divided phase space and rational frequencies in all elliptic islands. A further interesting family of toral maps with mixed dynamics are the *lazy baker maps* [28]. Here, the phase space is divided into countably many rational elliptic islands of total measure one; the hyperbolic dynamics takes place on a fractal set of Lebesgue measure zero.

In the case of Hamiltonian flows with partially ergodic phase space[4], Schubert [41] has proved a result analogous to our quantum ergodicity theorem (theorem 8.1), which however only holds for quasimode solutions of the Schrödinger equation and not necessarily for the eigenstates itself. The problem is that near-degeneracies in the spectrum of the quantum Hamiltonian (which in general cannot be ruled out) may create eigenstates that are extended across the entire phase space although the corresponding quasimodes are localized on the flow invariant components. For the quantum maps considered here, the spectrum can be controlled sufficiently well to rule out this possibility. A similar observation results from the analysis of the wave group on Riemannian manifolds where geodesic flow is periodic on some open invariant component, see the appendix by Zelditch.

It should be emphasized that our results do not address the question of the possible localization of eigenstates on sets of measure zero (such as scarred eigenstates, bouncing ball modes or the recently discovered hierarchical states [24]). Results in this direction have recently been obtained in the case of cat maps [6, 19, 20] and piecewise affine maps on the torus [12], which neatly complement the proofs of quantum unique ergodicity for cat maps [17, 25, 26], parabolic maps [31] and the modular surface [29, 42] (see the survey [39]).

[4] Examples of such systems are the billiard flows in Bunimovich's mushrooms [10].

Concrete examples of maps satisfying the axioms set out in section 2 are toral linked twist maps, see sections 3.1 and 3.2. The specific structure of these maps permits a more detailed asymptotic analysis, which will be presented elsewhere [35].

The axioms in section 2 are in fact sufficiently general to allow also applications to sequences of unitary matrices without quantum mechanical interpretation. In section 3.4, we discuss an application arising in the discretization of classical evolution operators.

2. Set-up

Let $\mathcal{M}$ be a d-dimensional compact smooth manifold and μ a probability measure on $\mathcal{M}$ which is absolutely continuous with respect to Lebesgue measure.

We fix an atlas of local charts $\phi_j : \mathcal{V}_j \to \mathbb{R}^d$, where the open subsets $\mathcal{V}_j$ cover $\mathcal{M}$. In the following, we thus identify subsets $\mathcal{S}$ of $\mathcal{M}$ with subsets Σ of $\mathbb{R}^d$ in the standard way. Let Σ be a subset of $\mathbb{R}^d$, and

$$\Sigma(\epsilon) = \{\xi \in \mathbb{R}^d : d(\xi, \Sigma) \leqslant \epsilon\} \tag{2.1}$$

its closed ϵ-neighbourhood, where $d(\cdot, \cdot)$ is the Euclidean metric on $\mathbb{R}^d$. The *s-dimensional upper Minkowski content* of Σ is defined as

$$\mathfrak{M}^{*s}(\Sigma) := \limsup_{\epsilon\to 0}(2\epsilon)^{s-d}\nu(\Sigma(\epsilon)), \tag{2.2}$$

where ν is the Lebesgue measure, see [32, section 5.5]. We say Σ *has Minkowski content zero* if $\mathfrak{M}^{*d}(\Sigma) = 0$. This is equivalent to saying that for every $\delta > 0$ we can cover Σ with equiradial Euclidean balls of total measure less than δ. We say *a subset $\mathcal{S}$ of $\mathcal{M}$ has Minkowski content zero* if each of the sets $\Sigma_j := \phi_j(\mathcal{S}|_{\mathcal{V}_j}) \subset \mathbb{R}^d$ has Minkowski content zero.

We consider piecewise smooth invertible maps $\Phi : \mathcal{M} \to \mathcal{M}$ that preserve μ. By *piecewise smooth* we mean here and in the following that there is a partitioning of $\mathcal{M}$ into countably many open sets $\mathcal{U}_i$, i.e. $\mathcal{M} = \overline{\bigcup_i \mathcal{U}_i}$ and $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$, so that $\Phi|_{\bar{\mathcal{U}}_i}$ is smooth[5], and the boundary set $\overline{\bigcup_i \partial\mathcal{U}_i}$ has Minkowski content zero. We will refer to this set as the *domain of discontinuity of* Φ and call its complement $\mathcal{M} - \overline{\bigcup_i \partial\mathcal{U}_i}$ the *domain of continuity of* Φ.

Let $\mathrm{M}_N(\mathbb{C})$ be the space of $N \times N$ matrices with complex coefficients. For a given infinite subset (*index set*) $\mathcal{I} \subset \mathbb{N}$, we say two sequences of matrices,

$$\boldsymbol{A} := \{A_N\}_{N\in\mathcal{I}}, \qquad \boldsymbol{B} := \{B_N\}_{N\in\mathcal{I}}, \tag{2.3}$$

are *semiclassically equivalent*, if

$$\|A_N - B_N\| \to 0 \tag{2.4}$$

as $N \in \mathcal{I}$ tends to infinity, where $\|\cdot\|$ denotes the usual operator norm

$$\|A\| := \sup_{\psi\in\mathbb{C}^N-\{0\}} \frac{\|A\psi\|}{\|\psi\|}. \tag{2.5}$$

We denote this equivalence relation by

$$\boldsymbol{A} \sim \boldsymbol{B}. \tag{2.6}$$

[5] This means that $\Phi|_{\bar{\mathcal{U}}_i}$ and all its derivatives are bounded continuous functions $\bar{\mathcal{U}}_i \to \mathcal{M}$; we allow for the possibility that those bounds are not uniform in i.

Lemma 2.1. *If* $\boldsymbol{A} \sim \boldsymbol{B}$ *then* $\operatorname{Tr} A_N = \operatorname{Tr} B_N + o(N)$.

Proof. We have

$$\frac{1}{N}|\operatorname{Tr} A_N - \operatorname{Tr} B_N| \leqslant \|A_N - B_N\| \to 0. \tag{2.7}$$

□

Let us define the product of two matrix sequences by $\boldsymbol{AB} = \{A_N B_N\}_{N\in\mathcal{I}}$, the inverse of $\boldsymbol{A}$ by $\boldsymbol{A}^{-1} = \{A_N^{-1}\}_{N\in\mathcal{I}}$ and its Hermitian conjugate by $\boldsymbol{A}^{\dagger} = \{A_N^{\dagger}\}_{N\in\mathcal{I}}$.

Axiom 2.1 (the correspondence principle for quantum observables). *Fix a measure* μ *as above. For some index set* $\mathcal{I} \subset \mathbb{N}$*, there is a sequence* $\mathbf{Op} := \{\mathrm{Op}_N\}_{N\in\mathcal{I}}$ *of linear maps,*

$$\mathrm{Op}_N : \mathrm{C}^{\infty}(\mathcal{M}) \to \mathrm{M}_N(\mathbb{C}), \qquad a \mapsto \mathrm{Op}_N(a),$$

so that

(a) for all $a \in \mathrm{C}^{\infty}(\mathcal{M})$,

$$\mathbf{Op}(\bar{a}) \sim \mathbf{Op}(a)^{\dagger};$$

(b) for all $a_1, a_2 \in \mathrm{C}^{\infty}(\mathcal{M})$,

$$\mathbf{Op}(a_1)\mathbf{Op}(a_2) \sim \mathbf{Op}(a_1 a_2);$$

(c) for all $a \in \mathrm{C}^{\infty}(\mathcal{M})$,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} \mathrm{Op}_N(a) = \int_{\mathcal{M}} a \, \mathrm{d}\mu.$$

Examples of quantum observables satisfying these conditions are given in section 3.1. In standard quantization recipes (such as the one discussed in section 3.1), one in addition has the property that

$$\mathbf{Op}(a_1)\mathbf{Op}(a_2) - \mathbf{Op}(a_2)\mathbf{Op}(a_1) \sim \frac{1}{2\pi \mathrm{i} N} \mathbf{Op}(\{a_1, a_2\}), \tag{2.8}$$

where $\{\,,\}$ is the Poisson bracket. This assumption is however not necessary for any of the results proved in this paper. The axioms (a)–(c) in fact apply to examples without quantum mechanical significance. One interesting case arises in the discretization of linked twist maps, where

$$\mathbf{Op}(a_1)\mathbf{Op}(a_2) = \mathbf{Op}(a_1 a_2) = \mathbf{Op}(a_2)\mathbf{Op}(a_1), \tag{2.9}$$

see section 3.4.

Axiom 2.2 (the correspondence principle for quantum maps). *There is a sequence of unitary matrices* $\boldsymbol{U}(\Phi) := \{U_N(\Phi)\}_{N\in\mathcal{I}}$ *such that for any* $a \in \mathrm{C}^{\infty}(\mathcal{M})$ *with compact support contained in the domain of continuity of* Φ*, we have*

$$\boldsymbol{U}(\Phi)^{-1}\mathbf{Op}(a)\boldsymbol{U}(\Phi) \sim \mathbf{Op}(a \circ \Phi).$$

In the following we consider maps Φ that may be periodic on a collection of disjoint sets $\mathcal{D}_\nu \subset \mathcal{M}$ ($\nu = 1, 2, \ldots$) of positive measure $\mu(\mathcal{D}_\mu) > 0$, with periods n_ν so that $\Phi^{n_\nu}\big|_{\mathcal{D}_\nu} = \mathrm{id}$. In addition to axioms 2.1 and 2.2, we will here stipulate that there are constants $\alpha_{\mathcal{D}_\nu} \in \mathbb{R}$ such that

$$\boldsymbol{U}(\Phi)^{n_\nu}\mathbf{Op}(a) \sim e(n_\nu \alpha_{\mathcal{D}_\nu})\,\mathbf{Op}(a), \qquad e(z) := \exp(2\pi \mathrm{i} z) \tag{2.10}$$

for any $a \in \mathrm{C}^\infty(\mathcal{M})$ with compact support contained in $\mathcal{D}_\nu$ and the domains of continuity of $\Phi, \Phi^2, \ldots, \Phi^{n_\nu}$. Whereas axioms 2.1 and 2.2 are satisfied by all standard quantization schemes, condition (2.10) is more restrictive: the constant $\alpha_{\mathcal{D}_\nu}$ could, for instance, be replaced by $\mathbf{Op}(\beta_\nu)$ where β_ν is a non-constant smooth function on $\mathcal{D}_\nu$; note the relation

$$\boldsymbol{U}(\Phi)^{n_\nu}\mathbf{Op}(a) \sim e(n_\nu \mathbf{Op}(\beta_\nu))\,\mathbf{Op}(a) \tag{2.11}$$

is still consistent with axiom 2.2. Condition (2.10) is however essential in the proofs of our main results since the spectrum of $e(\mathbf{Op}(\beta_\nu))$ may, in general, be dense on the unit circle in $\mathbb{C}$.

In sections 3.2 and 3.4, we discuss examples of semiclassical sequences of quantum maps satisfying axioms 2.1, 2.2 and condition (2.10).

3. Example: linked twist maps

In this section, we construct a well known example of quantum observables on the two-dimensional torus $\mathcal{M} = \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ satisfying axiom 2.1 (cf [16]), and corresponding examples of quantum linked twist maps satisfying axiom 2.2.

3.1. Quantum tori

It is convenient to represent a vector $\psi \in \mathbb{C}^N$ as a function $\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. Let us define the translation operators

$$[t_1\psi](Q) = \psi(Q+1) \tag{3.1}$$

and

$$[t_2\psi](Q) = e_N(Q)\psi(Q), \tag{3.2}$$

where $e_N(x) := e(x/N) = \exp(2\pi \mathrm{i} x/N)$. One easily checks that

$$t_1^{m_1} t_2^{m_2} = t_2^{m_2} t_1^{m_1} e_N(m_1 m_2) \qquad \forall m_1, m_2 \in \mathbb{Z}. \tag{3.3}$$

These relations are known as the *Weyl–Heisenberg commutation relations*. For $\boldsymbol{m} = (m_1, m_2) \in \mathbb{Z}^2$ put

$$T_N(\boldsymbol{m}) = e_N\left(\frac{m_1 m_2}{2}\right) t_2^{m_2} t_1^{m_1}. \tag{3.4}$$

Then,

$$T_N(\boldsymbol{m}) T_N(\boldsymbol{n}) = e_N\left(\frac{\omega(\boldsymbol{m},\boldsymbol{n})}{2}\right) T_N(\boldsymbol{m}+\boldsymbol{n}) \tag{3.5}$$

with the symplectic form

$$\omega(\boldsymbol{m},\boldsymbol{n}) = m_1 n_2 - m_2 n_1. \tag{3.6}$$

For any $a \in \mathrm{C}^\infty(\mathbb{T}^2)$, we define the quantum observable

$$\mathrm{Op}_N(a) = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \hat{a}(\boldsymbol{m}) T_N(\boldsymbol{m}), \tag{3.7}$$

where

$$\hat{a}(\boldsymbol{m}) = \int_{\mathbb{T}^2} a(\boldsymbol{\xi}) e(-\boldsymbol{\xi}\cdot\boldsymbol{m})\,\mathrm{d}\xi \tag{3.8}$$

are the Fourier coefficients of a. The observable $\mathrm{Op}_N(a)$ is also called the *Weyl quantization of a*. Axiom 2.1(a) is trivially satisfied. Axioms 2.1(b) and (c) follow from the following lemmas.

Lemma 3.1. *For all* $a_1, a_2 \in C^\infty(\mathbb{T}^2)$

$$\|\mathrm{Op}_N(a_1)\mathrm{Op}_N(a_2) - \mathrm{Op}_N(a_1a_2)\| \leqslant \frac{\pi}{N}\left(\sum_{\boldsymbol{m}\in\mathbb{Z}^2} \|\boldsymbol{m}\||\widehat{a_1}(\boldsymbol{m})|\right)\left(\sum_{\boldsymbol{n}\in\mathbb{Z}^2} \|\boldsymbol{n}\||\hat{a}_2(\boldsymbol{n})|\right). \tag{3.9}$$

Proof. Using the commutation relations (3.3) we find

$$\mathrm{Op}_N(a_1)\mathrm{Op}_N(a_2) = \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^2} \widehat{a_1}(\boldsymbol{m})\widehat{a_2}(\boldsymbol{n})T_N(\boldsymbol{m})T_N(\boldsymbol{n}), \tag{3.10}$$

$$= \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^2} e_N\left(\frac{\omega(\boldsymbol{m},\boldsymbol{n})}{2}\right)\widehat{a_1}(\boldsymbol{m})\widehat{a_2}(\boldsymbol{n})T_N(\boldsymbol{m}+\boldsymbol{n}), \tag{3.11}$$

$$= \sum_{\boldsymbol{m},\boldsymbol{k}\in\mathbb{Z}^2} e_N\left(\frac{\omega(\boldsymbol{m},\boldsymbol{k})}{2}\right)\widehat{a_1}(\boldsymbol{m})\widehat{a_2}(\boldsymbol{k}-\boldsymbol{m})T_N(\boldsymbol{k}) \tag{3.12}$$

with $\boldsymbol{k} = \boldsymbol{n} + \boldsymbol{m}$, and hence

$$\|\mathrm{Op}_N(a_1)\mathrm{Op}_N(a_2) - \mathrm{Op}_N(a_1a_2)\| \leqslant \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^2} \left|e_N\left(\frac{\omega(\boldsymbol{m},\boldsymbol{n})}{2}\right) - 1\right| |\widehat{a_1}(\boldsymbol{m})|\, |\widehat{a_2}(\boldsymbol{n})|. \tag{3.13}$$

The lemma now follows from

$$|e(x) - 1| \leqslant |2\pi x|, \qquad |\omega(\boldsymbol{m},\boldsymbol{n})| \leqslant \|\boldsymbol{m}\|\,\|\boldsymbol{n}\|. \tag{3.14}$$

□

Lemma 3.2. *For any* $a \in C^\infty(\mathbb{T}^2)$ *and* $R > 1$

$$\frac{1}{N}\mathrm{Tr}\,\mathrm{Op}_N(a) = \int_{\mathbb{T}^2} a\,\mathrm{d}\mu + O_{a,R}(N^{-R}). \tag{3.15}$$

Proof. Note that

$$\mathrm{Tr}\,T_N(\boldsymbol{m}) = \begin{cases} N & \text{if } \boldsymbol{m} = \boldsymbol{0} \bmod N\mathbb{Z}^2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

The lemma now follows from the rapid decay of the Fourier coefficients $\hat{a}(\boldsymbol{m})$ for $\|\boldsymbol{m}\| \to \infty$. □

Note that we have the alternative representation for $\mathrm{Op}_N(a)$,

$$[\mathrm{Op}_N(a)\psi](Q) = \sum_{m\in\mathbb{Z}} \tilde{a}\left(m, \frac{Q}{N} + \frac{m}{2N}\right)\psi(Q+m), \tag{3.17}$$

where

$$\tilde{a}(m,q) = \int_{\mathbb{R}/\mathbb{Z}} a(p,q)\, e(-pm)\,\mathrm{d}p, \tag{3.18}$$

which is sometimes useful. In fact (3.17) permits to quantize observables a, which are discontinuous in the q-variable. Note that if a is a smooth function of p and, for any $\nu \geqslant 0$, $(\mathrm{d}^\nu/\mathrm{d}p^\nu)a(p,q)$ is a bounded function on $\mathbb{T}^2$, then, for any $R > 1$, there is a constant C_R such that

$$|\tilde{a}(m,q)| \leqslant C_R(1+|m|)^{-R} \tag{3.19}$$

for all m, q. This fact is proved using integration by parts. Of course (3.19) holds in particular for smooth observables $a \in C^\infty(\mathbb{T}^2)$.

3.2. Quantum linked twist maps

A *twist map* Ψ_f is a map $\mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$\Psi_f : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p + f(q) \\ q \end{pmatrix} \quad \text{mod } \mathbb{Z}^2, \tag{3.20}$$

where $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is piecewise smooth, i.e. the domain of discontinuity of f in $\mathbb{R}/\mathbb{Z}$ has Minkowski content zero, cf section 2. Piecewise smooth functions of this type may be realized by taking a countable set of points $0 = \xi_0 < \xi_1 < \cdots < \xi_\infty = 1$ with finitely many accumulation points in [0, 1], and assume that $f \in C^\infty([\xi_i, \xi_{i+1}])$ for all $i = 0, 1, \ldots, \infty$.

Obviously Lebesgue measure $d\mu = dp\, dq$ is invariant under Ψ_f. A *linked twist map* Φ is now obtained by combining two twist maps, Ψ_{f_1} and Ψ_{f_2}, by setting

$$\Phi = R \circ \Psi_{f_1} \circ R^{-1} \circ \Psi_{f_2} \tag{3.21}$$

with the rotation

$$R : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} q \\ -p \end{pmatrix} \quad \text{mod } \mathbb{Z}^2. \tag{3.22}$$

Since Ψ_{f_1}, Ψ_{f_2} and R preserve μ, so does Φ. More explicitly, we have

$$R \circ \Psi_f \circ R^{-1} : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p \\ q - f(p) \end{pmatrix} \quad \text{mod } \mathbb{Z}^2 \tag{3.23}$$

and thus

$$\Phi : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p + f_2(q) \\ q - f_1(p + f_2(q)) \end{pmatrix} \quad \text{mod } \mathbb{Z}^2. \tag{3.24}$$

We define the quantization of the twist map Ψ_f by the unitary operator

$$[U_N(\Psi_f)\psi](Q) = e\left[-NV\left(\frac{Q}{N}\right)\right]\psi(Q), \tag{3.25}$$

where V is an arbitrary choice of a piecewise smooth function $\mathbb{R}/\mathbb{Z} \to \mathbb{R}$ satisfying $f = -V'$ and $V \in C^\infty$ on the domain of continuity of f[6].

Proposition 3.3. *For any $a \in C^\infty(\mathbb{T}^2)$ with compact support contained in the domain of continuity of Ψ_f, we have*

$$\|U_N(\Psi_f)^{-1}\mathrm{Op}_N(a)U_N(\Psi_f) - \mathrm{Op}_N(a \circ \Psi_f)\| = O(N^{-2}), \tag{3.26}$$

where the implied constant depends on a.

Proof. We have

$$[U_N(\Psi_f)^{-1}\mathrm{Op}_N(a)U_N(\Psi_f)\psi](Q) = \sum_{m\in\mathbb{Z}} \tilde{a}\left(m, \frac{Q}{N} + \frac{m}{2N}\right) \times e\left\{-N\left[V\left(\frac{Q+m}{N}\right) - V\left(\frac{Q}{N}\right)\right]\right\}\psi(Q+m) \tag{3.27}$$

and

$$[\mathrm{Op}_N(a \circ \Psi_f)\psi](Q) = \sum_{m\in\mathbb{Z}} \tilde{a}\left(m, \frac{Q}{N} + \frac{m}{2N}\right) e\left[mf\left(\frac{Q}{N} + \frac{m}{2N}\right)\right]\psi(Q+m), \tag{3.28}$$

[6] In the case of smooth twist maps $\Psi_f : \mathbb{T}^2 \to \mathbb{T}^2$ it is more convenient to view f as a C^∞ function $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, thus avoiding the introduction of artificial discontinuities in f. The potential V may now be defined as a C^∞ function $V : \mathbb{R} \to \mathbb{R}$ with $f = -V'$ locally, and the condition that $NV(Q/N + 1) = NV(Q/N)$ mod 1 for every Q, N. Examples are $f(q) = 2\tau q$ with $\tau \in \mathbb{Z}$, for which $V(q) = -\tau q^2$. In this case the correspondence principle stated in proposition 3.3 is in fact exact, i.e. the right-hand side of (3.26) is identically zero.

since

$$\widetilde{(a \circ \Psi_f)}(m, q) = e[mf(q)]\,\tilde{a}(m, q). \tag{3.29}$$

Therefore,

$$\|U_N(\Psi_f)^{-1}\mathrm{Op}_N(a)U_N(\Psi_f) - \mathrm{Op}_N(a \circ \Psi_f)\| \leqslant \max_q \sum_{m\in\mathbb{Z}} \left|\tilde{a}\left(m, q + \frac{m}{2N}\right)c_m(q, N)\right| \tag{3.30}$$

with

$$c_m(q, N) = e\left\{-N\left[V\left(q + \frac{m}{N}\right) - V(q)\right]\right\} - e\left[mf\left(q + \frac{m}{2N}\right)\right]. \tag{3.31}$$

Since $|c_m(q, N)| \leqslant 2$ and $|\tilde{a}(m, q)| \leqslant (1 + |m|)^{-5}$, we have

$$\max_q \sum_{|m|\geqslant N^{1/2}} \left|\tilde{a}\left(m, q + \frac{m}{2N}\right)c_m(q, N)\right| \leqslant N^{-2}. \tag{3.32}$$

Let us denote by *CS* the projection of the compact support of a onto the q axis. *CS* is a compact set which is in the domain of continuity of f.

For $|m| < N^{1/2}$, Taylor expansion around $x = q + m/2N$ yields (the second order terms cancel)

$$V\left(x + \frac{m}{2N}\right) - V\left(x - \frac{m}{2N}\right) = V'(x)\,\frac{m}{N} + O\left(\frac{m^3}{N^3}\right) \tag{3.33}$$

$$= -f(x)\,\frac{m}{N} + O\left(\frac{m^3}{N^3}\right) \tag{3.34}$$

uniformly for all $|m| < N^{1/2}$ and all $q \in CS$, provided N is sufficiently large so that $[q - N^{-1/2}, q + N^{-1/2}]$ is contained in the domain of continuity. Hence in this case

$$c_m(q, N) = O\left(\frac{m^3}{N^2}\right) \tag{3.35}$$

and

$$\max_q \sum_{|m|<N^{1/2}} \left|\tilde{a}\left(m, q + \frac{m}{2N}\right)c_m(q, N)\right| \leqslant O(N^{-2}) \max_q \sum_{m\in\mathbb{Z}} \left|m^3\tilde{a}\left(m, q + \frac{m}{2N}\right)\right| \tag{3.36}$$

$$= O(N^{-2}). \tag{3.37}$$

□

Proposition 3.4. *Suppose* $V(q) = v = \mathrm{const}$ *for all* q *in some open interval* $I \subset \mathbb{R}/\mathbb{Z}$*. Then, for any* $T > 1$ *and any* $a \in \mathrm{C}^\infty(\mathbb{T}^2)$ *with compact support contained in* $\mathbb{R}/\mathbb{Z} \times I \subset \mathbb{T}^2$*, we have*

$$\|U_N(\Psi_f)\mathrm{Op}_N(a) - e(-Nv)\mathrm{Op}_N(a)\| = O(N^{-T}), \tag{3.38}$$

where the implied constant depends on a *and* T*.*

Proof. We have

$$[(U_N(\Psi_f) - e(-Nv))\mathrm{Op}_N(a)\psi](Q) = \sum_{m\in\mathbb{Z}} \tilde{a}\left(m, \frac{Q}{N} + \frac{m}{2N}\right) \times \left\{e\left[-NV\left(\frac{Q+m}{N}\right)\right] - e(-Nv)\right\}\psi(Q + m). \tag{3.39}$$

As before we split the sum into two terms corresponding to $|m| < N^{1/2}$ and $|m| \geqslant N^{1/2}$. For N large enough, the first term vanishes since $\tilde{a}(m, q)$, as a function of q, is compactly supported inside the open interval I. The second term is bounded by

$$\max_q \sum_{|m|\geqslant N^{1/2}} \left| \tilde{a}\left(m, q + \frac{m}{2N}\right)\{e[-NV(q)] - e(-Nv)\}\right| \ll_{a,T} N^{-T} \tag{3.40}$$

for any $T > 1$ due to the rapid decay (3.19). □

The discrete Fourier transform $\mathcal{F}_N$ is a unitary operator defined by

$$[\mathcal{F}_N \psi](P) = \frac{1}{\sqrt{N}} \sum_{Q=0}^{N-1} \psi(Q) e_N(-QP). \tag{3.41}$$

Its inverse is given by the formula

$$[\mathcal{F}_N^{-1} \psi](Q) = \frac{1}{\sqrt{N}} \sum_{P=0}^{N-1} \psi(P) e_N(PQ). \tag{3.42}$$

Proposition 3.5. *For any* $a \in \mathrm{C}^\infty(\mathbb{T}^2)$

$$\mathcal{F}_N^{-1} \mathrm{Op}_N(a) \mathcal{F}_N = \mathrm{Op}_N(a \circ \mathrm{R}) \tag{3.43}$$

with the rotation R *as in (3.22).*

Proof. This follows from the identities $\mathcal{F}_N^{-1} t_1 \mathcal{F}_N = t_2^{-1}$ and $\mathcal{F}_N^{-1} t_2 \mathcal{F}_N = t_1$. □

The Fourier transform may therefore be viewed as a quantization of the rotation R that satisfies an *exact* correspondence principle, cf axiom 2.2.

The quantization of the linked twist map is now defined by

$$U_N(\Phi) = \mathcal{F}_N\, U_N(\Psi_{f_1})\, \mathcal{F}_N^{-1}\, U_N(\Psi_{f_2}). \tag{3.44}$$

Proposition 3.6. *For any* $a \in \mathrm{C}^\infty(\mathbb{T}^2)$ *with compact support in the domain of continuity of* Φ*, we have*

$$\|U_N(\Phi)^{-1} \mathrm{Op}_N(a) U_N(\Phi) - \mathrm{Op}_N(a \circ \Phi)\| = O(N^{-2}), \tag{3.45}$$

where the implied constant depends on a.

Proof. Apply propositions 3.3 and 3.5. □

The quantum map $U_N(\Phi)$ thus satisfies axiom 2.2.

Proposition 3.7. *Suppose* $V_1(p) = v_1 = \mathrm{const}$ *for all* p *in some open interval* $I_1 \subset \mathbb{R}/\mathbb{Z}$ *and* $V_2(q) = v_2 = \mathrm{const}$ *for all* q *in some open interval* $I_2 \subset \mathbb{R}/\mathbb{Z}$*. Then, for any* $T > 1$ *and any* $a \in \mathrm{C}^\infty(\mathbb{T}^2)$ *with compact support contained in the rectangle* $I_1 \times I_2 \subset \mathbb{T}^2$*, we have*

$$\|U_N(\Phi)\mathrm{Op}_N(a) - e[-N(v_1 + v_2)]\mathrm{Op}_N(a)\| = O(N^{-T}), \tag{3.46}$$

where the implied constant depends on a *and* T.

Proof. Apply propositions 3.4 and 3.5. □

In the examples considered in section 3.3 we have $v_1 = v_2 = 0$, and furthermore, for any $n \neq 0$, Φ^n acts as the identity precisely on the rectangle $I_1 \times I_2$. Hence, condition (2.10) [cf condition (c) of our central theorem 5.2] is satisfied in this case. In cases where $v_1 + v_2 \neq 0 \mod 1$ one may consider subsequences of $N \to \infty$ for which $\{N(v_1 + v_2)\} \to \alpha$ for any suitable fixed $\alpha \in [0, 1]$; here $\{\cdot\}$ denotes the fractional part.

3.3. Ergodic properties of linked twist maps

The ergodic properties of linked twist maps are well understood [11,37]. Let $[a_i, b_i]$ $(i = 1, 2)$ be subintervals of $\mathbb{R}/\mathbb{Z}$, and choose functions $f_i : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ with

(a) $f_i(q) = 0$ for $q \notin [a_i, b_i]$,
(b) $f_i(a_i) \in \mathbb{Z}$ and $f_i(b_i) - f_i(a_i) = k_i$ for some integer $k_i \in \mathbb{Z}$,
(c) $f_i \in C^2([a_i, b_i])$ with derivative $f_i'(q) \neq 0$ for all $q \in [a_i, b_i]$.

Let us define the constant

$$\gamma_i = \operatorname{sign}(k_i) \max_{q\in[a_i,b_i]} |f_i'(q)|. \tag{3.47}$$

Theorem 3.8. *Suppose either of the following conditions is satisfied*

(i) $\gamma_1\gamma_2 < 0$*;*
(ii) $|k_1|, |k_2| \geqslant 2$ *and* $\gamma_1\gamma_2 > C_0 \approx 17.244\,45$.

Then the map (3.21) acts ergodically (with respect to Lebesgue measure μ*) on the domain*

$$\mathcal{D}_0 = \{(p, q) \in \mathbb{T}^2 : p \in [a_1, b_1]\} \cup \{(p, q) \in \mathbb{T}^2 : q \in [a_2, b_2]\}. \tag{3.48}$$

The proofs of the two parts (i) and (ii) of this statement are due to Burton and Easton [11] and Przytycki [37], respectively. Both [11] and [37] in fact establish the Bernoulli property for the action of Φ on $\mathcal{D}_0$ under conditions (i) and (ii). We expect that these properties hold under weaker conditions, e.g. for smaller values of C_0. The continuity of the map at the lines $p = a_1, b_1$ and $q = a_2, b_2$, assumed in condition (b), is probably also not necessary.

For the linked twist map Φ used in figures 1–3 we have chosen

$$[a_1, b_1] = [a_2, b_2] = \left[\tfrac{1}{2}, 1\right], \tag{3.49}$$

$$f_1(p) = \begin{cases} -2p & \text{if } p \in \left[\frac{1}{2}, 1\right], \\ 0 & \text{if } p \notin \left[\frac{1}{2}, 1\right], \end{cases} \qquad V_1(p) = \begin{cases} p^2 & \text{if } p \in \left[\frac{1}{2}, 1\right], \\ 0 & \text{if } p \notin \left[\frac{1}{2}, 1\right], \end{cases} \tag{3.50}$$

and

$$f_2(q) = \begin{cases} 2q & \text{if } q \in \left[\frac{1}{2}, 1\right], \\ 0 & \text{if } q \notin \left[\frac{1}{2}, 1\right], \end{cases} \qquad V_2(q) = \begin{cases} -q^2 & \text{if } q \in \left[\frac{1}{2}, 1\right], \\ 0 & \text{if } q \notin \left[\frac{1}{2}, 1\right]. \end{cases} \tag{3.51}$$

More explicitly, this particular map Φ is obtained by first applying the twist

$$\begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{cases} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} & \text{if } q \in \left[\frac{1}{2}, 1\right], \\[2ex] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} & \text{if } q \notin \left[\frac{1}{2}, 1\right], \end{cases} \tag{3.52}$$

followed by

$$\begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{cases} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} & \text{if } p \in \left[\frac{1}{2}, 1\right], \\[2ex] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} & \text{if } p \notin \left[\frac{1}{2}, 1\right]. \end{cases} \tag{3.53}$$

Clearly, this Φ satisfies the above conditions (a)–(c) and (i).

3.4. Discretized linked twist maps

This section illustrates that the results of this paper may be applied to problems outside quantum mechanics, such as the discretization of maps on the torus, cf [3, 7] and references therein. A *discretization* of $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$ is defined as the invertible[7] map $\Phi_M : \mathbb{T}^2_M \to \mathbb{T}^2_M$ with $\mathbb{T}^2_M := (M^{-1}\mathbb{Z}/\mathbb{Z})^2$, where Φ_M is chosen in such a way that

$$\lim_{M\to\infty} \sup_{\xi\in\mathbb{T}^2_M} d(\Phi_M(\xi), \Phi(\xi)) = 0; \tag{3.54}$$

$d(\,\cdot\,,\,\cdot\,)$ denotes the Riemannian distance on $\mathbb{T}^2$. The discretized map induces a permutation matrix $U_N(\Phi) : \mathbb{C}^N \to \mathbb{C}^N$ with $N = M^2$ defined by

$$[U_N(\Phi)\psi](\xi) = \psi \circ \Phi_M^{-1}(\xi), \qquad \xi \in \mathbb{T}^2_M \tag{3.55}$$

(we represent vectors $\psi \in \mathbb{C}^N$ as functions $\psi : \mathbb{T}^2_M \to \mathbb{C}$).

The 'quantum' observables required in axiom 2.1 are simply defined as multiplication operators,

$$[\mathrm{Op}_N(a)\psi](\xi) = a(\xi)\,\psi(\xi) \tag{3.56}$$

and therefore trivially satisfy axioms 2.1(a) and (b). As to (c),

$$\frac{1}{N}\mathrm{Tr}\,\mathrm{Op}_N(a) = \frac{1}{M^2}\sum_{m_1,m_2=0}^{M-1} a\left(\frac{m_1}{M}, \frac{m_2}{M}\right) = \int_{\mathbb{T}^2} a\,\mathrm{d}\mu + O(M^{-1}). \tag{3.57}$$

It is easily checked that

$$U_N(\Phi)^{-1}\mathrm{Op}_N(a)U_N(\Phi) = \mathrm{Op}_N(a\circ\Phi_M) \tag{3.58}$$

and thus

$$\|U_N(\Phi)^{-1}\mathrm{Op}_N(a)U_N(\Phi) - \mathrm{Op}_N(a\circ\Phi)\| = \|\mathrm{Op}_N(a\circ\Phi_M) - \mathrm{Op}_N(a\circ\Phi)\| \tag{3.59}$$

$$\leqslant \sup_{\xi\in\mathbb{T}^2_M} |a(\Phi_M(\xi)) - a(\Phi(\xi))| \tag{3.60}$$

$$\ll \sup_{\xi\in\mathbb{T}^2_M} d(\Phi_M(\xi), \Phi(\xi)) \tag{3.61}$$

and hence axiom 2.2 is satisfied. Condition (2.10) follows from a similar argument, with $\alpha_{\mathcal{D}} = 0$.

A concrete discretization of a linked twist map is obtained by replacing each twist map Ψ_f by Ψ_{f_M}, with $f_M(q) = M^{-1}[Mf(q)]$, where $[x]$ denotes the integer part of x. Note that R preserves $\mathbb{T}^2_M$ and therefore requires no further discretization. In this case

$$\sup_{\xi\in\mathbb{T}^2_M} d(\Phi_M(\xi), \Phi(\xi)) = O(M^{-1}). \tag{3.62}$$

There is a simple geometric interpretation of the spectrum of $U_N(\Phi)$. The map Φ_M represents a permutation of $N = M^2$ elements, which can be written as a product of, say, ν cycles C_i of length ℓ_i, $i = 1, \ldots, \nu$. Each cycle corresponds to a periodic orbit of period ℓ_i for the action of Φ_M on $\mathbb{T}^2_M$. Let ξ_i be an arbitrary point on C_i. An orthonormal basis of eigenvectors of $U_N(\Phi)$ is then given by the functions

$$\varphi_{ij}(\xi) = \begin{cases} \ell_i^{-1/2} e\left(\dfrac{jk}{\ell_i}\right) & \text{if } \xi = \Phi_M^{-k}(\xi_i) \text{ for some } k = 0, \ldots, \ell_i - 1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.63}$$

[7] Invertability is not necessarily required in general discretization schemes. We assume it here to obtain a unitary representation.

with eigenvalue $\lambda_{ij} = e(j/\ell_i)$, where i runs over the cycles and j over the integers $0, 1, \ldots, \ell_i - 1$. Hence in particular $|\varphi_{ij}(\xi)|^2 = 1/\ell_i$ for every ξ on the periodic orbit and $|\varphi_{ij}(\xi)|^2 = 0$ otherwise.

Due to the commutativity of the observables $\mathbf{Op}(a)$, the proofs of some of the statements in later sections may be simplified—especially those in section 7.

4. Mollified characteristic functions

Let us now return to the general framework of section 2. Consider the characteristic function $\chi_{\mathcal{D}}$ of a domain $\mathcal{D} \subset \mathcal{M}$ with boundary of Minkowski content zero. An *ϵ-mollified characteristic function* $\tilde{\chi}_{\mathcal{D}} \in \mathrm{C}^\infty(\mathcal{M})$ has values in [0, 1] and $\tilde{\chi}_{\mathcal{D}}(x) = \chi_{\mathcal{D}}(x)$ on a set of x of measure $1 - \epsilon$. Since $\mathcal{D}$ has boundary of Minkowski content zero, we can construct such a smoothed function for any $\epsilon > 0$. Furthermore, we are able to construct ϵ-mollified $\tilde{\chi}_{\mathcal{D}}$ whose support is either contained in $\mathcal{D}$, or whose support contains $\mathcal{D}$, again for any $\epsilon > 0$. Note that if $\tilde{\chi}_{\mathcal{D}}$ is ϵ-mollified, so is $\tilde{\chi}_{\mathcal{D}}^n$ for any $n \in \mathbb{N}$ with the same ϵ.

After mollification, we may associate with a characteristic function $\chi_{\mathcal{D}}$ a quantum observable $\mathrm{Op}_N(\tilde{\chi}_{\mathcal{D}})$. Since $\mathrm{Op}_N(\tilde{\chi}_{\mathcal{D}})$ is in general not Hermitian, it is sometimes more convenient to consider the symmetrized version, the positive definite Hermitian matrix

$$\mathrm{Op}_N^{\mathrm{sym}}(\tilde{\chi}_{\mathcal{D}}) := \mathrm{Op}_N(\tilde{\chi}_{\mathcal{D}}^{1/2})\mathrm{Op}_N(\tilde{\chi}_{\mathcal{D}}^{1/2})^\dagger. \tag{4.1}$$

Note that $\tilde{\chi}_{\mathcal{D}}^{1/2} \in \mathrm{C}^\infty(\mathcal{M})$ since $\tilde{\chi}_{\mathcal{D}} \geqslant 0$. Furthermore, we have

$$\mathbf{Op}^{\mathrm{sym}}(\tilde{\chi}_{\mathcal{D}}) \sim \mathbf{Op}(\tilde{\chi}_{\mathcal{D}}). \tag{4.2}$$

The following proposition describes the distribution of eigenvalues of $\mathrm{Op}_N^{\mathrm{sym}}(\tilde{\chi}_{\mathcal{D}})$ and suggests that the operator may be viewed as an approximate projection operator onto a subspace of dimension $\sim N \times \mu(\mathcal{D})$.

Consider a sequence $\boldsymbol{J} := \{J_N\}_{N\in\mathcal{I}}$ of sets $J_N \subset \{1, \ldots, N\}$. The quantity

$$\Delta(\boldsymbol{J}) := \lim_{N\to\infty} \frac{\# J_N}{N}, \tag{4.3}$$

provided the limit exists, is called the *density of* $\boldsymbol{J}$.

Proposition 4.1. *Suppose $\tilde{\chi}_{\mathcal{D}}$ is an ϵ-mollified characteristic function and suppose $\mu_j \geqslant 0$ ($j = 1, \ldots, N$) are the eigenvalues of $\mathrm{Op}_N^{\mathrm{sym}}(\tilde{\chi}_{\mathcal{D}})$. Then there are set sequences $\boldsymbol{J} := \{J_N\}_{N\in\mathcal{I}}$ and $\boldsymbol{J}' := \{J'_N\}_{N\in\mathcal{I}}$ with densities*

$$\Delta(\boldsymbol{J}) = \mu(\mathcal{D}) + O(\epsilon^{1/3}), \qquad \Delta(\boldsymbol{J}') = 1 - \mu(\mathcal{D}) + O(\epsilon^{1/3}), \tag{4.4}$$

such that

(i) $\mu_j = 1 + O(\epsilon^{1/3})$ for all $j \in J_N$;
(ii) $\mu_j = O(\epsilon^{1/3})$ for all $j \in J'_N$.

Proof. By axiom 2.1, we have for every fixed integer $n \geqslant 1$,

$$\frac{1}{N}\mathrm{Tr}\big[\mathrm{Op}_N^{\mathrm{sym}}(\tilde{\chi}_{\mathcal{D}})^n\big] = \frac{1}{N}\mathrm{Tr}\,\mathrm{Op}_N^{\mathrm{sym}}(\tilde{\chi}_{\mathcal{D}}^n) + o_{\epsilon,n}(1) \tag{4.5}$$

$$= \int_{\mathcal{M}} \tilde{\chi}_{\mathcal{D}}^n \,\mathrm{d}\mu + o_{\epsilon,n}(1) \tag{4.6}$$

$$= \mu(\mathcal{D}) + O(\epsilon) + o_{\epsilon,n}(1), \tag{4.7}$$

where $O(\epsilon)$ does not depend on N and n. This implies for every $n \geqslant 1$,

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}\big[\mathrm{Op}_N^{\mathrm{sym}}(\tilde{\chi}_{\mathcal{D}})^n\big] = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \mu_j^n = \mu(\mathcal{D}) + O(\epsilon). \tag{4.8}$$

Therefore,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} (\mu_j^2 - \mu_j)^2 = O(\epsilon) \tag{4.9}$$

and thus

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\in H_N} (\mu_j - 1)^2 = O(\epsilon), \tag{4.10}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\notin H_N} \mu_j^2 = O(\epsilon), \tag{4.11}$$

where $H_N = \{j : \mu_j \geqslant 1/2\}$. By Chebyshev's inequality, (4.11) implies that

$$\lim_{N\to\infty} \frac{1}{N} \#\{j \notin H_N \ : \ \mu_j^2 > \gamma\} = O\left(\frac{\epsilon}{\gamma}\right), \tag{4.12}$$

for any $\gamma > 0$. This yields the bound

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\notin H_N} \mu_j = O\left(\gamma^{1/2} + \frac{\epsilon}{\gamma}\right), \tag{4.13}$$

since $0 \leqslant \mu_j < 1/2$. So

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N} \sum_{j\in H_N} 1 &= \lim_{N\to\infty} \frac{1}{N} \sum_{j\in H_N} [(\mu_j - 1)^2 - \mu_j^2 + 2\mu_j] && (4.14)\\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{j\in H_N} [-\mu_j^2 + 2\mu_j] + O(\epsilon) && (4.15)\\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} [-\mu_j^2 + 2\mu_j] + O(\epsilon) + O\left(\gamma^{1/2} + \frac{\epsilon}{\gamma}\right) && (4.16)\\
&= \mu(\mathcal{D}) + O(\epsilon^{1/3}), && (4.17)
\end{aligned}$$

if we choose $\gamma = \epsilon^{2/3}$, and hence the corresponding set sequence $\boldsymbol{H} := \{H_N\}_{N\in\mathcal{I}}$ has density $\Delta(\boldsymbol{H}) = \mu(\mathcal{D}) + O(\epsilon^{1/3})$. Once more in view of Chebyshev's inequality, (4.10) implies

$$\lim_{N\to\infty} \frac{1}{N} \#\{j \in H_N : (\mu_j - 1)^2 > \delta\} = O(\epsilon/\delta). \tag{4.18}$$

Choosing $\delta = \epsilon^{2/3}$ means that for a subsequence of $j \in H_N$ of density $\mu(\mathcal{D}) + O(\epsilon^{1/3})$ we have $\mu_j = 1 + O(\epsilon^{1/3})$. The corresponding result for $j \notin H_N$ follows by the same argument from (4.12). □

5. Trace asymptotics and Weyl's law

The following proposition is the key tool to understand the distribution of eigenvalues of $U_N(\Phi)$.

Proposition 5.1 (trace asymptotics). *Suppose* $\Phi : \mathcal{M} \to \mathcal{M}$ *is piecewise smooth, and*

(a) $\Phi^n|_{\mathcal{D}'} = \mathrm{id}$ *on some set* $\mathcal{D}' \subset \mathcal{M}$ *with boundary of Minkowski content zero;*
(b) the fixed points of Φ^n *on* $\mathcal{D} := \mathcal{M} - \mathcal{D}'$ *form a set of Minkowski content zero;*
(c) there is a constant $\alpha_{\mathcal{D}'} \in \mathbb{R}$ *such that for any* $a \in \mathrm{C}^\infty(\mathcal{M})$ *with compact support contained in* $\mathcal{D}'$ *and the domains of continuity of* Φ^n*, we have*

$$\boldsymbol{U}(\Phi)^n \mathbf{Op}(a) \sim e(n\alpha_{\mathcal{D}'})\, \mathbf{Op}(a).$$

Then

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}\, U_N(\Phi)^n = e(n\alpha_{\mathcal{D}'})\, \mu(\mathcal{D}'). \tag{5.1}$$

Proof. Given any $\epsilon > 0$, we can find an integer R and a partition of unity on $\mathcal{M}$ by ϵ-mollified characteristic functions,

$$1 = \tilde{\chi}_{\mathrm{bad}}(\xi) + \tilde{\chi}_{\mathcal{D}'}(\xi) + \sum_{r=1}^{R} \tilde{\chi}_r(\xi) \qquad \forall \xi \in \mathcal{M} \tag{5.2}$$

with the properties

(i) the interior of the support of $\tilde{\chi}_{\mathrm{bad}}$ contains the domains of discontinuity of Φ^ν in $\mathcal{M}$ for all $\nu = 1, \ldots, n$, and all fixed points of Φ^n in $\mathcal{D}$, and is chosen small enough so that $\int \tilde{\chi}_{\mathrm{bad}} d\mu < \epsilon$;
(ii) the support of $\tilde{\chi}_{\mathcal{D}'}$ is contained in $\mathcal{D}'$ and the domains of continuity of Φ^ν for all $\nu = 1, \ldots, n$, so that $\mu(\mathcal{D}') - \int \tilde{\chi}_{\mathcal{D}'}\, \mathrm{d}\mu < \epsilon$ and all fixed points of Φ^n are contained in the interior of the set $\mathrm{supp}\, \tilde{\chi}_{\mathrm{bad}} \cup \mathrm{supp}\, \tilde{\chi}_{\mathcal{D}'}$;
(iii) the support of $\tilde{\chi}_r$, with $r = 1, \ldots, R$, is chosen small enough, so that $\mathrm{supp}\, \tilde{\chi}_r \cap \Phi^n(\mathrm{supp}\, \tilde{\chi}_r) = \emptyset$ for all $\xi \in \mathcal{M}$.

Properties (i) and (ii) are possible since the fixed points in $\mathcal{D}$ and the discontinuities form sets of Minkowski content zero. To achieve (iii) note that the closure of $\mathcal{K} = \mathcal{M} - (\mathrm{supp}\, \tilde{\chi}_{\mathrm{bad}} \cup \mathrm{supp}\, \tilde{\chi}_{\mathcal{D}'})$ does not contain any fixed points, and Φ is continuous on $\mathcal{K}$. Hence, there is a sufficiently small radius $\eta = \eta(\epsilon)$ such that for all balls $\mathcal{B}_\eta \subset \mathcal{K}$ we have $\mathcal{B}_\eta \cap \Phi^n(\mathcal{B}_\eta) = \emptyset$.

By the linearity of Op, we have

$$\mathrm{Tr}\, U_N(\Phi)^n = \mathrm{Tr}[U_N(\Phi)^n \mathrm{Op}_N(\tilde{\chi}_{\mathrm{bad}})] + \mathrm{Tr}[U_N(\Phi)^n \mathrm{Op}_N(\tilde{\chi}_{\mathcal{D}'})] + \sum_{r=1}^{R} \mathrm{Tr}[U_N(\Phi)^n \mathrm{Op}_N(\tilde{\chi}_r)]. \tag{5.3}$$

We begin with the first term on the right-hand side:

$$\mathrm{Tr}[U_N(\Phi)^n \mathrm{Op}_N(\tilde{\chi}_{\mathrm{bad}})] = \mathrm{Tr}[U_N(\Phi)^n \mathrm{Op}_N^{\mathrm{sym}}(\tilde{\chi}_{\mathrm{bad}})] + o_\epsilon(N) \tag{5.4}$$

with the symmetrized $\mathrm{Op}_N^{\mathrm{sym}}(\tilde{\chi}_{\mathrm{bad}})$ as defined in (4.1). Suppose ψ_j and $\mu_j \geqslant 0$ are the (normalized) eigenstates and eigenvalues of $\mathrm{Op}_N^{\mathrm{sym}}(\tilde{\chi}_{\mathrm{bad}})$. Then

$$\begin{aligned} |\mathrm{Tr}[U_N(\Phi)^n \mathrm{Op}_N^{\mathrm{sym}}(\tilde{\chi}_{\mathrm{bad}})]| &= \left| \sum_{j=1}^{N} \mu_j \langle \psi_j, U_N(\Phi)^n \psi_j \rangle \right| \leqslant \sum_{j=1}^{N} \mu_j = \mathrm{Tr}\, \mathrm{Op}_N^{\mathrm{sym}}(\tilde{\chi}_{\mathrm{bad}}) \\ &= \mathrm{Tr}\, \mathrm{Op}_N(\tilde{\chi}_{\mathrm{bad}}) + o_\epsilon(N) = N\, O(\epsilon) + o_\epsilon(N). \end{aligned} \tag{5.5}$$

Since by condition (c), $\boldsymbol{U}(\Phi)^n \mathbf{Op}(\tilde{\chi}_{\mathcal{D}'}) \sim e(n\alpha_{\mathcal{D}'})\mathbf{Op}(\tilde{\chi}_{\mathcal{D}'})$, we find for the second term on the right-hand side of (5.3)

$$\mathrm{Tr}[U_N(\Phi)^n \mathrm{Op}_N(\tilde{\chi}_{\mathcal{D}'})] = e(n\alpha_{\mathcal{D}'})\mathrm{Tr}\,\mathrm{Op}_N(\tilde{\chi}_{\mathcal{D}'}) + o_\epsilon(N) \tag{5.6}$$

$$= Ne(n\alpha_{\mathcal{D}'}) \int \tilde{\chi}_{\mathcal{D}'}\,\mathrm{d}\mu + o_\epsilon(N) \tag{5.7}$$

$$= Ne(n\alpha_{\mathcal{D}'})\{\mu(\mathcal{D}') + O(\epsilon)\} + o_\epsilon(N), \tag{5.8}$$

where we have used (ii) in the last step.

For the last term in the sum (5.3) we have

$$\boldsymbol{U}(\Phi)^n \mathbf{Op}(\tilde{\chi}_r) \sim \boldsymbol{U}(\Phi)^n \mathbf{Op}(\tilde{\chi}_r^{1/2})\mathbf{Op}(\tilde{\chi}_r^{1/2}) \sim \mathbf{Op}(\tilde{\chi}_r^{1/2} \circ \Phi^{-n})\boldsymbol{U}(\Phi)^n \mathbf{Op}(\tilde{\chi}_r^{1/2}) \tag{5.9}$$

so

$$\mathrm{Tr}[U_N(\Phi)^n \mathrm{Op}_N(\tilde{\chi}_r)] = \mathrm{Tr}[\mathrm{Op}_N(\tilde{\chi}_r^{1/2} \circ \Phi^{-n})U_N(\Phi)^n \mathrm{Op}_N(\tilde{\chi}_r^{1/2})] + o_\epsilon(N) \tag{5.10}$$

$$= \mathrm{Tr}[\mathrm{Op}_N(\tilde{\chi}_r^{1/2})\mathrm{Op}_N(\tilde{\chi}_r^{1/2} \circ \Phi^{-n})U_N(\Phi)^n] + o_\epsilon(N) \tag{5.11}$$

$$= \mathrm{Tr}[\mathrm{Op}_N(\tilde{\chi}_r^{1/2} \cdot \tilde{\chi}_r^{1/2} \circ \Phi^{-n})U_N(\Phi)^n] + o_\epsilon(N) \tag{5.12}$$

$$= o_\epsilon(N), \tag{5.13}$$

since $\tilde{\chi}_r^{1/2} \cdot \tilde{\chi}_r^{1/2} \circ \Phi^{-n} = 0$ in view of (iii). Therefore,

$$\mathrm{Tr}\,U_N(\Phi)^n = Ne(n\alpha_{\mathcal{D}'})\{\mu(\mathcal{D}') + O(\epsilon)\} + o_\epsilon(N), \tag{5.14}$$

i.e.

$$\lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}\,U_N(\Phi)^n = e(n\alpha_{\mathcal{D}'})\,\mu(\mathcal{D}') + O(\epsilon), \tag{5.15}$$

which holds for every arbitrarily small $\epsilon > 0$. This concludes the proof. □

Theorem 5.2 (Weyl's law). *Suppose $\Phi : \mathcal{M} \to \mathcal{M}$ is piecewise smooth, and*

(a) there are disjoint sets $\mathcal{D}_\nu$ ($\nu = 1, 2, 3, \ldots$) with boundary of Minkowski content zero, on which Φ is periodic with period $n_\nu \geqslant 1$, i.e.

$$\Phi^{n_\nu}|_{\mathcal{D}_\nu} = \mathrm{id},$$

and, for every $0 < |n| < n_\nu$, the fixed points of Φ^n on $\mathcal{D}_\nu$ form a set of Minkowski content zero;

(b) for every $n \neq 0$, the fixed points of Φ^n on $\mathcal{D}_0 = \mathcal{M} - \bigcup_{\nu=1}^{\infty} \mathcal{D}_\nu$ form a set of Minkowski content zero;

(c) there is a constant $\alpha_{\mathcal{D}_\nu} \in \mathbb{R}$ such that for any $a \in \mathrm{C}^\infty(\mathcal{M})$ with compact support contained in $\mathcal{D}_\nu$ and the domains of continuity of $\Phi, \Phi^2, \ldots, \Phi^{n_\nu}$, we have

$$\boldsymbol{U}(\Phi)^{n_\nu} \mathbf{Op}(a) \sim e(n_\nu \alpha_{\mathcal{D}_\nu})\,\mathbf{Op}(a).$$

Then, for every continuous function $h : \mathbb{S}^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{C}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h(\theta_j) = \sum_{\nu=0}^{\infty} \mu(\mathcal{D}_\nu) \int_{\mathbb{S}^1} h(\theta)\,\rho_\nu(\theta)\,\mathrm{d}\theta, \tag{5.16}$$

where

$$\rho_0(\theta) = 1 \tag{5.17}$$

and, for $\nu \geqslant 1$,

$$\rho_\nu(\theta) = \frac{1}{n_\nu} \sum_{k=0}^{n_\nu - 1} \delta_{\mathbb{S}^1}\left(\theta - \frac{k}{n_\nu} - \alpha_{\mathcal{D}_\nu}\right). \tag{5.18}$$

Here,

$$\delta_{\mathbb{S}^1}(\theta) = \sum_{m\in\mathbb{Z}} \delta(\theta + m) \tag{5.19}$$

denotes the periodicized Dirac distribution.

Proof. For every ν such that n_ν divides n we have $\Phi^n|_{\mathcal{D}_\nu} = \mathrm{id}$. A simple modification of the proof of proposition 5.1 yields therefore, for $n \neq 0$,

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}\, U_N(\Phi)^n = \sum_{\substack{\nu=1\\ n_\nu|n}}^{\infty} e(n\alpha_{\mathcal{D}_\nu})\, \mu(\mathcal{D}_\nu). \tag{5.20}$$

The only difference in the proof is that $\mathcal{D}'$ is divided into the domains $\mathcal{D}_1, \mathcal{D}_2, \ldots$ with different integration constants $\alpha_{\mathcal{D}_\nu}$. For every $\epsilon > 0$ there is a $K = K_\epsilon$ such that

$$\mu\Bigg(\bigcup_{\nu=K+1}^{\infty} \mathcal{D}_\nu \Bigg) < \epsilon. \tag{5.21}$$

Hence, one effectively deals with only finitely many domains $\mathcal{D}_1, \ldots, \mathcal{D}_K$ and shows, following the steps in the proof of proposition 5.1, that

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}\, U_N(\Phi)^n = \sum_{\substack{\nu=1\\ n_\nu|n}}^{K} e(n\alpha_{\mathcal{D}_\nu})\, \mu(\mathcal{D}_\nu) + O(\epsilon), \tag{5.22}$$

which in turn yields (5.20).

Let us first assume that the test function h has only finitely many non-zero Fourier coefficients, i.e.

$$h(\theta) = \sum_{n\in\mathbb{Z}} \hat{h}(n) e(n\theta) \tag{5.23}$$

is a finite sum. We then have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h(\theta_j) = \lim_{N\to\infty} \frac{1}{N} \sum_{n\in\mathbb{Z}} \hat{h}(n) \mathrm{Tr}\, U_N(\Phi)^n \tag{5.24}$$

$$= \hat{h}(0) + \sum_{n\in\mathbb{Z}-\{0\}} \hat{h}(n) \sum_{\substack{\nu=1\\ n_\nu|n}}^{\infty} e(n\alpha_{\mathcal{D}_\nu})\, \mu(\mathcal{D}_\nu) \tag{5.25}$$

$$= \hat{h}(0) + \sum_{\nu=1}^{\infty} \mu(\mathcal{D}_\nu) \sum_{\substack{n\in\mathbb{Z}-\{0\}\\ n_\nu|n}} \hat{h}(n) e(n\alpha_{\mathcal{D}_\nu}). \tag{5.26}$$

Since

$$\sum_{\substack{n\in\mathbb{Z}\\ n_\nu|n}} \hat{h}(n) e(n\alpha_{\mathcal{D}_\nu}) = \frac{1}{n_\nu} \sum_{k=0}^{n_\nu-1} \sum_{n\in\mathbb{Z}} \hat{h}(n) e(n\alpha_{\mathcal{D}_\nu})\, e\bigg(\frac{kn}{n_\nu}\bigg) = \frac{1}{n_\nu} \sum_{k=0}^{n_\nu-1} h\bigg(\frac{k}{n_\nu} + \alpha_{\mathcal{D}_\nu}\bigg), \tag{5.27}$$

we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h(\theta_j) = \Bigg\{ \mu(\mathcal{D}_0)\, \hat{h}(0) + \sum_{\nu=1}^{\infty} \frac{\mu(\mathcal{D}_\nu)}{n_\nu} \sum_{k=0}^{n_\nu-1} h\bigg(\frac{k}{n_\nu} + \alpha_{\mathcal{D}_\nu}\bigg) \Bigg\}, \tag{5.28}$$

which proves the theorem for h with finite Fourier series. We now extend this result to test functions $h \in \mathrm{C}^1(\mathbb{S}^1)$. Let

$$h_K(\theta) = \sum_{\substack{n\in\mathbb{Z} \\ |n|\leqslant K}} \hat{h}(n)e(n\theta) \tag{5.29}$$

be the truncated Fourier series. Since $h \in \mathrm{C}^1(\mathbb{S}^1)$, its Fourier series converges absolutely and uniformly and hence, for any $\epsilon > 0$, there is a K such that $h_K(\theta) - \epsilon \leqslant h(\theta) \leqslant h_K(\theta) + \epsilon$ for all $\theta \in \mathbb{S}^1$. By (5.28), the limits of the left- and right-hand sides of

$$\frac{1}{N}\sum_{j=1}^{N} h_K(\theta_j) - \epsilon \leqslant \frac{1}{N}\sum_{j=1}^{N} h(\theta_j) \leqslant \frac{1}{N}\sum_{j=1}^{N} h_K(\theta_j) + \epsilon \tag{5.30}$$

exist and differ by less than 2ϵ, hence (5.28) holds also for the current h. The extension of (5.28) to h in $\mathrm{C}(\mathbb{S}^1)$ is achieved by the same argument, i.e. by approximating h pointwise by functions $h_\epsilon \in \mathrm{C}^1(\mathbb{S}^1)$ so that $h_\epsilon(\theta) - \epsilon \leqslant h(\theta) \leqslant h_\epsilon(\theta) + \epsilon$. □

6. Generalized Weyl's law

Proposition 6.1 (generalized trace asymptotics). *Choose Φ and $U_N(\Phi)$ as in proposition 5.1. Then for every $a \in \mathrm{C}^\infty(\mathcal{M})$ and $n \neq 0$,*

$$\lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}[\mathrm{Op}_N(a)U_N(\Phi)^n] = e(n\alpha_{\mathcal{D}'}) \int_{\mathcal{D}'} a\,\mathrm{d}\mu. \tag{6.1}$$

Proof. By linearity of the relation (6.1) we may assume without loss of generality that a is real and $\min_\xi a(\xi) \geqslant 0$. This implies that $a^{1/2} \in \mathrm{C}^\infty(\mathcal{M})$. Analogously to the proof of proposition 5.1, we have

$$\begin{aligned}\mathrm{Tr}[\mathrm{Op}_N(a)U_N(\Phi)^n] &= \mathrm{Tr}[U_N(\Phi)^n\mathrm{Op}_N(\tilde{\chi}_{\mathrm{bad}}\cdot a)] \\ &\quad + \mathrm{Tr}[U_N(\Phi)^n\mathrm{Op}_N(\tilde{\chi}_{\mathcal{D}'}\cdot a)] + \sum_{r=1}^{R}\mathrm{Tr}[U_N(\Phi)^n\mathrm{Op}_N(\tilde{\chi}_r\cdot a)] + o_\epsilon(N).\end{aligned} \tag{6.2}$$

The proof is concluded in the same way as the proof of proposition 5.1, with all mollified characteristic functions $\tilde{\chi}$ replaced by $\tilde{\chi}\cdot a$. □

Theorem 6.2 (generalized Weyl's law). *Choose Φ and $U_N(\Phi)$ as in theorem 5.2. Let $\varphi_j \in \mathbb{C}^N$ ($j = 1, \ldots, N$) be an orthonormal basis of eigenstates of $U_N(\Phi)$, with corresponding eigenphases $\theta_j \in \mathbb{S}^1$. Then, for every $a \in \mathrm{C}^\infty(\mathcal{M})$ and every continuous function $h : \mathbb{S}^1 \to \mathbb{C}$,*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} h(\theta_j)\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j\rangle = \sum_{\nu=0}^{\infty}\int_{\mathcal{D}_\nu} a\,\mathrm{d}\mu \int_{\mathbb{S}^1} h(\theta)\,\rho_\nu(\theta)\,\mathrm{d}\theta. \tag{6.3}$$

Proof. We may assume again without loss of generality that a is real and $\min_\xi a(\xi) \geqslant 0$. In view of proposition 6.1 and the proof of theorem 5.2 we have for every h_K with finite Fourier expansion (as in (5.23))

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} h_K(\theta_j)\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j\rangle = \sum_{\nu=0}^{\infty}\int_{\mathcal{D}_\nu} a\,\mathrm{d}\mu \int_{0}^{1} h_K(\theta)\,\rho_\nu(\theta)\,\mathrm{d}\theta. \tag{6.4}$$

For any $h \geqslant 0$ we have

$$\left| \sum_{j=1}^{N} h(\theta_j) \langle \mathrm{Op}_N(a)\varphi_j, \varphi_j \rangle - \sum_{j=1}^{N} h(\theta_j) \|\mathrm{Op}_N(a^{1/2})\varphi_j\|^2 \right| \leqslant \sup h \left| \mathrm{Tr}[\mathrm{Op}_N(a) - \mathrm{Op}_N(a^{1/2})\mathrm{Op}_N(a^{1/2})^\dagger] \right| = o(N) \sup h. \tag{6.5}$$

Hence (6.4) is equivalent to

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h_K(\theta_j) \|\mathrm{Op}_N(a^{1/2})\varphi_j\|^2 = \sum_{\nu=0}^{\infty} \int_{\mathcal{D}_\nu} a \, \mathrm{d}\mu \int_0^1 h_K(\theta)\, \rho_\nu(\theta)\, \mathrm{d}\theta. \tag{6.6}$$

We now use the same approximation argument as in the proof of theorem 5.2, for $h \in \mathrm{C}^1(\mathbb{S}^1)$. Given any ϵ, there is a K such that $h_K(\theta) - \epsilon \leqslant h(\theta) \leqslant h_K(\theta) + \epsilon$ for all $\theta \in \mathbb{S}^1$. The limits of the left- and right-hand sides of

$$\frac{1}{N} \sum_{j=1}^{N} [h_K(\theta_j) - \epsilon] \|\mathrm{Op}_N(a^{1/2})\varphi_j\|^2 \leqslant \frac{1}{N} \sum_{j=1}^{N} h(\theta_j) \|\mathrm{Op}_N(a^{1/2})\varphi_j\|^2 \tag{6.7}$$

$$\leqslant \frac{1}{N} \sum_{j=1}^{N} [h_K(\theta_j) + \epsilon] \|\mathrm{Op}_N(a^{1/2})\varphi_j\|^2 \tag{6.8}$$

differ by less than

$$2\epsilon \sup h_K \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \|\mathrm{Op}_N(a^{1/2})\varphi_j\|^2 \leqslant 2\epsilon \sup h_K \lim_{N\to\infty} \frac{1}{N} \mathrm{Tr}[\mathrm{Op}_N(a^{1/2})\mathrm{Op}_N(a^{1/2})^\dagger] \tag{6.9}$$

$$= 2\epsilon \sup h_K \int_{\mathcal{M}} a \, \mathrm{d}\mu, \tag{6.10}$$

which can be arbitrarily small for $\epsilon \to 0$. Thus,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h(\theta_j) \|\mathrm{Op}_N(a^{1/2})\varphi_j\|^2 = \sum_{\nu=0}^{\infty} \int_{\mathcal{D}_\nu} a \, \mathrm{d}\mu \int_0^1 h(\theta)\, \rho_\nu(\theta)\, \mathrm{d}\theta. \tag{6.11}$$

A similar approximation argument shows that (6.11) holds also for all continuous h. In view of (6.5), the relation (6.11) is equivalent to (6.3). The assumption $h \geqslant 0$ can be removed by using the linearity of (6.3) in h. □

7. Localization

The sequence $\psi := \{\psi_N\}_{N\in\mathcal{I}}$ of vectors $\psi_N \in \mathbb{C}^N - \{\mathbf{0}\}$ is said to be *semiclassically localized in the domain* $\mathcal{D}$ if for every $a \in \mathrm{C}^\infty(\mathcal{M})$ with $a|_{\mathcal{D}} = 0$, we have

$$\lim_{N\to\infty} \frac{\langle \mathrm{Op}_N(a)\psi, \psi \rangle}{\|\psi\|^2} = 0. \tag{7.1}$$

Theorem 7.1. *Choose* Φ *and* $U_N(\Phi)$ *as in theorem 5.2, and let* $\varphi_1, \ldots, \varphi_N \in \mathbb{C}^N$ *be an orthonormal basis of eigenstates of* $U_N(\phi)$. *Then there are set sequences* $\boldsymbol{J} := \{J_N\}_{N\in\mathcal{I}}$ *and* $\boldsymbol{J}' := \{J'_N\}_{N\in\mathcal{I}}$ *with densities*

$$\Delta(\boldsymbol{J}) = \mu(\mathcal{D}_0), \qquad \Delta(\boldsymbol{J}') = 1 - \mu(\mathcal{D}_0), \tag{7.2}$$

such that

(i) for any $a \in C^\infty(\mathcal{M})$ *with* $a|_{\mathcal{D}_0} = 0$*,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\in J_N} \left|\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j\rangle\right|^2 = 0;$$

(ii) for any $a \in C^\infty(\mathcal{M})$ *with* $a|_{\mathcal{M}-\mathcal{D}_0} = 0$*,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\in J'_N} |\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j\rangle|^2 = 0.$$

Proof. For any given $\epsilon > 0$, there is a constant $K = K_\epsilon$ such that

$$\mu\left(\bigcup_{\nu=K+1}^{\infty} \mathcal{D}_\nu\right) < \epsilon. \tag{7.3}$$

Consider the following subset of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$,

$$\Theta_\epsilon := \bigcup_{\nu=1}^{K} \bigcup_{k=0}^{n_\nu-1} \left[\frac{k}{n_\nu} + \alpha_{\mathcal{D}_\nu} - \frac{\epsilon}{Kn_\nu}, \frac{k}{n_\nu} + \alpha_{\mathcal{D}_\nu} + \frac{\epsilon}{Kn_\nu}\right] + \mathbb{Z}. \tag{7.4}$$

We may construct a continuous function $h = h_\epsilon$ with values in [0, 1] such that

$$h(\theta) = \begin{cases} 1 & \text{if } \theta \notin \Theta_{2\epsilon}, \\ 0 & \text{if } \theta \in \Theta_\epsilon. \end{cases} \tag{7.5}$$

Consider the set $J_{N,\epsilon}$ of j, for which $\theta_j \notin \Theta_{2\epsilon}$. The corresponding set sequence $\boldsymbol{J}_\epsilon = \{J_{N,\epsilon}\}_{N\in\mathcal{I}}$ has, in view of Weyl's law (theorem 5.2), density $\Delta(\boldsymbol{J}_\epsilon) = \mu(\mathcal{D}_0) + O(\epsilon)$, where the implied constant is independent of K, since the measure of $\Theta_{2\epsilon}$ is at most

$$\sum_{\nu=1}^{K} \sum_{k=0}^{n_\nu-1} \frac{4\epsilon}{Kn_\nu} = 4\epsilon. \tag{7.6}$$

Now,

$$\sum_{j\in J_{N,\epsilon}} |\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j\rangle|^2 \leqslant \sum_{j=1}^{N} h(\theta_j)|\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j\rangle|^2 \tag{7.7}$$

$$\leqslant \sum_{j=1}^{N} h(\theta_j)\|\mathrm{Op}_N(a)\varphi_j\|^2 \tag{7.8}$$

$$= \sum_{j=1}^{N} h(\theta_j)\langle \mathrm{Op}_N(|a|^2)\varphi_j, \varphi_j\rangle + o_\epsilon(N). \tag{7.9}$$

Hence, by theorem 6.2,

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{j\in J_{N,\epsilon}} |\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j\rangle|^2 \leqslant \sum_{\nu=0}^{\infty} \int_{\mathcal{D}_\nu} |a|^2 \,\mathrm{d}\mu \int_0^1 h(\theta)\, \rho_\nu(\theta)\, \mathrm{d}\theta. \tag{7.10}$$

Now, under assumption (i) of the theorem,

$$\int_{\mathcal{D}_0} |a|^2 \,\mathrm{d}\mu = 0 \tag{7.11}$$

and furthermore

$$\sum_{\nu=1}^{K} \int_{\mathcal{D}_\nu} |a|^2 \, \mathrm{d}\mu \int_0^1 h(\theta)\, \rho_\nu(\theta) \, \mathrm{d}\theta = 0 \tag{7.12}$$

since h is supported outside the support of $\rho_1, \ldots, \rho_K$. For the remaining sum,

$$\left| \sum_{\nu=K+1}^{\infty} \int_{\mathcal{D}_\nu} |a|^2 \, \mathrm{d}\mu \int_0^1 h(\theta)\, \rho_\nu(\theta) \, \mathrm{d}\theta \right| \leqslant \left| \sum_{\nu=K+1}^{\infty} \int_{\mathcal{D}_\nu} |a|^2 \, \mathrm{d}\mu \right| \leqslant \epsilon \max|a|^2 \tag{7.13}$$

by (7.3). Hence,

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{j \in J_{N,\epsilon}} |\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j \rangle|^2 = O(\epsilon) \tag{7.14}$$

with $\epsilon > 0$ arbitrarily small. Therefore, there is a sequence of values $\epsilon = \epsilon_N$ such that $\epsilon_N \to 0$ as $N \to \infty$, and

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{j \in J_N} |\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j \rangle|^2 = 0, \tag{7.15}$$

where $J_N := J_{N,\epsilon_N}$. The proof for case (i) is complete.

As to case (ii), define $J'_{N,\epsilon}$ as the set of j, for which $\theta_j \in \Theta_\epsilon$. The corresponding set sequence $\boldsymbol{J}'_\epsilon = \{J'_{N,\epsilon}\}_{N\in\mathcal{I}}$ has density $\Delta(\boldsymbol{J}'_\epsilon) = 1 - \mu(\mathcal{D}_0) + O(\epsilon)$, recall Weyl's law (theorem 5.2). Then, analogous to (7.10),

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{j \in J'_{N,\epsilon}} |\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j \rangle|^2 \leqslant \sum_{\nu=0}^{\infty} \int_{\mathcal{D}_\nu} |a|^2 \, \mathrm{d}\mu \int_0^1 [1 - h(\theta)]\, \rho_\nu(\theta) \, \mathrm{d}\theta, \tag{7.16}$$

where all but the $\nu = 0$ term vanish, since $a|_{\mathcal{M}-\mathcal{D}_0} = 0$. So

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{j \in J'_{N,\epsilon}} |\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j \rangle|^2 \leqslant \int_{\mathcal{D}_0} |a|^2 \, \mathrm{d}\mu \int_0^1 [1 - h(\theta)] \, \mathrm{d}\theta = O(\epsilon). \tag{7.17}$$

□

Corollary 7.2. *There are set sequences $\boldsymbol{I} := \{I_N\}_{N\in\mathcal{I}}$ and $\boldsymbol{I}' := \{I'_N\}_{N\in\mathcal{I}}$ with densities*

$$\Delta(\boldsymbol{I}) = \mu(\mathcal{D}_0), \qquad \Delta(\boldsymbol{I}') = 1 - \mu(\mathcal{D}_0), \tag{7.18}$$

such that

(i) the eigenstates φ_j, $j \in I_N$, are semiclassically localized in $\mathcal{D}_0$;
(ii) the eigenstates φ_j, $j \in I'_N$, are semiclassically localized in $\mathcal{M} - \mathcal{D}_0$.

Proof. This is a straightforward consequence of theorem 7.1 and Chebyshev's inequality. □

Theorem 7.3. *If in addition to the assumptions of theorem 7.1 the phases α_ν ($\nu = 1, 2, 3, \ldots$) are linearly independent over $\mathbb{Q}$, then there are set sequences $\boldsymbol{J}^\nu := \{J_N^\nu\}_{N\in\mathcal{I}}$ with densities*

$$\Delta(\boldsymbol{J}^\nu) = \mu(\mathcal{D}_\nu), \tag{7.19}$$

such that for any $a \in \mathrm{C}^\infty(\mathcal{M})$ with $a|_{\mathcal{D}_\nu} = 0$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j \in J_N^\nu} |\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j \rangle|^2 = 0. \tag{7.20}$$

Proof. This follows from a slight modification of the proof of theorem 7.1, where the set J_N^ν is approximated by the set $J_{N,\epsilon}^\nu$ comprising those j for which

$$\theta_j \in \bigcup_{k=0}^{n_\nu-1} \left[\frac{k}{n_\nu} + \alpha_{\mathcal{D}_\nu} - \frac{\epsilon\delta}{Kn_\nu}, \frac{k}{n_\nu} + \alpha_{\mathcal{D}_\nu} + \frac{\epsilon\delta}{Kn_\nu} \right] \mod 1. \tag{7.21}$$

The crucial observation is that, by the linear independence of the α_ν over $\mathbb{Q}$, there exists a $\delta = \delta_{\epsilon,K} > 0$ small enough, such that sets corresponding to different $\nu = 1, 2, 3, \ldots, K$ are disjoint. □

Thus, by Chebyshev's inequality, for a subsequence of $j \in J_N^\nu$ of density $\mu(\mathcal{D}_\nu)$ the eigenstates φ_j are semiclassically localized in $\mathcal{D}_\nu$.

8. Quantum ergodicity

Let us now turn to the question of quantum ergodicity for maps that have one ergodic component $\mathcal{D}_0$ and are periodic on a remaining countable collection $\mathcal{D}_1, \mathcal{D}_2, \ldots$ of domains. Examples of linked twist maps with this property are discussed in section 3.3.

Theorem 8.1. *Choose Φ and $U_N(\Phi)$ as in theorem 5.2, and suppose Φ acts ergodically on $\mathcal{D}_0$. Let $\varphi_1, \ldots, \varphi_N \in \mathbb{C}^N$ be an orthonormal basis of eigenstates of $U_N(\phi)$. Then, for any $a \in \mathrm{C}^\infty(\mathcal{M})$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j\in J_N} \left| \langle \mathrm{Op}_N(a)\varphi_j, \varphi_j\rangle - \int_{\mathcal{D}_0} a \, \mathrm{d}\mu \right|^2 = 0. \tag{8.1}$$

Proof. We may assume without loss of generality that $\int_{\mathcal{D}_0} a \, \mathrm{d}\mu = 0$ and $|a| \leqslant 1$. It is then sufficient to show[8]

$$S_2(a, N) := \frac{1}{N} \sum_{j\in J_N} |\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j\rangle|^2 \to 0 \tag{8.2}$$

as $N \to \infty$.

For any given $T \geqslant 1$, we may write

$$a = a_T + a_T' + a_T^{\mathrm{bad}}, \tag{8.3}$$

where

(i) $a_T \in \mathrm{C}^\infty$ has compact support contained in $\mathcal{D}_0$ and the domain of continuity of $\Phi, \Phi^2, \ldots, \Phi^T$, and furthermore $\int a_T \, \mathrm{d}\mu = 0$, $|a_T| \leqslant 1$;
(ii) $a_T' \in \mathrm{C}^\infty$ is supported inside $\mathcal{M} - \mathcal{D}_0$, and $|a_T'| \leqslant 1$;
(iii) $\int_{\mathcal{M}} |a_T^{\mathrm{bad}}|^2 \, \mathrm{d}\mu < T^{-1}$.

By the triangle inequality,

$$S_2(a, N)^{1/2} \leqslant S_2(a_T, N)^{1/2} + S_2(a_T', N)^{1/2} + S_2(a_T^{\mathrm{bad}}, N)^{1/2}. \tag{8.4}$$

By theorem 7.1,

$$\lim_{N\to\infty} S_2(a_T', N) = 0. \tag{8.5}$$

[8] The argument presented here is inspired by the proof of quantum ergodicity for cat maps [14, 38], cf also [48].

Furthermore, by the Cauchy–Schwartz inequality,

$$S_2(a_T^{\mathrm{bad}}, N) \leqslant \frac{1}{N}\sum_{j=1}^{N} \left\| \mathrm{Op}_N(a_T^{\mathrm{bad}})\varphi_j \right\|^2 \tag{8.6}$$

$$= \frac{1}{N}\mathrm{Tr}\,\mathrm{Op}_N(|a_T^{\mathrm{bad}}|^2) + o_T(1) \tag{8.7}$$

and hence

$$\limsup_{N\to\infty} S_2(a_T^{\mathrm{bad}}, N) < T^{-1}. \tag{8.8}$$

As to the remaining term,

$$S_2(a_T, N) = \frac{1}{N}\sum_{j\in J_N} |\langle \mathrm{Op}_N(a_T)\varphi_j, \varphi_j\rangle|^2, \tag{8.9}$$

it remains to be proved that the limsup of (8.9) can be made arbitrarily small for sufficiently large T. To this end define the ergodic average of a_T by

$$a_T^T := \frac{1}{T}\sum_{n=1}^{T} a_T \circ \Phi^n. \tag{8.10}$$

Since φ_j are the eigenfunctions of $U_N(\Phi)$ we have

$$S_2(a_T, N) = \frac{1}{N}\sum_{j\in J_N} \left| \frac{1}{T}\sum_{n=1}^{T} \left\langle U_N(\Phi)^{-n}\mathrm{Op}_N(a_T)U_N(\Phi)^n\varphi_j, \varphi_j\right\rangle \right|^2 \tag{8.11}$$

$$\leqslant \frac{1}{N}\sum_{j=1}^{N} \left| \frac{1}{T}\sum_{n=1}^{T} \left\langle U_N(\Phi)^{-n}\mathrm{Op}_N(a_T)U_N(\Phi)^n\varphi_j, \varphi_j\right\rangle \right|^2 \tag{8.12}$$

$$\leqslant \frac{1}{N}\sum_{j=1}^{N} \left\| \frac{1}{T}\sum_{n=1}^{T} U_N(\Phi)^{-n}\mathrm{Op}_N(a_T)U_N(\Phi)^n\varphi_j \right\|^2 \tag{8.13}$$

$$= \frac{1}{N}\sum_{j=1}^{N} \left\| \mathrm{Op}_N(a_T^T)\varphi_j \right\|^2 + o_T(1) \tag{8.14}$$

by axiom 2.2. Now

$$\frac{1}{N}\sum_{j=1}^{N} \left\| \mathrm{Op}_N(a_T^T)\varphi_j \right\|^2 = \frac{1}{N}\sum_{j=1}^{N} \left\langle \mathrm{Op}_N(a_T^T)^{\dagger}\mathrm{Op}_N(a_T^T)\varphi_j, \varphi_j \right\rangle \tag{8.15}$$

$$= \frac{1}{N}\sum_{j=1}^{N} \langle \mathrm{Op}_N(|a_T^T|^2)\varphi_j, \varphi_j\rangle + o_T(1) \tag{8.16}$$

$$= \int_{\mathcal{D}_0} |a_T^T|^2\, \mathrm{d}\mu + o_T(1). \tag{8.17}$$

We have

$$\left(\int_{\mathcal{D}_0} |a_T^T|^2\, \mathrm{d}\mu\right)^{1/2} \leqslant \left(\int_{\mathcal{D}_0} |a^T|^2\, \mathrm{d}\mu\right)^{1/2} + \left(\int_{\mathcal{D}_0} |a_T^{\mathrm{bad}T}|^2\, \mathrm{d}\mu\right)^{1/2}, \tag{8.18}$$

where

$$a^T := \frac{1}{T}\sum_{n=1}^{T} a \circ \Phi^n, \qquad a_T^{\mathrm{bad}T} := \frac{1}{T}\sum_{n=1}^{T} a_T^{\mathrm{bad}} \circ \Phi^n. \tag{8.19}$$

We have by Cauchy–Schwartz

$$\int_{\mathcal{D}_0} |a_T^{\mathrm{bad}T}|^2 \,\mathrm{d}\mu = \frac{1}{T^2} \sum_{j,k=1}^{T} \int_{\mathcal{D}_0} (a_T^{\mathrm{bad}} \circ \Phi^j)\overline{(a_T^{\mathrm{bad}} \circ \Phi^k)} \,\mathrm{d}\mu \tag{8.20}$$

$$\leqslant \frac{1}{T^2} \sum_{j,k=1}^{T} \left(\int_{\mathcal{D}_0} |a_T^{\mathrm{bad}} \circ \Phi^j|^2 \,\mathrm{d}\mu \right)^{1/2} \left(\int_{\mathcal{D}_0} |a_T^{\mathrm{bad}} \circ \Phi^k|^2 \,\mathrm{d}\mu \right)^{1/2} \tag{8.21}$$

$$= \int_{\mathcal{D}_0} |a_T^{\mathrm{bad}}|^2 \,\mathrm{d}\mu < \frac{1}{T} \tag{8.22}$$

using the Φ-invariance of μ and assumption (iii). Therefore,

$$\limsup_{N\to\infty} S_2(a_T, N) \leqslant \int_{\mathcal{D}_0} |a^T|^2 \,\mathrm{d}\mu + O(T^{-1}). \tag{8.23}$$

Since Φ acts ergodically on $\mathcal{D}_0$, we have a mean ergodic theorem for test functions $a \in \mathrm{L}^2(\mathcal{M})$, i.e.

$$\lim_{T\to\infty} \int_{\mathcal{D}_0} |a^T|^2 \,\mathrm{d}\mu = 0 \tag{8.24}$$

and hence $\limsup_{N\to\infty} S_2(a_T, N)$ becomes arbitrarily small for T sufficiently large. □

Corollary 8.2. *There is a set sequence* $\boldsymbol{I} := \{I_N\}_{N\in\mathcal{I}}$ *with density* $\Delta(\boldsymbol{I}) = \mu(\mathcal{D}_0)$ *such that*

$$\langle \mathrm{Op}_N(a)\varphi_j, \varphi_j\rangle \to \int_{\mathcal{D}_0} a \,\mathrm{d}\mu \tag{8.25}$$

for all $j \in I_N$, $N \to \infty$.

Proof. Apply Chebyshev's inequality with the variance given in (8.1). □

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Appendix. Converse quantum ergodicity (by Steve Zelditch)

The purpose of this appendix is to briefly make some connections between the preceding article of Marklof and O'Keefe and some results in our articles [46, 47] concerning the distribution of eigenfunctions and eigenvalues on certain Riemannian manifolds with an open invariant component where the geodesic flow is periodic.

At the same time, we wish to emphasize the relevance of all of the results to the *converse quantum ergodicity problem*: are quantum ergodic systems necessarily classically ergodic? As discussed in the preceding article, it is well known that classical ergodicity implies quantum ergodicity. However, there are few results in the converse direction. There could exist non-ergodic classical systems with quantum ergodic quantizations, because the invariant sets might not be quantizable in any suitable sense.

When the system is periodic in an open invariant set, one expects to be able to quantize this feature and prove that the system cannot be quantum ergodic. Examples where this has been carried out are given in the preceding article and in [47] (see also [41]). Aside from fully integrable systems, these appear to be the only examples where quantizations of classically non-ergodic systems have been proved to be non-quantum ergodic. It is very plausible that many other (if not all) classically non-ergodic systems are quantum non-ergodic, e.g. quantizations of KAM systems, but no rigorous proofs of this exist at this time except in the partially periodic case.

The examples studied in [47] were special Riemannian manifolds, which we might informally call 'pimpled spheres': we take the standard sphere (S^n, g_0) and deform the metric, and possibly the topology, in a polar cap $B_r(x_0)$ of some small radius r to obtain a new Riemannian manifold (M, g). We assume that $M_r := M\backslash B_r(x_0) \equiv S^n_r := S^n\backslash B_r(x_0)$ as Riemannian manifolds. We then denote by $\mathcal{M}_r \subset S^*M$ the smooth manifold with boundary formed by the closed geodesics of (S^n, g_0) that lie in M_r. The boundary consists of closed geodesics which intersect $\partial B_r(x_0)$ tangentially. Thus, the boundary between the invariant periodic component and the remaining component is smooth and in fact the geodesic flow is 'almost clean' in the sense of [46]. This condition resembles (but is more restrictive than) the Minkowski content zero condition on the boundaries of the mixed phase space components in the preceding paper. Also, we did not fix the metric or the type of dynamics of the geodesic flow in the non-periodic component. In [46], theorem 3.20, we used the method of moments to determine the Szegö limit measure of the wave group in these examples, i.e. the limit measure given in Marklof–O'Keefe's theorem 5.2. As in the their paper, we used these results to obtain eigenfunction distribution results. The following is a corollary of theorem A(b) proved in [47].

Proposition A.1. *The Laplacian* Δ *of a pimpled sphere* (M, g) *is never quantum ergodic.*

Let us sketch the proof combining the argument of [47] and that of the preceding paper. We assume the reader is familiar with pseudodifferential and Fourier integral operators, which are the observables and quantum maps in the Riemannian setting.

Proof. Let A be a zeroth order pseudodifferential operator on M with essential support in M_r, i.e. assume the complete symbol of A is supported in $\mathcal{M}_r$. Since the geodesic flow of (M, g) is periodic of period 2π in $\mathcal{M}_r$, $A\mathrm{e}^{2\pi\mathrm{i}\sqrt{\Delta}}$ is a zeroth order pseudodifferential operator on M. Using the calculation of the symbol of the wave group in [18], it is simple to see that the principal symbol of $A\mathrm{e}^{2\pi\mathrm{i}\sqrt{\Delta}}$ equals $\mathrm{e}^{\mathrm{i}\alpha}\sigma_A$ for a constant $\mathrm{e}^{\mathrm{i}\alpha}$ (a Maslov phase). Here, σ_A is the principal symbol of A. Thus, $\mathrm{e}^{2\pi\mathrm{i}\sqrt{\Delta}}$ satisfies the hypothesis (2.10) of the preceding paper.

Now argue by contradiction. If Δ were quantum ergodic, we would have

$$\langle B\phi_j, \phi_j\rangle \to \int_{S^*M} \sigma_B \,\mathrm{d}\mathcal{L} \tag{A.1}$$

along a subsequence of density one of any orthonormal basis $\{\phi_j\}$ of Δ-eigenfunctions. Here, $\mathcal{L}$ is normalized Liouville measure and B is a zeroth order pseudodifferential operator. Letting $B = A\mathrm{e}^{2\pi\mathrm{i}\sqrt{\Delta}}$, we would obtain

$$\langle A\mathrm{e}^{2\pi\mathrm{i}\sqrt{\Delta}}\phi_j, \phi_j\rangle \to \mathrm{e}^{\mathrm{i}\alpha} \int_{S^*M} \sigma_A \,\mathrm{d}\mathcal{L}. \tag{A.2}$$

However, we also have

$$\langle A e^{2\pi i\sqrt{\Delta}}\phi_j, \phi_j\rangle = e^{2\pi i\lambda_j}\langle A\phi_j, \phi_j\rangle \sim e^{2\pi i\lambda_j}\int_{S^*M}\sigma_A \, d\mathcal{L}. \tag{A.3}$$

Here, $\{\lambda_j\}$ are the eigenvalues of $\sqrt{\Delta}$ associated to ϕ_j. It follows that $e^{2\pi i\lambda_j} \to e^{i\alpha}$ along a subsequence of eigenvalues of density one.

But this implies that all of the eigenvalues of $e^{2\pi i\sqrt{\Delta}}$ cluster around $e^{i\alpha}$, i.e.

$$d\mu_\lambda := \frac{1}{N(\lambda)}\sum_{j:\lambda_j<\lambda}\delta_{e^{2\pi i\lambda_j}} \to \delta_{e^{i\alpha}}. \tag{A.4}$$

Here, $N(\lambda) = \#\{j : \lambda_j < \lambda\}$. But the weak limit of $d\mu_\lambda$ was calculated in [46], theorem 3.20 (cf also theorem 5.2 in the preceding article), and shown to be

$$\frac{\mathcal{L}(\mathcal{M}_r)}{\mathcal{L}(S^*M)}\delta_{e^{i\alpha}} + \left(1 - \frac{\mathcal{L}(\mathcal{M}_r)}{\mathcal{L}(S^*M)}\right) d\theta. \tag{A.5}$$

This contradiction proves that (M, g) is not quantum ergodic. □

In comparison, the argument in [47] used a trace formula for $\mathrm{Tr}\, A\Pi_\lambda$ where Π_λ is the orthogonal projection onto the span of ϕ_j with $\lambda_j \leqslant \lambda$. The above argument based on individual elements seems more vivid.

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