

# A FIVE DISTANCE THEOREM FOR KRONECKER SEQUENCES

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**ABSTRACT.** The three distance theorem (also known as the three gap theorem or Steinhaus problem) states that, for any given real number  $\alpha$  and integer  $N$ , there are at most three values for the distances between consecutive elements of the Kronecker sequence  $\alpha, 2\alpha, \dots, N\alpha \bmod 1$ . In this paper we consider a natural generalisation of the three distance theorem to the higher dimensional Kronecker sequence  $\vec{\alpha}, 2\vec{\alpha}, \dots, N\vec{\alpha}$  modulo an integer lattice. We prove that in two dimensions there are at most five values that can arise as a distance between nearest neighbors, for all choices of  $\vec{\alpha}$  and  $N$ . Furthermore, for almost every  $\vec{\alpha}$ , five distinct distances indeed appear for infinitely many  $N$  and hence five is the best possible general upper bound. In higher dimensions we have similar explicit, but less precise, upper bounds. For instance in three dimensions our bound is 13, though we conjecture the truth to be seven. We furthermore study the number of possible distances from a point to its nearest neighbor in a restricted cone of directions. This may be viewed as a generalisation of the gap length in one dimension. For large cone angles we use geometric arguments to produce explicit bounds directly analogous to the three distance theorem. For small cone angles we use ergodic theory of homogeneous flows in the space of unimodular lattices to show that the number of distinct lengths is (a) unbounded for almost all  $\vec{\alpha}$  and (b) bounded for  $\vec{\alpha}$  that satisfy certain Diophantine conditions.

## 1. INTRODUCTION

Consider a finite set  $S_N$  comprising  $N$  distinct points  $\xi_1, \dots, \xi_N$  on the unit torus  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ . The points in  $S_N$  partition  $\mathbb{T}$  into  $N$  intervals, representing the *gaps* of  $S_N$ . We denote by  $\delta_{n,N}$  the size of the  $n$ th gap, i.e., the distance between  $\xi_n$  and its nearest neighbor *to the right*. We denote by  $g_N = |\{\delta_{n,N} \mid 1 \leq n \leq N\}|$  the number of distinct gap sizes. For a generic choice of  $S_N$  one has  $g_N = N$ , since all gap lengths are generically distinct. A striking observation, known as the three distance (or three gap) theorem, is that for the Kronecker sequence  $\xi_n = n\alpha + \mathbb{Z}$ , one has  $g_N \leq 3$ , for any  $\alpha \in \mathbb{R}$  and  $N \in \mathbb{N}$ ; see [36, 37, 38, 39] for the original proofs and [23, 25, 29, 31, 33, 34, 35] for alternative approaches. Natural extensions to return maps for billiards in rectangles and interval exchange transformations are discussed in [16, 19, 20, 21] and [40], respectively. There are various generalisations of the three gap theorem to higher dimensions, several of which require Diophantine conditions on the choice of parameter. In the present paper we discuss natural extensions of the three distance theorem to higher dimensional Kronecker sequences, which represent

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translations of a multidimensional torus by a vector  $\vec{\alpha}$ . We will here consider nearest neighbor distances as well as distances to neighbors in restricted directions. The former may be viewed as a special case of the setting of Biringer and Schmidt [2], who considered the number of distinct nearest neighbor distances for an orbit generated by an isometry of a general Riemannian manifold. If distances are measured by the maximum norm rather than a Riemannian metric, Chevallier [8, Corollaire 1.2] showed that there are at most five distinct distances for Kronecker sequences on two-dimensional tori. Related studies in this context include the papers by Chevallier [9, 10, 12] and by Vijay [41]. For other higher dimensional variants of the three distance theorem, see [1, 3, 4, 5, 11, 13, 18, 22, 24, 25, 26, 27, 30].

Our setting is as follows. Let  $\mathcal{L}$  be a unimodular lattice in  $\mathbb{R}^d$  (one example to keep in mind is  $\mathbb{Z}^d$ ) and consider a point set  $S_N = \{\xi_1, \dots, \xi_N\}$  on the  $d$ -dimensional torus  $\mathbb{T}^d = \mathbb{R}^d/\mathcal{L}$ . The point set  $S_N$  that we are interested in is the  $d$ -dimensional Kronecker sequence,

$$(1.1) \quad S_N = S_N(\vec{\alpha}, \mathcal{L}) = \{\xi_n = n\vec{\alpha} + \mathcal{L} \mid 1 \leq n \leq N\} \subset \mathbb{T}^d,$$

for given  $\vec{\alpha} \in \mathbb{R}^d$ . Note that the  $\xi_n$  are not necessarily distinct if  $\vec{\alpha} \in \mathbb{Q}\mathcal{L}$ ; in this case we remove all multiple occurrences, and as a result the number of elements in  $S_N$  remains bounded as  $N \rightarrow \infty$ .

Let  $\delta_{n,N}$  be the distance of  $\xi_n$  to its nearest neighbor with respect to the standard flat Riemannian metric on  $\mathbb{T}^d$ . As above,  $g_N = g_N(\vec{\alpha}, \mathcal{L})$  denotes the number of distinct nearest neighbor distances  $\delta_{n,N}$ . Biringer and Schmidt [2] proved that, for all choices of  $\mathcal{L}$ ,  $\vec{\alpha}$ , and  $N$ ,

$$(1.2) \quad g_N(\vec{\alpha}, \mathcal{L}) \leq 3^d + 1.$$

Our first theorem improves this bound, and also gives the best possible result in dimension  $d = 2$ .

**Theorem 1.** *For every unimodular lattice  $\mathcal{L}$ ,  $\vec{\alpha} \in \mathbb{R}^d$  and  $N \in \mathbb{N}$  we have that*

$$(1.3) \quad g_N(\vec{\alpha}, \mathcal{L}) \leq \begin{cases} 5 & \text{if } d = 2, \\ \sigma_d + 1 & \text{if } d \geq 3, \end{cases}$$

where  $\sigma_d$  is the kissing number for  $\mathbb{R}^d$ .

Recall that the *kissing number*  $\sigma_d$  for  $\mathbb{R}^d$  is the maximum number of non-overlapping spheres of radius one in  $\mathbb{R}^d$  which can be arranged so that they all touch the unit sphere in exactly one point. The study of kissing numbers has a long and interesting history, and is connected to many areas of mathematics (see [6] and [32] for surveys of results). It is interesting to note that the three distance theorem in dimension  $d = 1$  is compatible with the bound  $g_N \leq \sigma_1 + 1 = 3$ , but that already in dimension  $d = 2$  this becomes suboptimal, since here  $\sigma_2 + 1 = 7 > 5$ . A table of known bounds for kissing numbers in dimensions  $d \leq 24$  is provided in Figure 1. Our corresponding bounds for  $g_N$  in dimensions  $d = 3, \dots, 10$  are therefore 13, 25, 46, 79, 135, 241, 365, 555. We do not claim that these bounds are optimal for any dimension  $d \geq 3$ . In particular, we conjecture that  $g_N(\vec{\alpha}, \mathcal{L}) \leq 7$  for  $d = 3$ . Figures 2 and 3 show examples where 5 (for  $d = 2$ ) and 7 (for  $d = 3$ ) distinct distances are obtained.

$d$	Upper bound for $\sigma_d$	$d$	Upper bound for $\sigma_d$
1	<b>2</b>	13	2069
2	<b>6</b>	14	3183
3	<b>12</b>	15	4866
4	<b>24</b>	16	7355
5	45	17	11072
6	78	18	16572
7	134	19	24812
8	<b>240</b>	20	36764
9	364	21	54584
10	554	22	82340
11	870	23	124416
12	1357	24	<b>196560</b>

FIGURE 1. A table of known bounds for kissing numbers. All bounds are taken from [6], and bounds listed in bold face are known to be best possible.

For general  $d$ , it follows from Theorem 1 together with an estimate for  $\sigma_d$  due to Kabatiansky and Levenshtein (see [28, Theorem 4, Corollary 1] or [14, Chapter 9]) that

$$(1.4) \quad g_N(\vec{\alpha}, \mathcal{L}) \leq 2^{0.401d(1+o(d))} \quad \text{as } d \rightarrow \infty.$$

The rate of convergence of the  $o(d)$  term in this estimate can be made more precise and explicit (non-asymptotic) upper bounds can also be obtained by applying [28, Equation (52)].

Values of  $\vec{\alpha}$  for which  $g_N(\vec{\alpha}, \mathcal{L}) = 5$  are surprisingly rare. Our first computer search, which took 1000 randomly and uniformly selected numbers  $\vec{\alpha} \in [0, 1]^2$  and checked  $g_N(\vec{\alpha}, \mathbb{Z}^2)$  for all  $N \leq 10^4$ , found only five values of  $\vec{\alpha}$  for which there was an  $N$  with  $g_N(\vec{\alpha}, \mathbb{Z}^2) = 5$ .

Nevertheless, as we will now see, every numerical example gives rise to a lower bound for an infinite sequence of  $N$  and for almost every  $\vec{\alpha}$ . In what follows, we say a sequence  $N_1 < N_2 < N_3 < \dots$  of integers is *sub-exponential* if

$$(1.5) \quad \lim_{i \rightarrow \infty} \frac{N_{i+1}}{N_i} = 1.$$

**Theorem 2.** *Let  $\mathcal{L}$  and  $\mathcal{L}_0$  be unimodular lattices. There is a  $P \subset \mathbb{R}^d$  of full Lebesgue measure, such that for every  $\vec{\alpha} \in P$ ,  $\vec{\alpha}_0 \in \mathbb{R}^d$ , and for every sub-exponential sequence  $(N_i)_i$ , we have*

$$(1.6) \quad \limsup_{i \rightarrow \infty} g_{N_i}(\vec{\alpha}, \mathcal{L}) \geq \sup_{N \in \mathbb{N}} g_N(\vec{\alpha}_0, \mathcal{L}_0).$$

In dimension  $d = 2$ , the choice  $\vec{\alpha}_0 = \left(\frac{19}{50}, \frac{33}{250}\right)$ ,  $N = 9$  and  $\mathcal{L}_0 = \mathbb{Z}^2$  (this is the example in Figure 2, left) produces precisely five distinct distances given by

$$\frac{\sqrt{74}}{125} \approx 0.0688, \quad \frac{\sqrt{\frac{3793}{2}}}{125} \approx 0.3484, \quad \frac{\sqrt{1901}}{125} \approx 0.3488, \quad \frac{\sqrt{\frac{157}{2}}}{25} \approx 0.3544, \quad \frac{3\sqrt{221}}{125} \approx 0.3568.$$

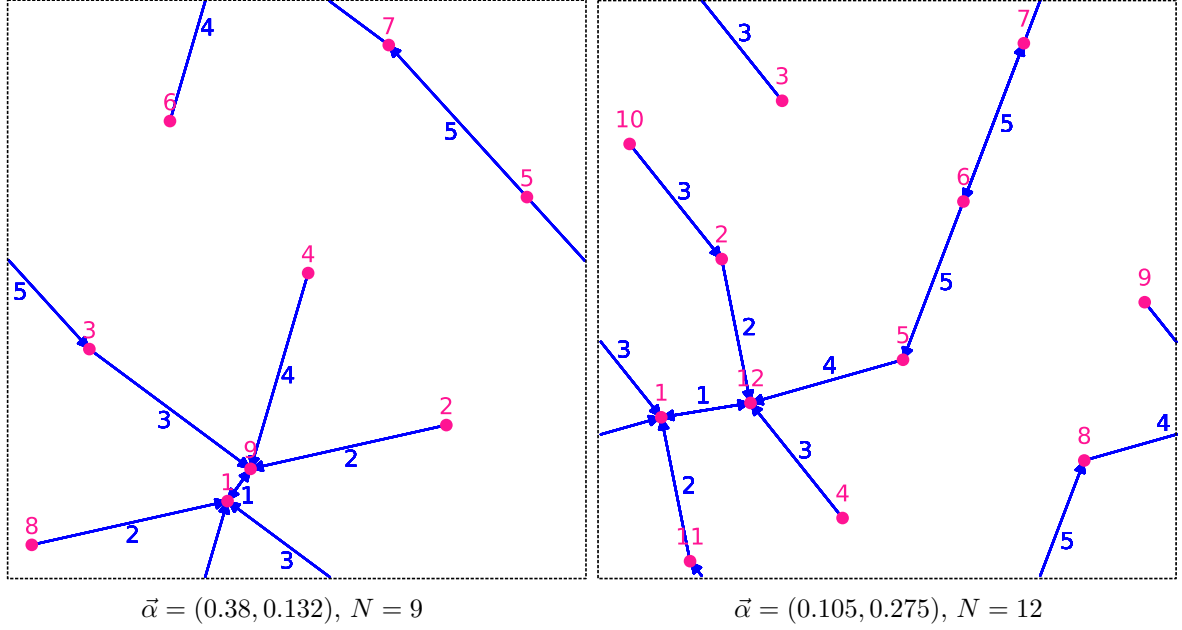


FIGURE 2. The nearest neighbor graph for the Kronecker sequence  $n\vec{\alpha} + \mathbb{Z}^2$  ( $n = 1, \dots, N$ ) in the torus  $\mathbb{R}^2/\mathbb{Z}^2$  (the unit square with opposite sides identified), for  $N = 9$  (left) and  $N = 12$  (right) and different choices of  $\vec{\alpha}$ . The vertex representing  $n\vec{\alpha} + \mathbb{Z}^2$  is labeled by  $n$  and colored in pink. The blue directed edges point from a vertex to its nearest neighbour(s). The blue edge labels correspond to the indices of each of the five distinct distances. Note that in each example there is a vertex with two nearest neighbours: vertex  $n = 5$  on the left and  $n = 6$  on the right.

Moreover, for  $d = 3$ ,  $\vec{\alpha}_0 = (\frac{46}{125}, \frac{107}{500}, \frac{43}{500})$ ,  $N = 15$  and  $\mathcal{L}_0 = \mathbb{Z}^3$  (cf. Figure 3), we have seven distinct distances,

$$\begin{aligned} \frac{17\sqrt{\frac{7}{2}}}{125} \approx 0.2544, \quad \frac{\sqrt{\frac{13513}{2}}}{250} \approx 0.3288, \quad \frac{21\sqrt{\frac{37}{2}}}{250} \approx 0.3613, \quad \frac{\sqrt{\frac{177}{2}}}{25} \approx 0.3763, \\ \frac{\sqrt{\frac{19237}{2}}}{250} \approx 0.3923, \quad \frac{\sqrt{2866}}{125} \approx 0.4283, \quad \frac{\sqrt{\frac{23577}{2}}}{250} \approx 0.4343. \end{aligned}$$

Applying these data to Theorem 2, combined with the upper bound of 5 in Theorem 1, we immediately obtain the following result.

**Theorem 3.** *Let  $d = 2$  or  $3$ . For any unimodular lattice  $\mathcal{L}$  in  $\mathbb{R}^d$ , there is a set  $P \subset \mathbb{R}^d$  of full Lebesgue measure, such that for every  $\vec{\alpha} \in P$ , and for every sub-exponential sequence  $(N_i)_i$ , we have that*

$$(1.7) \quad \limsup_{i \rightarrow \infty} g_{N_i}(\vec{\alpha}, \mathcal{L}) \begin{cases} = 5 & \text{if } d = 2, \\ \geq 7 & \text{if } d = 3. \end{cases}$$

Thus the upper bound of 5 in Theorem 1 is indeed optimal for every  $\mathcal{L}$ , almost every  $\vec{\alpha}$  and infinitely many  $N$ .

In dimension  $d = 1$  nearest neighbour distances do not necessarily coincide with the set of gap lengths, as gaps are the distance to the nearest neighbor in a fixed direction.

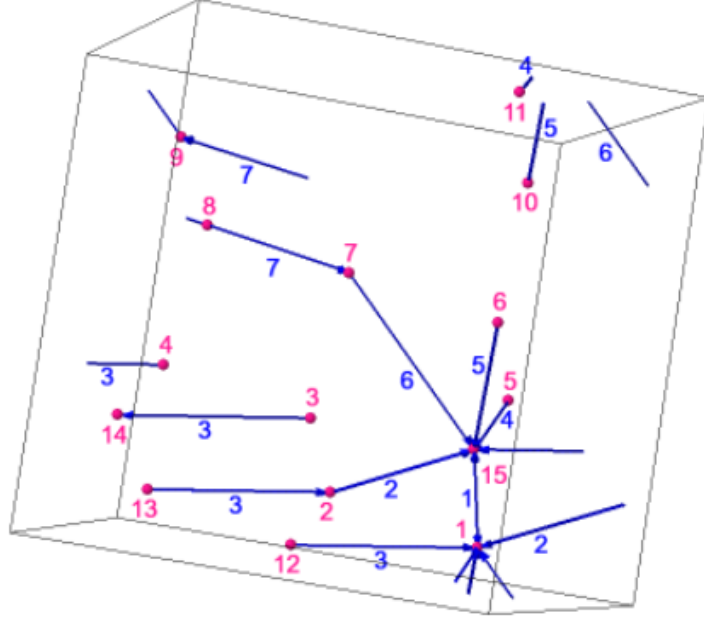


FIGURE 3. The nearest neighbor graph for the Kronecker sequence  $n\vec{\alpha} + \mathbb{Z}^3$  ( $n = 1, \dots, 15$ ) in the torus  $\mathbb{R}^3/\mathbb{Z}^3$  (the unit cube with opposite faces identified), for  $\vec{\alpha} = (\frac{46}{125}, \frac{107}{500}, \frac{43}{500})$ . The vertex representing  $n\vec{\alpha} + \mathbb{Z}^3$  is labeled by  $n$  and colored in pink. The blue directed edges point from a vertex to its nearest neighbour(s). The blue edge labels correspond to the indices of each of the seven distinct distances.

To generalise this interpretation of a gap to higher dimensions, fix a subset  $\mathcal{D}$  of the unit sphere  $\mathbb{S}_1^{d-1}$ , and denote by  $\delta_{n,N}(\mathcal{D})$  the distance from  $\xi_n$  to its nearest neighbor *in the direction of  $\mathcal{D}$* . More precisely, denote by  $\vec{\xi}_n \in \mathbb{R}^d$  a fixed representative of the coset  $\xi_n \bmod \mathcal{L}$  so that  $\xi_n = \vec{\xi}_n + \mathcal{L}$  and define

$$(1.8) \quad \delta_{n,N}(\mathcal{D}) = \min\{|\vec{\xi}_m - \vec{\xi}_n + \vec{\ell}| \mid \vec{\xi}_m - \vec{\xi}_n + \vec{\ell} \in \mathbb{R}_{>0}\mathcal{D}, 1 \leq m \leq N, \vec{\ell} \in \mathcal{L}\},$$

where  $|\cdot|$  is the standard Euclidean norm in  $\mathbb{R}^d$ . For  $\mathcal{D} = \mathbb{S}_1^{d-1}$  we recover the nearest neighbor distance

$$(1.9) \quad \delta_{n,N}(\mathbb{S}_1^{d-1}) = \delta_{n,N} = \min\{|\vec{\xi}_m - \vec{\xi}_n + \vec{\ell}| > 0 \mid 1 \leq m \leq N, \vec{\ell} \in \mathcal{L}\}.$$

In particular, note that the nearest neighbor of  $\xi_n$  might be  $\xi_n$  itself; in this case  $\delta_{n,N} = |\vec{\ell}|$  for suitable non-zero  $\ell \in \mathcal{L}$ .

As an illustration consider dimension  $d = 1$ . The circle of radius one is  $\mathbb{S}_1^0 = \{-1, 1\}$  and the choice  $\mathcal{D} = \{1\}$  produces the distance to the nearest neighbor to the right (which leads to the classical three distance theorem). On the other hand  $\mathcal{D} = \{-1, 1\}$  yields the distance to the nearest neighbor (in both directions). Since we are in dimension one, the set of distances to nearest neighbors is contained in the set of distances to nearest neighbors to the right, but it is not necessarily equal to it. However, for the case of the Kronecker sequence, the three distance theorem also holds also for nearest neighbors. In other words, there are examples of  $\alpha$  and  $N$  for which the number of distinct nearest neighbor distances is equal to three (e.g. take  $\alpha = e$  and  $N = 5$ ).

The central object of our study is the number  $g_N(\mathcal{D})$  of distinct nearest neighbor distances in direction  $\mathcal{D} \subseteq \mathbb{S}_1^{d-1}$ ,

$$(1.10) \quad g_N(\mathcal{D}) = |\{\delta_{n,N}(\mathcal{D}) \mid 1 \leq n \leq N\}|,$$

for  $d \geq 2$ . We will also write  $g_N(\mathcal{D}) = g_N(\mathcal{D}, \vec{\alpha}, \mathcal{L})$  to highlight the dependence on vector and lattice.

We first present detailed results for dimension  $d = 2$ . In this case  $\mathbb{S}_1^{d-1}$  is the unit circle in  $\mathbb{R}^2$ . The following theorem deals with the case when  $\mathcal{D} \subseteq \mathbb{S}_1^1$  is an interval of arclength  $\tau > \pi$ . (The case  $\tau = 2\pi$  has already been covered in Theorem 1 and we include it here for completeness.)

**Theorem 4.** *Let  $d = 2$ , and assume  $\mathcal{D} \subseteq \mathbb{S}_1^1$  is a half-open interval of arclength  $\tau > \pi$ . Then for any unimodular lattice  $\mathcal{L}$ ,  $\vec{\alpha} \in \mathbb{R}^2$  and  $N \in \mathbb{N}$  we have that*

$$(1.11) \quad g_N(\mathcal{D}, \vec{\alpha}, \mathcal{L}) \leq \begin{cases} 5 & \text{if } \tau = 2\pi, \\ 9 & \text{if } 5\pi/3 < \tau < 2\pi, \\ 8 & \text{if } \tau = 5\pi/3, \\ 9 & \text{if } 4\pi/3 < \tau < 5\pi/3, \\ 8 & \text{if } \tau = 4\pi/3, \\ 12 + \left\lfloor \frac{\sin(\tau/2 + \pi/6)}{\sin(\tau - \pi)} \right\rfloor & \text{if } \pi < \tau < 4\pi/3. \end{cases}$$

Related results in various settings have been obtained independently by Chevallier [9, 10, 12] and Vijay [41], but the precise description of the nearest neighbor problem which we give here does not appear to have been considered. However, the problem studied by Vijay in his paper ‘‘Eleven Euclidean distances are enough’’ [41] is roughly comparable to the special case of  $\tau = 3\pi/2$  in Theorem 4 above.

By way of contrast with the upper bounds above, our next result demonstrates that the restriction in Theorems 1 and 4 to a ‘‘large’’ set of directions  $\mathcal{D}$  is essential. Note that the upper bound for  $g_N$  in the final case of Theorem 4 tends to infinity as  $\tau \rightarrow \pi^+$ . In fact, in dimension  $d = 2$  intervals  $\mathcal{D} \subset \mathbb{S}_1^1$  of lengths  $\tau < \pi$  produce unbounded numbers of distinct distances, for almost every  $\vec{\alpha}$ . This is part of the content of the following theorem, which also deals with analogous regions  $\mathcal{D}$  in higher dimensions.

**Theorem 5.** *Let  $d \geq 2$  and  $\mathcal{L}$  a unimodular lattice. There exists a set  $P \subset \mathbb{R}^d$  of full Lebesgue measure, such that for every  $\mathcal{D} \subset \mathbb{S}_1^{d-1}$  with non-empty interior and closure contained in an open hemisphere, for every  $\vec{\alpha} \in P$ , and for every sub-exponential sequence  $(N_i)_i$ , we have*

$$(1.12) \quad \sup_i g_{N_i}(\mathcal{D}, \vec{\alpha}, \mathcal{L}) = \infty, \quad \liminf_i g_{N_i}(\mathcal{D}, \vec{\alpha}, \mathcal{L}) < \infty.$$

Our final observation is that the finite distance phenomenon is recovered in any dimension, for general test sets  $\mathcal{D}$ , if we impose Diophantine conditions on  $\vec{\alpha}$ . We say that  $\vec{\alpha} \in \mathbb{R}^d$  is *badly approximable by  $\mathbb{Q}\mathcal{L}$*  if there is a constant  $c > 0$  such that  $|n\vec{\alpha} - \vec{\ell}|_\infty > cn^{-1/d}$  for all  $\vec{\ell} \in \mathcal{L}$ ,  $n \in \mathbb{N}$ . Here  $|\cdot|_\infty$  denotes the maximum norm.

**Theorem 6.** *Let  $d \geq 2$ ,  $\mathcal{L}$  a unimodular lattice and  $\vec{\alpha} \in \mathbb{R}^d$  badly approximable by  $\mathbb{Q}\mathcal{L}$ . For  $\mathcal{D} \subseteq \mathbb{S}_1^{d-1}$  with non-empty interior, we have that*

$$(1.13) \quad \sup_{N \in \mathbb{N}} g_N(\mathcal{D}, \vec{\alpha}, \mathcal{L}) < \infty.$$

For comparison, the more precise bounds in Theorems 1 and 4 hold for all  $\vec{\alpha}$ , but only for a restricted class of  $\mathcal{D}$ .

To relate the above Diophantine condition on  $\vec{\alpha}$  to the standard notion of *badly approximable by  $\mathbb{Q}^d$*  (which corresponds to the special case  $\mathcal{L} = \mathbb{Z}^d$ ), take  $M_0 \in \mathrm{SL}(d, \mathbb{R})$  so that  $\mathcal{L} = \mathbb{Z}^d M_0$ . We then see that  $\vec{\alpha} \in \mathbb{R}^d$  is badly approximable by  $\mathbb{Q}\mathcal{L}$  if and only if  $\vec{\alpha}_0 = \vec{\alpha} M_0^{-1}$  is badly approximable by  $\mathbb{Q}^d$ . (The positive constants  $c$  appearing in both definitions are not necessarily the same.) Furthermore, by Khintchine's transference principle (see the Corollary to Theorem II in [7, Chapter V]), the vector  $\vec{\alpha}_0$  is badly approximable by  $\mathbb{Q}^d$  if and only if there is  $c > 0$  such that  $\|\vec{m} \cdot \vec{\alpha}_0\|_{\mathbb{R}/\mathbb{Z}} > c|\vec{m}|^{-d}$  for all non-zero  $\vec{m} \in \mathbb{Z}^d$ . Here  $\|x\|_{\mathbb{R}/\mathbb{Z}} = \min_{k \in \mathbb{Z}} |x + k|$  denotes the distance to the nearest integer.

The key strategy of the proofs of the above theorems is to express the quantities  $S_N(\vec{\alpha})$ ,  $\delta_{n,N}(\mathcal{D})$  and  $g_N(\mathcal{D})$  in terms of functions on the space  $\mathrm{SL}(d+1, \mathbb{Z}) \setminus \mathrm{SL}(d+1, \mathbb{R})$  of unimodular lattices in  $\mathbb{R}^{d+1}$ . This is explained in detail in Section 2. Once this connection is established, the proofs of Theorems 1 and 4 reduce to geometric arguments involving lattices and sphere coverings, which are laid out in Sections 3-5 in dimension  $d = 2$  and in Section 6 for dimensions  $d \geq 3$ . The proofs of Theorems 2, 5, and 6 require upper and lower bounds for the relevant functions on the space of lattices, combined with the same ergodic-theoretic arguments used in [26]. This material is presented in Section 7.

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## 2. REFORMULATION IN TERMS OF LATTICES

This section follows the approach developed for the three gap theorem [31] and higher dimensional variants concerning gaps in values taken by linear forms, and hitting times for toral rotations [26].

By substituting  $k = m - n$  in equation (1.8), we find that the distance from  $\xi_n$  to its nearest neighbor in the direction of  $\mathcal{D}$  is given by

$$(2.1) \quad \begin{aligned} \delta_{n,N}(\mathcal{D}) &= \min\{|k\vec{\alpha} + \vec{\ell}| \mid k\vec{\alpha} + \vec{\ell} \in \mathbb{R}_{>0}\mathcal{D}, -n < k \leq N - n, \vec{\ell} \in \mathcal{L}\} \\ &= \min\{|k\vec{\alpha} + \vec{\ell}| \mid k\vec{\alpha} + \vec{\ell} \in \mathbb{R}_{>0}\mathcal{D}, -n < k < N_+ - n, \vec{\ell} \in \mathcal{L}\}, \end{aligned}$$

where  $N_+ := N + \frac{1}{2}$ . Select  $M_0 \in \mathrm{SL}(d, \mathbb{R})$  so that  $\mathcal{L} = \mathbb{Z}^d M_0$ , and let

$$(2.2) \quad A_N(\vec{\alpha}) = A_N(\vec{\alpha}, \mathcal{L}) = \begin{pmatrix} 1 & 0 \\ 0 & M_0 \end{pmatrix} \begin{pmatrix} 1 & \vec{\alpha} \\ 0 & \mathbf{1}_d \end{pmatrix} \begin{pmatrix} N^{-1} & 0 \\ 0 & N^{1/d} \mathbf{1}_d \end{pmatrix}.$$

Then we have that

$$(2.3) \quad \delta_{n,N}(\mathcal{D}) = N_+^{-1/d} \min \left\{ |\vec{v}| \mid (u, \vec{v}) \in \mathbb{Z}^{d+1} A_{N_+}(\vec{\alpha}), \right. \\ \left. -\frac{n}{N_+} < u < 1 - \frac{n}{N_+}, \vec{v} \in \mathbb{R}_{>0}\mathcal{D} \right\},$$

for all  $1 \leq n \leq N$ . To cast this in a more general setting, let  $G = \mathrm{SL}(d+1, \mathbb{R})$  and  $\Gamma = \mathrm{SL}(d+1, \mathbb{Z})$ . Then for general  $M \in G$  and  $t \in (0, 1)$ , define

$$(2.4) \quad \mathcal{Q}_{\mathcal{D}}(M, t) = \{(u, \vec{v}) \in \mathbb{Z}^{d+1}M \mid -t < u < 1-t, \vec{v} \in \mathbb{R}_{>0}\mathcal{D}\}$$

and

$$(2.5) \quad F_{\mathcal{D}}(M, t) = \min \{ |\vec{v}| \mid (u, \vec{v}) \in \mathcal{Q}_{\mathcal{D}}(M, t) \}.$$

In this notation it is clear that

$$(2.6) \quad \delta_{n,N}(\mathcal{D}) = N_+^{-1/d} F_{\mathcal{D}} \left( A_{N_+}(\vec{\alpha}), \frac{n}{N_+} \right).$$

Before proceeding further, we first establish the following basic result.

**Proposition 1.** *If  $\mathcal{D} \subseteq \mathbb{S}_1^{d-1}$  has non-empty interior, then  $F_{\mathcal{D}}$  is well-defined as a function  $\Gamma \backslash G \times (0, 1) \rightarrow \mathbb{R}_{>0}$ .*

*Proof.* We first show that the set  $\mathcal{Q}_{\mathcal{D}}(M, t)$  is non-empty for all  $M \in G$  and  $t \in (0, 1)$ . Fix any  $\vec{w} \in \mathcal{D}^\circ$ , and denote by  $\Sigma_\perp \subset \mathbb{R}^d$  the  $(d-1)$ -dimensional hyperplane perpendicular to  $\vec{w}$ . Denote by  $\vec{v}_\perp$  the orthogonal projection of  $\vec{v}$  to  $\Sigma_\perp$ . Given  $(M, t)$ , let  $\epsilon > 0$  be sufficiently small so that (i)  $\epsilon < \min\{t, 1-t\}$ , (ii) there is no non-zero lattice point in  $\mathbb{Z}^{d+1}M$  within  $\epsilon$ -distance to the origin. Furthermore, fix  $\delta > 0$  sufficiently small so that

$$(2.7) \quad \{\vec{v} \in \mathbb{R}^d \mid \vec{w} \cdot \vec{v} > \epsilon, |\vec{v}_\perp| < \delta\} \subset \mathbb{R}_{>0}\mathcal{D}.$$

Such a  $\delta$  exists since  $\vec{w} \in \mathcal{D}^\circ$ . By construction

$$(2.8) \quad \{(u, \vec{v}) \in \mathbb{Z}^{d+1}M \mid |u| < \epsilon, \vec{w} \cdot \vec{v} > \epsilon, |\vec{v}_\perp| < \delta\} \subset \mathcal{Q}_{\mathcal{D}}(M, t)$$

In view of Minkowski's theorem, the symmetric, convex set  $\{(u, \vec{v}) \in \mathbb{R}^{d+1} \mid |u| < \epsilon, |\vec{v}_\perp| < \delta\}$  contains a non-zero element of  $\mathbb{Z}^{d+1}M$ . By construction there is no non-zero lattice point in  $\mathbb{Z}^{d+1}M$  within  $\epsilon$ -distance to the origin, and furthermore  $\mathbb{Z}^{d+1}M$  is symmetric under reflection at the origin. We can therefore conclude that the set in (2.8), and therefore also  $\mathcal{Q}_{\mathcal{D}}(M, t)$ , is non-empty. Hence by the uniform discreteness of  $\mathbb{Z}^{d+1}M$  the minimum value in the definition of  $F_{\mathcal{D}}$  exists.

Finally, note that  $F_{\mathcal{D}}(\gamma M, t) = F_{\mathcal{D}}(M, t)$  for  $\gamma \in \Gamma$ , and hence  $F$  is well-defined on  $\Gamma \backslash G \times (0, 1)$ .  $\square$

Note that for  $R \in \mathrm{SO}(d)$  we have

$$(2.9) \quad F_{\mathcal{D}R} \left( M \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, t \right) = F_{\mathcal{D}}(M, t).$$



Let us define

$$(2.10) \quad \mathcal{Q}_{\mathcal{D}}(M) = \bigcup_{t \in (0,1)} \mathcal{Q}_{\mathcal{D}}(M, t) = \{(u, \vec{v}) \in \mathbb{Z}^{d+1}M \mid |u| < 1, \vec{v} \in \mathbb{R}_{>0}\mathcal{D}\}.$$

The set

$$(2.11) \quad \mathcal{M}_{\mathcal{D}}(M) = \{|\vec{v}| \mid (u, \vec{v}) \in \mathcal{Q}_{\mathcal{D}}(M)\}$$

contains the set of values taken by the function  $t \mapsto F_{\mathcal{D}}(M, t)$ . It is a locally finite subset of  $\mathbb{R}_{>0}$ , i.e., there are at most finitely many points in any bounded interval. It follows that for fixed  $M$ , the function  $t \mapsto F_{\mathcal{D}}(M, t)$  is piecewise constant.

We denote by

$$(2.12) \quad \mathcal{G}_{\mathcal{D}}(M) = |\{F_{\mathcal{D}}(M, t) \mid 0 < t < 1\}|$$

the number of distinct values attained by the function  $t \mapsto F_{\mathcal{D}}(M, t)$ . For  $N > 0$ , let

$$(2.13) \quad \mathcal{G}_{\mathcal{D},N}(M) = |\{F_{\mathcal{D}}(M, \frac{n}{N_+}) \mid 1 \leq n \leq N\}|.$$

We have  $\mathcal{G}_{\mathcal{D},N}(M) \leq \mathcal{G}_{\mathcal{D}}(M)$ , and so in particular

$$(2.14) \quad g_N(\mathcal{D}) = \mathcal{G}_{\mathcal{D},N}(A_{N_+}(\vec{\alpha})) \leq \mathcal{G}_{\mathcal{D}}(A_{N_+}(\vec{\alpha})).$$

### 3. GEOMETRIC LEMMAS IN DIMENSION $d = 2$

To fix notation for our subsequent discussion, we define a representative set of vectors  $(u_i, \vec{v}_i) \in \mathcal{Q}_{\mathcal{D}}(M)$ , for which the lengths  $|\vec{v}_i|$  are distinct, and each of which corresponds to an element in the set

$$(3.1) \quad \mathcal{F}_{\mathcal{D}}(M) = \{F_{\mathcal{D}}(M, t) \mid 0 < t < 1\}.$$

To be specific, for each  $M \in \Gamma \backslash G$  we fix vectors  $(u_1, \vec{v}_1), \dots, (u_K, \vec{v}_K) \in \mathcal{Q}_{\mathcal{D}}(M)$  with  $K = \mathcal{G}_{\mathcal{D}}(M)$ , so that the following conditions hold:

(V1)  $0 < |\vec{v}_1| < |\vec{v}_2| < \dots < |\vec{v}_K|$ .

(V2) For each  $\delta \in \mathcal{F}_{\mathcal{D}}(M)$  there exists an  $1 \leq i \leq K$  such that  $\delta = |\vec{v}_i|$ .

(V3) For each  $1 \leq i \leq K$ , there exists a  $t \in (0, 1)$  such that  $(u_i, \vec{v}_i) \in \mathcal{Q}_{\mathcal{D}}(M, t)$  and  $|\vec{v}_i| = F_{\mathcal{D}}(M, t)$ .

Let us now focus on the case  $d = 2$ . In the following we identify the unit circle  $\mathbb{S}_1^1$  with the interval  $[-\pi, \pi) \bmod 2\pi$ , so that 0 corresponds to direction  $(1, 0) \in \mathbb{S}_1^1$ . In view of the rotation invariance (2.9) we may assume without loss of generality that  $\mathcal{D} \subseteq \mathbb{S}_1^1$  is centered at  $\theta = 0$ , i.e.,  $\mathcal{D} = [-\tau/2, \tau/2)$ , which we still view as a subset of  $\mathbb{S}_1^1$  (not  $\mathbb{R}$ ).

Our proof of Theorem 4, or rather the proofs of the more general Theorems 7 and 8 below, will be divided into three main cases, which together cover all possible angles  $\tau \in (\pi, 2\pi]$  described in (1.11). In each of these cases we will partition  $\mathcal{D}$  into subsets, consisting of a *symmetric* set  $\mathcal{S} \subset \mathcal{D}$  (symmetric with respect to the rotation  $\theta \mapsto \theta + \pi \bmod 2\pi$ ) and up to three *asymmetric* subsets. For notational convenience, let us set

$$(3.2) \quad \psi = 2\pi - \tau \in [0, \pi) \quad \text{and} \quad \phi = \tau - \pi \in (0, \pi].$$

First we specify our definitions of the asymmetric subsets in each of the three main cases.

**Case (C1):** If  $5\pi/3 \leq \tau \leq 2\pi$  then we define one asymmetric subset  $\mathcal{A}_0 \subset \mathcal{D}$  by

$$(3.3) \quad \mathcal{A}_0 = [-\psi/2, \psi/2).$$

(Note that this is the empty set if  $\tau = 2\pi$ .)

**Case (C2):** If  $4\pi/3 \leq \tau < 5\pi/3$  then we define two asymmetric subsets  $\mathcal{A}_{-1}$  and  $\mathcal{A}_1$  by

$$(3.4) \quad \mathcal{A}_{-1} = [-\psi/2, 0), \quad \mathcal{A}_1 = [0, \psi/2).$$

**Case (C3):** If  $\pi < \tau < 4\pi/3$  then we define three asymmetric subsets  $\mathcal{A}_{-1}$ ,  $\mathcal{A}_0$ , and  $\mathcal{A}_1$  by

$$(3.5) \quad \mathcal{A}_{-1} = [-\psi/2, -\pi/6), \quad \mathcal{A}_0 = [-\pi/6, \pi/6), \quad \mathcal{A}_1 = [\pi/6, \psi/2).$$

In all three cases, we define the symmetric subset  $\mathcal{S}$  by

$$(3.6) \quad \mathcal{S} = [-\tau/2, -\psi/2) \cup [\psi/2, \tau/2).$$

It is clear that  $\mathcal{S}$  is the largest symmetric subset of  $\mathcal{D}$ , that  $\mathcal{D}$  is the disjoint union of  $\mathcal{S}$  and its asymmetric subsets, and that each asymmetric subset is a half-open interval of length at most  $\pi/3$ .

Now we will establish several propositions which will help streamline the proofs of Theorems 7 and 8 below (which in turn will imply Theorem 4). First we will need the following elementary fact which, for future reference, we state for arbitrary dimension  $d \geq 2$ .

**Proposition 2.** *If  $d \geq 2$  and if the angle between two non-zero vectors  $\vec{w}_1, \vec{w}_2 \in \mathbb{R}^d$  is less than  $\pi/3$ , then*

$$(3.7) \quad |\vec{w}_1 - \vec{w}_2| < \max\{|\vec{w}_1|, |\vec{w}_2|\}.$$

*Furthermore, this inequality also holds if the angle between  $\vec{w}_1, \vec{w}_2 \in \mathbb{R}^d$  is equal to  $\pi/3$ , as long as  $|\vec{w}_1| \neq |\vec{w}_2|$ .*

*Proof.* For the first part of the proposition, suppose without loss of generality that  $|\vec{w}_1| \leq |\vec{w}_2|$ . Then the vectors  $\vec{w}_1/|\vec{w}_2|$  and  $\vec{w}_2/|\vec{w}_2|$  lie on the unit sphere and are separated by an angle less than  $\pi/3$ . Therefore

$$(3.8) \quad \left| \frac{\vec{w}_1}{|\vec{w}_2|} - \frac{\vec{w}_2}{|\vec{w}_2|} \right| < 1,$$

and the result follows. Furthermore, under the assumptions of the second part of the proposition, we draw the same conclusion.  $\square$

Next we will prove several propositions which place various restrictions on the integer  $K = \mathcal{G}_{\mathcal{D}}(M)$  defined at the start of this section; recall (V1)–(V3).

**Proposition 3.** *If for some integer  $1 \leq i \leq K$ , we have that  $u_i \in (-1/2, 1/2)$  and  $\vec{v}_i \in \mathbb{R}_{>0}\mathcal{S}$ , then we must have that  $i = K$ .*

*Proof.* Suppose first that  $u_i \in [0, 1/2)$ . Then for any  $0 < t < 1 - u_i$ , we have that  $(u_i, \vec{v}_i) \in \mathcal{Q}_{\mathcal{D}}(M, t)$  and thus  $|\vec{v}_i| \geq F_{\mathcal{D}}(M, t)$ . By the symmetry of  $\mathcal{S}$  we have  $(-u_i, -\vec{v}_i) \in \mathcal{Q}_{\mathcal{D}}(M)$ . Thus for any  $u_i < t < 1$ , we have that  $(-u_i, -\vec{v}_i) \in \mathcal{Q}_{\mathcal{D}}(M, t)$  and thus  $|\vec{v}_i| \geq F_{\mathcal{D}}(M, t)$ . Since  $u_i < 1/2$ , we conclude that  $|\vec{v}_i| \geq F_{\mathcal{D}}(M, t)$  for all  $0 < t < 1$ , which proves  $i = K$ . The case  $u_i \in (-1/2, 0]$  follows from the same argument.  $\square$

**Proposition 4.** *If for some  $i$  and  $j$  with  $1 \leq i, j \leq K$ , we have that  $u_i \in (-1/2, 0]$  and  $u_j \in [0, 1/2)$ , then  $i = K$  or  $j = K$ .*

*Proof.* Under the hypotheses of the proposition, we have  $|\vec{v}_i| \geq F_{\mathcal{D}}(M, t)$  for  $-u_i < t < 1$  and  $|\vec{v}_j| \geq F_{\mathcal{D}}(M, t)$  for  $0 < t < 1 - u_j$ . This covers all possible values of  $t \in (0, 1)$ , and shows that  $i$  or  $j$  must equal  $K$ .  $\square$

**Proposition 5.** *If  $1 \leq i, j \leq K$  and  $u_i = u_j$ , then  $i = j$ .*

*Proof.* Suppose by way of contradiction that  $i \neq j$ , and without loss of generality that  $i < j$ . Then for any  $t \in (0, 1)$  satisfying  $-t < u_i = u_j \leq 1 - t$ , we would have by (V1) that

$$(3.9) \quad F_{\mathcal{D}}(M, t) \leq |\vec{v}_i| < |\vec{v}_j|.$$

However this contradicts condition (V3), so we must have that  $i = j$ .  $\square$

**Proposition 6.** *Let  $1 \leq i \leq K$  and  $(u, \vec{v}) \in \mathcal{Q}_{\mathcal{D}}(M)$ . If*

$$(3.10) \quad -1 < u_i \leq u \leq 0 \quad \text{or} \quad 0 \leq u \leq u_i < 1$$

*then  $|\vec{v}_i| \leq |\vec{v}|$ .*

*Proof.* Suppose that  $0 \leq u \leq u_i < 1$ ; the other case follows by symmetry. By (V3) there exist  $t_i \in (0, 1 - u_i)$  such that  $F_{\mathcal{D}}(M, t_i) = |\vec{v}_i|$ . Furthermore  $F_{\mathcal{D}}(M, t) \leq |\vec{v}|$  for all  $t \in (0, 1 - u)$ . Thus, taking  $t = t_i \in (0, 1 - u)$ , we have  $|\vec{v}_i| \leq |\vec{v}|$ .  $\square$

**Proposition 7.** *Let  $1 \leq i, j \leq K$ . If*

$$(3.11) \quad -1 < u_i < u_j \leq 0 \quad \text{or} \quad 0 \leq u_j < u_i < 1$$

*then  $|\vec{v}_i| < |\vec{v}_j|$  and  $i < j$ .*

*Proof.* In view of (V1), this is a direct consequence of Proposition 6.  $\square$

**Proposition 8.** *Let  $\tau \geq \pi$ ,  $1 \leq j \leq K$  and  $(u, \vec{v}) \in \mathcal{Q}_{\mathcal{D}}(M)$  such that  $\vec{v}_j \neq \vec{v}$  and  $0 < |\vec{v}| \leq |\vec{v}_j|$ . Suppose the angle between the vectors  $\vec{v}_j$  and  $\vec{v}$  is less than  $\pi/3$ . If*

$$(3.12) \quad -1 < u \leq u_j \leq -1/2 \quad \text{or} \quad 1/2 \leq u_j \leq u < 1,$$

*then  $j = K$ .*

*Proof.* We consider the case  $1/2 \leq u_j \leq u < 1$ ; the proof for the alternative follows from the same argument by symmetry. By the assumption on the angle between the vectors  $\vec{v}_j$  and  $\vec{v}$ , we have by Proposition 2 that  $|\vec{v}_j - \vec{v}| < |\vec{v}_j|$ . Since  $\vec{v}_j \neq \vec{v}$  and  $\tau \geq \pi$ , we have that at least one of  $\vec{v}_j - \vec{v}$ ,  $\vec{v} - \vec{v}_j$  is in  $\mathbb{R}_{>0}\mathcal{D}$ .

First suppose that  $\vec{v} - \vec{v}_j \in \mathbb{R}_{>0}\mathcal{D}$ . Then, since

$$0 \leq u - u_j < 1/2 \leq u_j < 1,$$

we have  $(u - u_j, \vec{v} - \vec{v}_j) \in \mathcal{Q}_{\mathcal{D}}(M)$  and by Proposition 6 that  $|\vec{v} - \vec{v}_j| \geq |\vec{v}_j|$ . This is a contradiction, so we conclude that  $\vec{v} - \vec{v}_j \notin \mathbb{R}_{>0}\mathcal{D}$ .

The only other possibility is that  $\vec{v}_j - \vec{v} \in \mathbb{R}_{>0}\mathcal{D}$ . In this case,  $(u_j - u, \vec{v}_j - \vec{v}) \in \mathcal{Q}_{\mathcal{D}}(M)$ ,

$$(3.13) \quad u_j - u \leq 0 \quad \text{and} \quad u - u_j < 1 - u_j.$$

It follows from this that, for  $u - u_j < t < 1$ , we have  $F_{\mathcal{D}}(M, t) \leq |\vec{v} - \vec{v}_j| < |\vec{v}_j|$  and for  $0 < t < 1 - u_j$  (which in particular holds for all  $t$  with  $0 < t \leq u - u_j$ ), we have  $F_{\mathcal{D}}(M, t) \leq |\vec{v}_j|$ . Therefore  $j = K$ .  $\square$

**Proposition 9.** *Let  $\tau \geq \pi$  and  $1 \leq i, j \leq K$ . Suppose the angle between the vectors  $\vec{v}_i$  and  $\vec{v}_j$  is less than  $\pi/3$ . If*

$$(3.14) \quad -1 < u_i < u_j \leq -1/2 \quad \text{or} \quad 1/2 \leq u_j < u_i < 1,$$

then  $j = K$ .

*Proof.* This is a direct corollary of Proposition 8 (take  $u = u_i$ ).  $\square$

**Proposition 10.** *Let  $\tau \geq \pi$  and  $1 \leq i, j \leq K$ . Suppose the angle between the vectors  $\vec{v}_i$  and  $\vec{v}_j$  is less than  $\pi/3$ . If*

$$(3.15) \quad -1/2 < u_i < u_j \leq 0 \quad \text{or} \quad 0 \leq u_j < u_i < 1/2,$$

then  $\vec{v}_j - \vec{v}_i \notin \mathbb{R}_{>0}\mathcal{D}$  and

$$(3.16) \quad |\vec{v}_i| \leq |\vec{v}_i - \vec{v}_j| < |\vec{v}_j|.$$

*Proof.* We assume  $0 \leq u_j < u_i < 1/2$ ; the other case follows by symmetry. It follows from Proposition 7 that  $i < j$  and  $|\vec{v}_i| < |\vec{v}_j|$ , and it follows from Proposition 2 that  $|\vec{v}_i - \vec{v}_j| < |\vec{v}_j|$ .

Suppose, contrary to what we are trying to prove, that  $\vec{v}_j - \vec{v}_i \in \mathbb{R}_{>0}\mathcal{D}$ . Then, since  $0 < u_i - u_j < 1/2$ , we have  $(u_j - u_i, \vec{v}_j - \vec{v}_i) \in \mathcal{Q}_{\mathcal{D}}(M)$  and hence for  $u_i - u_j < t < 1$  we have  $F_{\mathcal{D}}(M, t) \leq |\vec{v}_i - \vec{v}_j| < |\vec{v}_j|$ . Furthermore, for  $0 < t < 1 - u_i$  we have  $F_{\mathcal{D}}(M, t) \leq |\vec{v}_i| < |\vec{v}_j|$ . Now  $u_i - u_j < 1 - u_i$ , so  $F_{\mathcal{D}}(M, t) < |\vec{v}_j|$  for all  $t \in (0, 1)$ . But by (V3) there is  $t \in (0, 1)$  such that  $F_{\mathcal{D}}(M, t) = |\vec{v}_j|$ . This is a contradiction, so we conclude that  $\vec{v}_j - \vec{v}_i \notin \mathbb{R}_{>0}\mathcal{D}$ .

It remains to show that  $|\vec{v}_i| \leq |\vec{v}_i - \vec{v}_j|$ . Since  $\vec{v}_j - \vec{v}_i \notin \mathbb{R}_{>0}\mathcal{D}$  and  $\tau \geq \pi$  we have  $\vec{v}_i - \vec{v}_j \in \mathbb{R}_{>0}\mathcal{D}$ . Then  $(u_i - u_j, \vec{v}_i - \vec{v}_j) \in \mathcal{Q}_{\mathcal{D}}(M)$  with  $0 < u_i - u_j < 1/2$ . Thus for  $0 < t < 1 - (u_i - u_j)$  we have  $F_{\mathcal{D}}(M, t) \leq |\vec{v}_i - \vec{v}_j|$ . Note that by (V3) there exists a  $t_i \in (0, 1 - u_i)$  such that  $F_{\mathcal{D}}(M, t_i) = |v_i|$ . Now  $1 - u_i \leq 1 - (u_i - u_j)$  and so  $F_{\mathcal{D}}(M, t_i) = |v_i| \leq |\vec{v}_i - \vec{v}_j|$  as needed.  $\square$

The previous proposition will be used in the proof of Theorem 7 in conjunction with the following two elementary geometric propositions.

**Proposition 11.** *Suppose that*

- we are in case (C1) or (C3) and  $\vec{w}_1, \vec{w}_2 \in \mathbb{R}_{>0}\mathcal{A}_0$ , or
- we are in case (C2) and  $\vec{w}_1, \vec{w}_2 \in \mathbb{R}_{>0}\mathcal{A}_{-1}$ , or
- we are in case (C2) and  $\vec{w}_1, \vec{w}_2 \in \mathbb{R}_{>0}\mathcal{A}_1$ .

If

$$(3.17) \quad |\vec{w}_1| \leq |\vec{w}_1 - \vec{w}_2| < |\vec{w}_2|,$$

then  $\vec{w}_2 - \vec{w}_1 \in \mathbb{R}_{>0}\mathcal{D}$ .

*Proof.* First suppose that we are in case (C2), that  $\vec{w}_1$  and  $\vec{w}_2$  point in direction  $\mathcal{A}_1$ , and that (3.17) holds. We will argue using Figure 4.

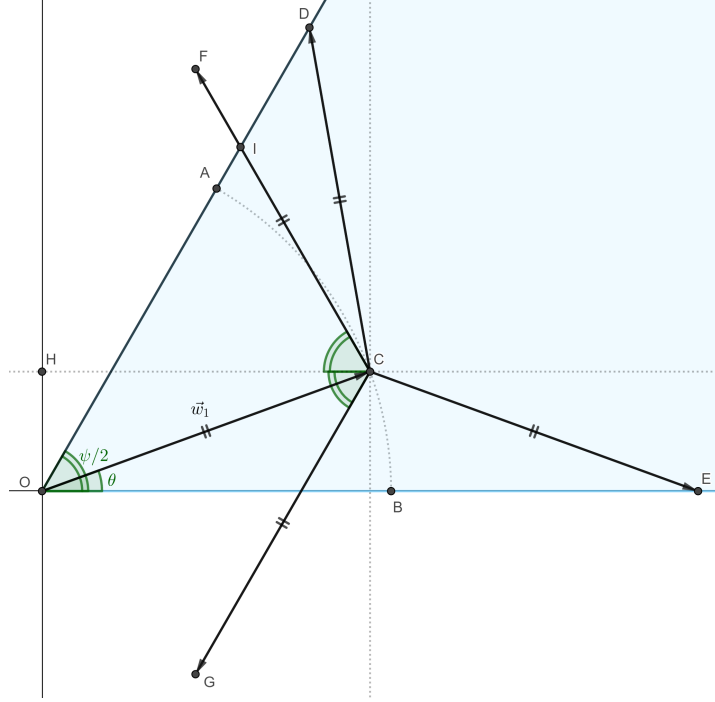


FIGURE 4. Diagram of  $\mathcal{A}_1$  in case (C2)

In the figure, angle AOB measures  $\psi/2$  and sweeps out  $\mathcal{A}_1$ , and the line through H and C is parallel to the  $x$ -axis. The vector  $\vec{w}_1$  is shown, and the angles HCF and HCG also measure  $\psi/2$ . By condition (3.17), vector  $\vec{w}_2$  has to lie outside of both the circle of radius  $|\vec{w}_1|$  centered at O, and the circle of radius  $|\vec{w}_1|$  centered at C. The circle of radius  $|\vec{w}_1|$  centered at C intersects the boundary of  $\mathbb{R}_{>0}\mathcal{A}_1$  at the three points D, O, and E, and the points F and G have also been chosen so that they lie on this circle. We will show (as indicated in the figure) that F and G lie outside of  $\mathbb{R}_{>0}\mathcal{A}_1$ . This will complete the proof in this sub-case since, if  $\vec{w}_2 - \vec{w}_1$  were not in  $\mathbb{R}_{>0}\mathcal{D}$  then  $\vec{w}_2$  would have to lie in the cone swept out by angle FCG, above the ray originating from C and passing through G, and on or below the ray originating from C and passing through F. This, together with the condition that it lies outside of the circle of radius  $|\vec{w}_1|$  centered at C, would force it to lie outside of  $\mathbb{R}_{>0}\mathcal{A}_1$ , which is contradictory to our hypotheses.

It is clear from the fact the  $\psi/2 \leq \pi/3$  that G lies below the  $x$ -axis, so it cannot be in  $\mathbb{R}_{>0}\mathcal{A}_1$  (in fact we only need  $\psi/2 < \pi/2$  for this to hold). To see why F is not in  $\mathbb{R}_{>0}\mathcal{A}_1$ , first note that angle OCH has measure  $\theta$ , from which it follows that angle CIO has measure  $\pi - \psi$  (to avoid circular reasoning, the point I is defined as the intersection

of the line through C and F with the line through O and A). On the other hand, angle CDO has measure  $\psi/2 - \theta$  and, since  $3\psi/2 \leq \pi$ , we have that

$$(3.18) \quad \psi/2 - \theta \leq \pi - \psi.$$

This implies that the point F lies on or to the left of the line through O and D, therefore it is not in  $\mathbb{R}_{>0}\mathcal{A}_1$ . The proof for case (C2) when  $\vec{w}_1$  and  $\vec{w}_2$  point in direction  $\mathcal{A}_{-1}$  follows by symmetry.

Next suppose that we are in case (C1), that  $\vec{w}_1$  and  $\vec{w}_2$  point in direction  $\mathcal{A}_0$ , and that (3.17) holds. Here the proof is similar, and we will argue using Figure 5.

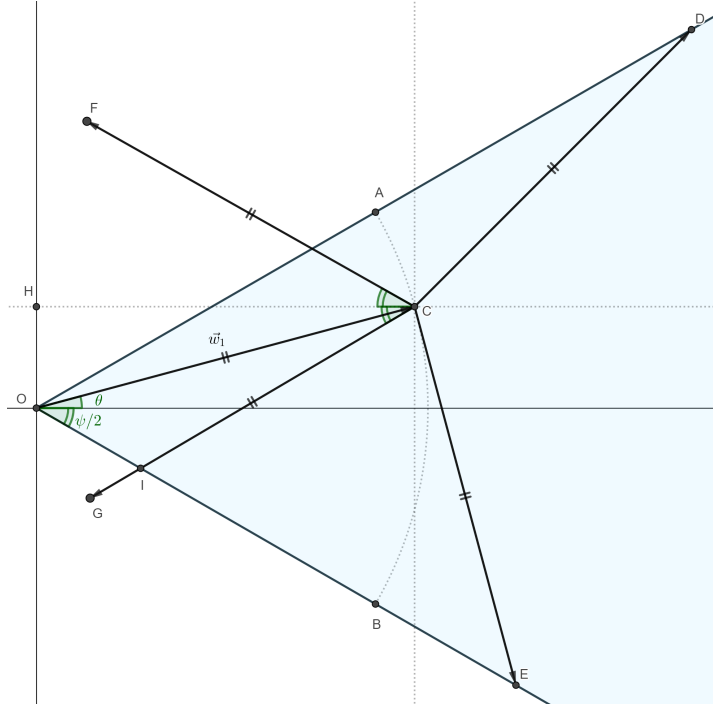


FIGURE 5. Diagram of  $\mathcal{A}_0$  in case (C1)

First assume that  $0 \leq \theta < \psi/2$ . Since  $\psi/2 \leq \pi/6$ , the argument given above implies again that F lies outside of  $\mathbb{R}_{>0}\mathcal{A}_0$ . Angle CIO has measure  $\pi - \psi$ , while angle CEO has measure  $\psi/2 + \theta$ , and since

$$(3.19) \quad \psi/2 + \theta < \pi - \psi,$$

this implies that G lies outside of  $\mathbb{R}_{>0}\mathcal{A}_0$ . This argument actually works for all  $\theta$  and  $\psi$  satisfying  $0 \leq \theta < \psi/2 \leq \pi/4$ , and a symmetrical argument applies when  $-\pi/4 \leq -\psi/2 < \theta < 0$ . The proof for case (C3), when  $\vec{w}_1$  and  $\vec{w}_2$  point in direction  $\mathcal{A}_0$ , follows from the same argument.  $\square$

**Proposition 12.** *Suppose that we are in case (C3) and that  $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$  are any vectors which all point in direction  $\mathcal{A}_{-1}$ , or which all point in direction  $\mathcal{A}_1$ . If, for each  $1 \leq i \leq n - 1$ , we have that  $\vec{w}_{i+1} - \vec{w}_i \notin \mathbb{R}_{>0}\mathcal{D}$  and that*

$$(3.20) \quad |\vec{w}_i| \leq |\vec{w}_i - \vec{w}_{i+1}| < |\vec{w}_{i+1}|,$$

then we must have that

$$(3.21) \quad n \leq 1 + \left\lfloor \frac{\sin(\tau/2 + \pi/6)}{\sin(\tau - \pi)} \right\rfloor.$$

*Proof.* Suppose  $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$  point in direction  $\mathcal{A}_1$  and consider Figure 6.

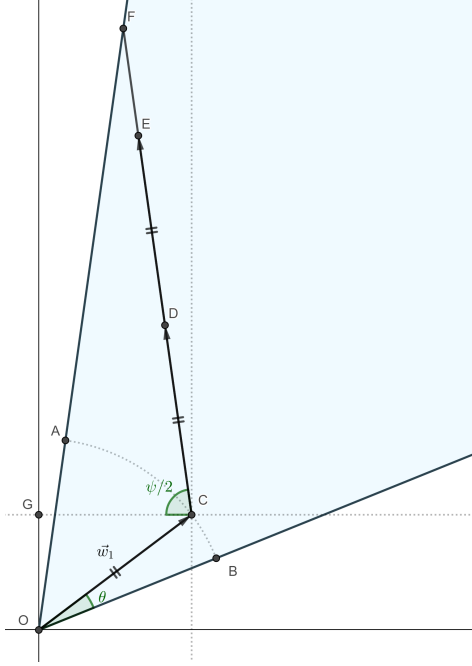


FIGURE 6. Diagram of  $\mathcal{A}_1$  in case (C3)

In the figure, angle AOB measures  $\psi/2 - \pi/6 < \pi/3$  and sweeps out  $\mathcal{A}_1$ , and the line through G and C is parallel to the  $x$ -axis. For each  $1 \leq i \leq n - 1$ , we have that  $\vec{w}_{i+1} - \vec{w}_i \notin \mathbb{R}_{>0}\mathcal{D}$  and

$$(3.22) \quad |\vec{w}_{i+1} - \vec{w}_i| \geq |\vec{w}_1|,$$

so it follows that

$$(3.23) \quad n - 1 \leq \left\lfloor \frac{|\text{CF}|}{|\vec{w}_1|} \right\rfloor.$$

Line segment CF is longest when  $\theta = 0$ , which gives the bound

$$(3.24) \quad |\text{CF}| \leq \frac{|\vec{w}_1| \sin(\psi/2 - \pi/6)}{\sin(\pi - \psi)}.$$

Substituting  $\psi = 2\pi - \tau$  gives

$$(3.25) \quad \frac{|\text{CF}|}{|\vec{w}_1|} \leq \frac{\sin(\tau/2 + \pi/6)}{\sin(\tau - \pi)}.$$

Combining this with (3.23) completes the proof of the proposition.  $\square$

Finally, to obtain the bounds reported in some of the cases of Theorem 4, we will need to gather together a few more facts.

**Proposition 13.** *Let  $1 \leq i, j \leq K$ ,  $i \neq j$ , and let  $\vec{v}_j \in \mathbb{R}_{>0}\mathcal{S}$ . If*

$$(3.26) \quad -1 < u_i \leq -u_j \leq 0 \quad \text{or} \quad 0 \leq -u_j \leq u_i < 1,$$

*then  $|\vec{v}_i| < |\vec{v}_j|$  and  $i < j$ .*

*Proof.* The vector  $-\vec{v}_j$  is in  $\mathcal{S}$ , and hence  $(-u_j, -\vec{v}_j) \in \mathcal{Q}_{\mathcal{D}}(M)$ . Proposition 6 then yields the statement.  $\square$

**Proposition 14.** *Let  $1 \leq i, j \leq K$ , and let  $\vec{v}_j \in \mathbb{R}_{>0}\mathcal{S}$ . Assume the angle between  $\vec{v}_i$  and  $-\vec{v}_j$  is less than  $\pi/3$ . If*

$$(3.27) \quad -1 < u_i \leq -u_j \leq -1/2 \quad \text{or} \quad 1/2 \leq -u_j \leq u_i < 1,$$

*then  $j = K$ .*

*Proof.* The proof is similar to that of Proposition 8, which would directly apply if we had assumed  $\vec{v}_i \in \mathbb{R}_{>0}\mathcal{S}$  rather than  $\vec{v}_j \in \mathbb{R}_{>0}\mathcal{S}$ .

The assumption on the angle implies  $i \neq j$ . We consider the case  $1/2 \leq -u_j \leq u_i < 1$ ; the proof for the alternative follows from the same argument by symmetry. By the assumption on the angle between the vectors  $\vec{v}_i$  and  $-\vec{v}_j$ , we have by Proposition 2 that  $|\vec{v}_i + \vec{v}_j| < |\vec{v}_j|$ . Since  $\vec{v}_i \neq \vec{v}_j$  and  $\tau \geq \pi$ , we have that at least one of  $\pm(\vec{v}_i + \vec{v}_j)$  is in  $\mathbb{R}_{>0}\mathcal{D}$ . Suppose  $\vec{v}_i + \vec{v}_j \in \mathbb{R}_{>0}\mathcal{D}$ .

Then, since  $-(u_j, \vec{v}_j) \in \mathcal{Q}_{\mathcal{D}}(M)$  and

$$0 \leq u_i + u_j < 1/2 \leq -u_j < 1,$$

we have  $(u_i + u_j, \vec{v}_i + \vec{v}_j) \in \mathcal{Q}_{\mathcal{D}}(M)$  and by Proposition 6 that  $|\vec{v}_i + \vec{v}_j| \geq |\vec{v}_j|$ , a contradiction. Therefore  $\vec{v}_i + \vec{v}_j \notin \mathbb{R}_{>0}\mathcal{D}$  and we must have  $-(\vec{v}_i + \vec{v}_j) \in \mathbb{R}_{>0}\mathcal{D}$ . This means  $-(u_i + u_j, \vec{v}_i + \vec{v}_j) \in \mathcal{Q}_{\mathcal{D}}(M)$  and

$$(3.28) \quad 0 \leq u_i + u_j < 1 + u_j.$$

It follows from this that, for all  $t$  with  $u_i + u_j < t < 1$ , we have  $F_{\mathcal{D}}(M, t) \leq |\vec{v}_i + \vec{v}_j| < |\vec{v}_j|$  and for  $0 < t < 1 + u_j$  (which in particular holds for all  $t$  with  $0 < t \leq u_i + u_j$ ), we have  $F_{\mathcal{D}}(M, t) \leq |\vec{v}_j|$ . Therefore  $j = K$ .  $\square$

The previous proposition allows us to deduce the following simple and useful result. Recall that  $\phi = \tau - \pi$ , so that  $\tau = \psi + 2\phi$ .

**Proposition 15.** *Let  $S_2$  denote the number of integers  $i$  with  $1 \leq i \leq K$ ,  $\vec{v}_i \in \mathbb{R}_{>0}\mathcal{S}$ , and  $u_i \in (-1, -1/2] \cup [1/2, 1)$ . Then*

$$(3.29) \quad S_2 \leq \begin{cases} 1 + 2 \left\lceil \frac{\phi}{\pi/3} \right\rceil & \text{if } \vec{v}_K \in \mathbb{R}_{>0}\mathcal{S} \text{ and } u_K \in (-1, -1/2] \cup [1/2, 1), \\ 2 \left\lceil \frac{\phi}{\pi/3} \right\rceil & \text{otherwise.} \end{cases}$$

*Proof.* The quantity  $\phi$  is the angle swept out by the part of  $\mathcal{S}$  which lies above the  $x$ -axis. The maximum number of vectors which can be placed in this region, so that the angles between any two vectors is at least  $\pi/3$ , is  $\lceil \phi/(\pi/3) \rceil$ . The upper bound in (3.29) therefore follows from combining the results of Propositions 9 and 14.  $\square$



4. EXPLICIT UPPER BOUNDS IN DIMENSION  $d = 2$ , PART 1

Throughout this section we take  $d = 2$ . In view of (2.14), the following statement directly implies all cases of Theorem 4, except the case when  $\mathcal{D} = \mathbb{S}_1^1$  (which is handled in the next section).

**Theorem 7.** *Let  $d = 2$ , and assume  $\mathcal{D} \subseteq \mathbb{S}_1^1$  is a half-open interval of arclength  $\tau > \pi$ . Then for any  $M \in \text{SL}(3, \mathbb{R})$  we have that*

$$(4.1) \quad \mathcal{G}_{\mathcal{D}}(M) \leq \begin{cases} 9 & \text{if } 5\pi/3 < \tau < 2\pi, \\ 8 & \text{if } \tau = 5\pi/3, \\ 9 & \text{if } 4\pi/3 < \tau < 5\pi/3, \\ 8 & \text{if } \tau = 4\pi/3, \\ 12 + \left\lfloor \frac{\sin(\tau/2 + \pi/6)}{\sin(\tau - \pi)} \right\rfloor & \text{if } \pi < \tau < 4\pi/3. \end{cases}$$

The remainder of this section is dedicated to the proof of this theorem. For the proof, we will apply the propositions from the previous section to each of the five cases described in (4.1). To summarize the main points of our arguments:

- (i) Let  $S_1$  denote the number of  $1 \leq i \leq K$  with  $u_i \in (-1/2, 1/2)$  and  $\vec{v}_i \in \mathbb{R}_{>0}\mathcal{S}$ . Proposition 3 guarantees that  $S_1 \leq 1$ , and that if  $S_1 = 1$  then the corresponding value of  $i$  equals  $K$ .
- (ii) Let  $S_2$  denote the number of  $1 \leq i \leq K$  with  $u_i \in (-1, -1/2] \cup [1/2, 1)$  and  $\vec{v}_i \in \mathbb{R}_{>0}\mathcal{S}$ . Proposition 15 gives an upper bound for  $S_2$ .
- (iii) Let  $A_1$  denote the number of  $1 \leq i \leq K$  with  $u_i \in (-1/2, 1/2)$  and with  $\vec{v}_i$  in direction of any asymmetric subset. Propositions 4 and 10-12 give upper bounds for  $A_1 + S_1$ .
- (iv) Let  $A_2$  denote the number of  $1 \leq i \leq K$  with  $u_i \in (-1, -1/2] \cup [1/2, 1)$  and with  $\vec{v}_i$  in direction of any asymmetric subset. Propositions 5-9 give upper bounds for  $A_2 + S_2$ .

In all cases, we have that  $\mathcal{G}_{\mathcal{D}}(M) = K = S_1 + A_1 + S_2 + A_2$ . In what follows, recall that  $\psi = 2\pi - \tau$  and  $\phi = \tau - \pi$ .

**Case (C1),  $\tau = 2\pi$ :** This is actually not one of the cases considered in Theorem 7, but we include it as a demonstration of the proof technique, and to provide an easy argument that  $\mathcal{G}_{\mathcal{D}}(M) \leq 6$ . This bound will be improved in the next section, by a slightly more complicated argument, to show that  $\mathcal{G}_{\mathcal{D}}(M) \leq 5$ .

In this case  $\psi = 0$  and  $\phi = \pi$ , so it is clear that  $A_1 = A_2 = 0$ . We claim that  $S_2 \leq 5$ . Suppose by way of contradiction that  $S_2 \geq 6$ . Since  $\tau = 2\pi$ , we may assume without loss of generality that the vectors  $(u_i, \vec{v}_i)$  have been chosen so that  $u_i \in [0, 1)$  for each  $i$ . Then we can find  $1 \leq i < j \leq K$  with  $1/2 \leq u_i, u_j < 1$ , and for which the angle between  $\vec{v}_i$  and  $\vec{v}_j$  is less than or equal to  $\pi/3$ . By the second part of Proposition 2, we then have that

$$(4.2) \quad |\vec{v}_i - \vec{v}_j| < |\vec{v}_j|.$$

However, since  $|u_i - u_j| < 1/2$  and  $\vec{v}_i - \vec{v}_j \in \mathbb{R}_{>0}\mathcal{S}$ , this implies that

$$(4.3) \quad F_{\mathcal{D}}(M, t) \leq |\vec{v}_i - \vec{v}_j| < |\vec{v}_j|,$$

for all  $t \in (0, 1)$ , which contradicts assumption (V3). Therefore  $S_2 \leq 5$ . Since  $S_1 \leq 1$ , this gives the bound  $\mathcal{G}_{\mathcal{D}}(M) \leq 6$ .

**Case (C1)**,  $5\pi/3 < \tau < 2\pi$ : In this case  $0 < \psi < \pi/3$  and  $2\pi/3 < \phi < \pi$ . There is only one asymmetric subset, namely  $\mathcal{A}_0$ .

A. Assume  $S_1 = 1$ . Then by Proposition 3 we must have that

$$(4.4) \quad u_K \in (-1/2, 1/2).$$

Proposition 4 implies that we cannot have  $i$  and  $j$  with  $1 \leq i, j < K$ ,  $u_i \in (-1/2, 0]$ , and  $u_j \in [0, 1/2)$ . Therefore, by Propositions 10 and 11, we have that  $A_1 \leq 1$ .

Since  $\lceil \frac{\phi}{\pi/3} \rceil = 3$  and (4.4) holds, Proposition 15 implies that  $S_2 \leq 6$ . We claim that, in this sub-case,  $S_2 + A_2 \leq 7$ . It is clear from Proposition 9 that, since (4.4) holds, we must have that  $A_2 \leq 2$ , and that if  $A_2 = 2$  then one element has its first component in  $(-1, -1/2]$  and the other in  $[1/2, 1)$ . In order to establish our claim we only need to consider what happens when  $S_2 \geq 5$ . If  $S_2 \geq 5$  then there must be at least three values of  $1 \leq i < K$  with  $u_i \in (-1, -1/2]$  and  $\vec{v}_i \in \mathbb{R}_{>0}\mathcal{S}$ , or at least three values with  $u_i \in [1/2, 1)$  and  $\vec{v}_i \in \mathbb{R}_{>0}\mathcal{S}$  (or possibly both, if  $S_2 = 6$ ). The argument in both cases is the same, so suppose without loss of generality that the former condition holds. Then, there are at least 2 values  $1 \leq i < j < K$  with  $u_i, u_j \in (-1, -1/2]$ ,  $\vec{v}_i, \vec{v}_j \in \mathbb{R}_{>0}\mathcal{S}$ , and with  $\vec{v}_i$  and  $\vec{v}_j$  either both above or both below the  $x$ -axis. If there were also a value of  $1 \leq k \leq K$  with  $u_k \in (-1, -1/2]$  and  $\vec{v}_k \in \mathcal{A}_0$  then, since  $\phi + \psi = \pi$ , at least one pair of the vectors  $\vec{v}_i, \vec{v}_j$ , and  $\vec{v}_k$  would be separated by an angle of less than  $\pi/3$ . This, together with Proposition 9, would contradict (4.4), therefore such a  $k$  cannot exist. This means  $A_2 \leq 1$ , and  $S_2 + A_2 \leq 6 + 1 = 7$ .

Combining the above bounds gives that  $\mathcal{G}_{\mathcal{D}}(M) \leq 1 + 1 + 6 + 1 = 9$ .

B. Assume  $S_1 = 0$ . Then by Propositions 10 and 11, we have that  $A_1 \leq 2$ . If  $A_1 = 2$  then by Proposition 4 we again have that (4.4) holds. The same argument as in the previous Case A yields  $S_2 + A_2 \leq 6 + 1 = 7$ , and therefore  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 2 + 6 + 1 = 9$ .

Assume now  $A_1 \leq 1$ . Since  $\lceil \frac{\phi}{\pi/3} \rceil = 3$  we have by Proposition 15 that  $S_2 \leq 7$ , and the case  $S_2 = 7$  can only arise if  $\vec{v}_K \in \mathbb{R}_{>0}\mathcal{S}$  and  $u_K \in (-1, -1/2] \cup [1/2, 1)$ .

(a) Assume first  $S_2 = 7$ . Then there exist vectors  $\vec{v}_{i_1}, \dots, \vec{v}_{i_5} \in \mathbb{R}_{>0}\mathcal{S}$  with indices

$$(4.5) \quad 1 \leq i_1 < i_2 < \dots < i_5 < K$$

chosen so that the vectors have the smallest possible lengths. There must be at least 2 of these vectors, say  $\vec{v}_{i_1}$  and  $\vec{v}_{i_2}$ , which both lie above or below the  $x$ -axis, and with corresponding  $u_i$  values either both in  $(-1, -1/2]$  or both in  $[1/2, 1)$ . If there were also a value of  $1 \leq k \leq K$  with  $u_k \in (-1, -1/2]$  and  $\vec{v}_k \in \mathcal{A}_0$  then, since  $\phi + \psi = \pi$ , at least one pair of the vectors  $\vec{v}_{i_1}, \vec{v}_{i_2}$ , and  $\vec{v}_k$  would be separated by an angle of less than  $\pi/3$ . Proposition 9 implies that one of the

indices has to be equal to  $K$ . Since  $i, j < K$ , we have  $k = K$ , contradicting the fact that  $\vec{v}_K \in \mathbb{R}_{>0}\mathcal{S}$ . We have therefore  $A_2 \leq 1$ , so  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 1 + 7 + 1 = 9$ .

- (b) Assume now  $S_2 = 5$  or  $6$ . In this case we claim that  $A_2 \leq 2$ . To see why this is true, suppose that  $A_2 \geq 3$ . Then, by Proposition 9, we have  $1 \leq i < j < K$  such that  $\vec{v}_i, \vec{v}_j, \vec{v}_K \in \mathcal{A}_0$  and  $u_i, u_j, u_K \notin (-1/2, 1/2)$  with  $u_i$  and  $u_j$  having opposite sign. Since  $S_2 \geq 5$ , there are at least five values of  $\ell$  with  $1 \leq \ell < K$ ,  $\vec{v}_\ell \in \mathbb{R}_{>0}\mathcal{S}$ , and  $u_\ell \notin (-1/2, 1/2)$ . By the same arguments above, at least two of these would have to lie either above or below the  $x$ -axis, and have  $u_\ell$  values both in  $(-1, -1/2]$  or both in  $[1/2, 1)$ . This, by Proposition 9, would then imply that one of  $i$  or  $j$ , or one of these  $\ell$  values, equals  $K$ , which is a contradiction. Therefore  $A_2 \leq 2$ ,  $S_2 \leq 6$  and so  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 1 + 6 + 2 = 9$ .
- (c) Finally, assume  $S_2 \leq 4$ . By Proposition 9 we have that  $A_2 \leq 3$ , so  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 1 + 4 + 3 = 8$ .

**Case (C1)**,  $\tau = 5\pi/3$ : In this case  $\psi = \pi/3$  and  $\phi = 2\pi/3$ , and there is only one asymmetric subset, namely  $\mathcal{A}_0$ . In this case the argument is similar to the previous case. The key improvement is in the application of Proposition 15, since now  $\lceil \frac{\phi}{\pi/3} \rceil = 2$ .

- A. If  $S_1 = 1$  then (4.4) holds and we conclude as in the previous Case A that  $A_1 \leq 1$ ,  $S_2 \leq 4$ . Proposition 9 and (4.4) imply that  $A_2 \leq 2$ , and so  $\mathcal{G}_{\mathcal{D}}(M) \leq 1 + 1 + 4 + 2 = 8$ .
- B. If  $S_1 = 0$  then  $A_1 \leq 2$  as in Case B above. If  $A_1 = 2$  then (4.4) holds, so  $S_2 \leq 4$  by Proposition 15 and we conclude  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 2 + 4 + 2 = 8$ .

Now assume  $A_1 \leq 1$ . Proposition 15 gives  $S_2 \leq 5$ . If  $S_2 = 5$  then, by the argument in Case B (b) above,  $A_2 \leq 2$ . We conclude  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 1 + 5 + 2 = 8$ . If  $S_2 \leq 4$  then, by the same argument as in Case B (c) above,  $A_2 \leq 3$  and  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 1 + 4 + 3 = 8$ .

**Case (C2)**,  $4\pi/3 < \tau < 5\pi/3$ : In this case  $\pi/3 < \psi < 2\pi/3$  and  $\pi/3 < \phi < 2\pi/3$ . There are two asymmetric subsets,  $\mathcal{A}_{-1}$  and  $\mathcal{A}_1$ . The argument here is similar to that given above, except that now when we use Propositions 10 and 11, they must be applied to both asymmetric subsets.

- A. Assume  $S_1 = 1$ . Then (4.4) holds, and Propositions 10 and 11 imply that  $A_1 \leq 2$  (the same argument as in Case A above, now applied to each of the two asymmetric cones). Furthermore  $\lceil \frac{\phi}{\pi/3} \rceil = 2$  and thus Proposition 15 yields  $S_2 \leq 4$ .
- (a) We claim that, if  $S_2 \geq 3$ , then  $A_2 \leq 2$ . Suppose by way of contradiction that  $S_2 \geq 3$ , that  $1 \leq i, j, k < K$ ,  $u_i, u_j, u_k \notin (-1/2, 1/2)$ , and that  $\vec{v}_i, \vec{v}_j, \vec{v}_k$  are distinct vectors in direction  $\mathcal{A}_{-1} \cup \mathcal{A}_1$ . It could not be the case that either  $u_i, u_j, u_k \in (-1, -1/2]$  or that  $u_i, u_j, u_k \in [1/2, 1)$ , otherwise we would have to have by Proposition 9 that  $i, j$ , or  $k$  equals  $K$ . Therefore two of the numbers  $u_i, u_j$ , and  $u_k$  lie in one of the intervals  $(-1, -1/2]$  or  $[1/2, 1)$ , and the other number lies in the other interval. Without loss of generality (the argument is the same in all cases) let us suppose that  $u_i, u_j \in (-1, -1/2]$  and that  $u_k \in [1/2, 1)$ . Then there cannot be any values of  $1 \leq \ell < K$  with  $u_\ell \in (-1, -1/2]$  and  $\vec{v}_\ell \in \mathbb{R}_{>0}\mathcal{S}$ . If there were then, again since  $\psi + \phi = \pi$ , at least one pair of the three vectors  $\vec{v}_i, \vec{v}_j$ , and  $\vec{v}_\ell$  would be separated by an angle of less than  $\pi/3$ ,

giving the contradiction that  $i, j$ , or  $\ell$  equals  $K$ . This means that there are at least three values of  $1 \leq \ell < K$  for which  $u_\ell \in [1/2, 1)$  and  $\vec{v}_\ell \in \mathbb{R}_{>0}\mathcal{S}$ . At least two of these vectors lie either above or below the  $x$ -axis. Since  $u_k$  is also in  $[1/2, 1)$ , this means (by the same argument just given) that either  $u_k$ , or one of these  $u_\ell$  values, is  $u_K$ . This is also a contradiction, so we conclude that  $A_1 \leq 2$ . Putting this all together, we have that  $\mathcal{G}_{\mathcal{D}}(M) \leq 1 + 2 + 4 + 2 = 9$ .

- (b) If  $S_2 = 1$  or  $2$ , then we must have that  $A_2 \leq 3$ . To see why, suppose  $A_2 \geq 4$  and choose  $1 \leq i_1 < i_2 < i_3 < i_4 < K$  so that  $u_{i_1}, \dots, u_{i_4} \notin (-1/2, 1/2)$  and  $\vec{v}_{i_1}, \dots, \vec{v}_{i_4} \notin \mathcal{S}$ . By the same argument as before, we cannot have three of these  $u_i$  values in either  $(-1, -1/2]$  or in  $[1/2, 1)$ . Therefore two of them must be in one of these intervals, and the other two must be in the other interval. There is at least one value of  $1 \leq \ell < K$  with  $u_\ell \notin (-1/2, 1/2)$  and  $\vec{v}_\ell \in \mathbb{R}_{>0}\mathcal{S}$ , and as before, this implies that one of the vectors we have just listed is  $\vec{v}_K$ ; a contradiction with (4.4). Therefore  $A_2 \leq 3$ . This shows that  $\mathcal{G}_{\mathcal{D}}(M) \leq 1 + 2 + 2 + 3 = 8$ .
- (c) Assume  $S_2 = 0$ . By Proposition 9, we have that  $A_2 \leq 5$ , and so  $\mathcal{G}_{\mathcal{D}}(M) \leq 1 + 2 + 0 + 5 = 8$ .

B. Assume  $S_1 = 0$ . Then  $A_1 \leq 4$  by Propositions 10 and 11. But  $A_1 = 4$  contradicts Proposition 4, so in fact  $A_1 \leq 3$ . If  $A_1 = 3$  then by Proposition 4 we have that (4.4) holds and, for the problem of bounding  $A_2 + S_2$ , we are in the same position as we just were in Case A. By exactly the same arguments, we therefore have that  $A_2 + S_2 \leq 6$ , and that  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 3 + 6 = 9$ .

If  $A_1 \leq 2$  then  $S_2 \leq 5$  by Proposition 15, and we again break the problem into cases.

- (a) If  $S_2 = 5$  then  $\vec{v}_K \in \mathbb{R}_{>0}\mathcal{S}$ , and we claim that  $A_2 \leq 2$ . To see why, suppose that  $1 \leq u_1 < \dots < u_4 < K$  are chosen so that  $\vec{v}_{i_1}, \dots, \vec{v}_{i_4}$  are in direction  $\mathcal{S}$ . Suppose that at least three of these vectors all have their  $u_i$  values either in  $(-1, -1/2]$  or in  $[1/2, 1)$  and without loss of generality, suppose these values lie in  $(-1, -1/2]$ . Then as before, since at least two of these vectors both lie either above or below the  $x$ -axis, there cannot be a value of  $1 \leq j < K$  with  $u_j \in (-1, -1/2]$  and with  $\vec{v}_j \notin \mathbb{R}_{>0}\mathcal{S}$ . Furthermore, by Proposition 9, there can be at most two values of  $1 \leq j < K$  with  $u_j \in [1/2, 1)$  and  $\vec{v}_j \notin \mathcal{S}$ , which gives  $A_2 \leq 2$ . The other possibility is that two of values of  $u_{i_1}, \dots, u_{i_4}$  lie in  $(-1, -1/2]$ , and two lie in  $[1/2, 1)$ . In this case, by the same arguments as above, there cannot be two values of  $1 \leq j < K$  with  $\vec{v}_j \notin \mathbb{R}_{>0}\mathcal{S}$  and with  $u_j \in (-1, -1/2]$ , and neither can there be two values with  $u_j \in [1/2, 1)$ . This again gives  $A_2 \leq 2$ . This shows that if  $A_1 \leq 2$  and  $S_2 = 5$  then  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 2 + 5 + 2 = 9$ .
- (b) If  $S_2 = 3$  or  $4$  then we claim that  $A_2 \leq 3$ . To see why this is true, suppose by way of contradiction that  $S_2 = 3$  or  $4$  and that  $\vec{v}_{i_1}, \dots, \vec{v}_{i_4} \notin \mathbb{R}_{>0}\mathcal{S}$  are distinct vectors with corresponding  $u_i$  values all in  $(-1, -1/2] \cup [1/2, 1)$ . We cannot have four of these values all in  $(-1, -1/2]$  or all in  $[1/2, 1)$ . If three of them all lie in one of these intervals, then without loss of generality let us suppose that the interval is  $(-1, -1/2]$ , the corresponding indices are  $i_2, i_3$ , and  $i_4$  and that the largest of these indices is  $i_4$ . We must then have that  $i_4 = K$ . We have that  $u_{i_1}$  lies in  $[1/2, 1)$ , so to avoid contradiction there can be at most two

values of  $1 \leq \ell < K$  with  $u_\ell \in [1/2, 1)$  and  $\vec{v}_\ell \in \mathbb{R}_{>0}\mathcal{S}$ . Then, there must be at least one value of  $\ell$  with  $u_\ell \in (-1, -1/2]$  and  $\vec{v}_\ell \in \mathbb{R}_{>0}\mathcal{S}$ . However, we then conclude by previous arguments that either this  $u_\ell$  value, or one of  $u_{i_1}$  or  $u_{i_2}$ , must equal  $u_K$ . This is a contradiction. We are left with the possibility that two of the numbers  $u_{i_1}, \dots, u_{i_4}$  lie in  $(-1, -1/2]$ , and that the other two lie in  $[1/2, 1)$ . This then implies that there is at most one value of  $1 \leq \ell \leq K$  with  $\vec{v}_\ell \in \mathbb{R}_{>0}\mathcal{S}$  and  $u_\ell \in (-1, -1/2]$ , and similarly at most one with  $\vec{v}_\ell \in \mathbb{R}_{>0}\mathcal{S}$  and  $u_\ell \in [1/2, 1)$ . This contradicts the assumption that  $S_2 = 3$  or  $4$ , so we conclude that in this case  $A_2 \leq 3$ . This gives that  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 2 + 4 + 3 = 9$ .

(c) Finally, if  $S_2 \leq 2$  we use the bound  $A_2 \leq 5$  and obtain  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 2 + 2 + 5 = 9$ .

**Case (C2)**,  $\tau = 4\pi/3$ : In this case  $\psi = 2\pi/3$  and  $\phi = \pi/3$ , and so  $\lceil \frac{\phi}{\pi/3} \rceil = 1$ . There are two asymmetric cones,  $\mathcal{A}_{-1}$  and  $\mathcal{A}_1$ .

A. Assume  $S_1 = 1$ . Then (4.4) holds, and as in the previous Case A we have  $A_1 \leq 2$  and  $S_2 \leq 2$ . If  $S_2 = 2$  then choose  $1 \leq i, j \leq K-1$  with  $i \neq j$  and  $\vec{v}_i, \vec{v}_j \in \mathbb{R}_{>0}\mathcal{S}$ . If  $u_i$  and  $u_j$  both lie in  $(-1, -1/2]$  then there can be at most one value of  $1 \leq k \leq K-1$  with  $\vec{v}_k \notin \mathbb{R}_{>0}\mathcal{S}$  and with  $u_k \in (-1, -1/2]$ , and at most two values with  $u_k \in [1/2, 1)$ , which gives  $A_2 \leq 3$ . Similarly if  $u_i$  and  $u_j$  both lie in  $[1/2, 1)$ . If  $u_i \in (-1, -1/2]$  and  $u_j \in [1/2, 1)$ , or vice-versa, then there can be at most one value of  $1 \leq k \leq K-1$  with  $\vec{v}_k \notin \mathbb{R}_{>0}\mathcal{S}$  and with  $u_k \in (-1, -1/2]$ , and also at most one value with  $u_k \in [1/2, 1)$ , so in this case  $A_2 \leq 2$ . This gives the bound  $\mathcal{G}_{\mathcal{D}}(M) \leq 1 + 2 + 2 + 3 = 8$ .

If  $S_2 = 0$  or  $1$  we use the bound  $A_2 \leq 4$ , which follows from Proposition 9 and (4.4). Thus  $\mathcal{G}_{\mathcal{D}}(M) \leq 1 + 2 + 1 + 4 = 8$ .

B. Assume  $S_1 = 0$ . Then  $A_1 \leq 3$ . If  $A_1 = 3$  then (4.4) holds and, for the problem of bounding  $A_2 + S_2$ , we may follow exactly the same arguments just used in Case A to obtain the bound  $A_2 + S_2 \leq 5$ . Therefore  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 3 + 5 = 8$ .

If  $A_1 \leq 2$  we use the bound  $S_2 \leq 3$  from Proposition 15. By arguments similar to those in the previous Case B (b) and (c): If  $S_2 = 3$  then  $A_2 \leq 3$  and  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 2 + 3 + 3 = 8$ . If  $S_2 = 1$  or  $2$  then  $A_2 \leq 4$  and  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 2 + 2 + 4 = 8$ . If  $S_2 = 0$  then  $A_2 \leq 5$  and  $\mathcal{G}_{\mathcal{D}}(M) \leq 0 + 2 + 0 + 5 = 7$ .

**Case (C3)**,  $\pi < \tau < 4\pi/3$ : In this case  $2\pi/3 < \psi < \pi$  and  $0 < \phi < \pi/3$ . Now there are three asymmetric subsets,  $\mathcal{A}_{-1}$ ,  $\mathcal{A}_0$ , and  $\mathcal{A}_1$ . The argument is similar to those given previously, except that to bound the number of vectors  $\vec{v}_i \in \mathcal{A}_0$  with  $u_i \in (-1/2, 1/2)$ , we must now use Proposition 12. Also, we do not try to optimize the argument as much, since the bound in Proposition 12 is probably already sub-optimal.

A. If  $S_1 = 1$  then (4.4) holds, and Propositions 10-12 imply that

$$(4.6) \quad A_1 \leq 3 + \left\lceil \frac{\sin(\tau/2 + \pi/6)}{\sin(\tau - \pi)} \right\rceil.$$

Since  $\lceil \frac{\phi}{\pi/3} \rceil = 1$ , Proposition 15 implies that  $S_2 \leq 2$ . Also, Proposition 9 (applied to each of the three asymmetric sets) implies that  $A_2 \leq 6$ , so we have that

$$(4.7) \quad \mathcal{G}_{\mathcal{D}}(M) \leq 12 + \left\lfloor \frac{\sin(\tau/2 + \pi/6)}{\sin(\tau - \pi)} \right\rfloor.$$

B. Finally, suppose that  $S_1 = 0$ . In this case, again by Propositions 10-12, there can be at most

$$(4.8) \quad 3 + \left\lfloor \frac{\sin(\tau/2 + \pi/6)}{\sin(\tau - \pi)} \right\rfloor$$

indices  $1 \leq i \leq K$  with  $\vec{v}_i$  in an asymmetric set and with  $u_i \in [0, 1/2)$ . In addition to these vectors, if there is any other vector  $\vec{v}_j$  in an asymmetric set, with  $u_j \in (-1/2, 0)$ , then it would follow from Proposition 4 that  $j = K$ . Therefore there can be at most one such vector  $\vec{v}_j$ . The same argument applies with the intervals  $[0, 1/2)$  and  $(-1/2, 0)$  replaced by  $(-1/2, 0]$  and  $(0, 1/2)$ , respectively, and this gives the bound

$$(4.9) \quad A_1 \leq 4 + \left\lfloor \frac{\sin(\tau/2 + \pi/6)}{\sin(\tau - \pi)} \right\rfloor.$$

If equality holds in this inequality then we know that (4.4) holds, we are in the same situation just encountered in Case A, and we again use the bounds  $S_2 \leq 2$  and  $A_2 \leq 6$ , arriving at the same bound (4.7) for  $\mathcal{G}_{\mathcal{D}}(M)$ .

If there is strict inequality in (4.9) then Proposition 15 implies that  $S_2 \leq 3$ , and it also tells us that if  $S_2 = 3$  then  $\vec{v}_K$  lies in a symmetric set. Similarly, we have from Proposition 15 that  $A_2 \leq 7$ , and that if  $A_2 = 7$  then  $\vec{v}_K$  lies in an asymmetric set. Therefore we cannot have both  $S_2 = 3$  and  $A_2 = 7$ . This gives that  $S_2 + A_2 \leq 9$ , which again implies that (4.7) holds.

## 5. EXPLICIT UPPER BOUNDS IN DIMENSION $d = 2$ , PART 2

Throughout this section we set  $d = 2$  and  $\mathcal{D} = \mathbb{S}_1^1$ . In this case the upper bound of 6 obtained in the previous section falls just short of establishing the claimed five distance theorem. We will first deduce a little more information about the possible distances which can occur in this case. The goal of this section is to prove the following theorem, which will thereby complete the proof of Theorem 4 and the  $d = 2$  case of Theorem 1.

**Theorem 8.** *For any  $M \in \mathrm{SL}_3(\mathbb{R})$  we have that*

$$(5.1) \quad \mathcal{G}_{\mathbb{S}_1^1}(M) \leq 5.$$

Since we are dealing with the case  $\mathcal{D} = \mathbb{S}_1^1$  we can assume, by replacing each vector  $(u_i, \vec{v}_i)$  by its negative if necessary, that  $u_i \geq 0$  for  $1 \leq i \leq K$ . For simplicity we make this assumption for the duration of this section. It is convenient at this point to gather together some additional properties which must be satisfied by the vectors  $\vec{v}_i$ .

**Proposition 16.** *If  $\mathcal{G}_{\mathbb{S}_1^1}(M) = K$  then, for all  $1 \leq i < j \leq K - 1$ ,*

$$(5.2) \quad |\vec{v}_i - \vec{v}_j| > |\vec{v}_j|,$$

and for all  $1 \leq i < j < k \leq K - 1$ ,

$$(5.3) \quad |\vec{v}_i - \vec{v}_j - \vec{v}_k| \geq |\vec{v}_k|.$$

*Proof.* Let  $1 \leq i < j \leq K - 1$ . To prove (5.2), first note that

$$(5.4) \quad 0 < u_i - u_j < 1/2.$$

Since  $\mathcal{G}_{\mathbb{S}_1^1}(M) = K$  we must have that  $u_j \geq 1/2$ . Therefore, applying Proposition 6 to the vector  $(u_i - u_j, \vec{v}_i - \vec{v}_j) \in \mathcal{Q}_{\mathcal{D}}(\mathbb{S}_1^1)$  gives that

$$(5.5) \quad |\vec{v}_i - \vec{v}_j| \geq |\vec{v}_j|.$$

If there were equality in this inequality then, by the argument used in the proof of Proposition 3, we would have that  $j = K$ , which is a contradiction. Therefore, the strict inequality (5.2) holds.

Next, to prove (5.3), let  $1 \leq i < j < k \leq K - 1$ . We have that  $1/2 \leq u_k < u_j < u_i < 1$ , and therefore that

$$(5.6) \quad 0 < u_k + u_j - u_i < u_k.$$

Applying Proposition 6 to the vector  $(u_k + u_j - u_i, \vec{v}_k + \vec{v}_j - \vec{v}_i) \in \mathcal{Q}_{\mathcal{D}}(\mathbb{S}_1^1)$  gives that

$$(5.7) \quad |\vec{v}_k + \vec{v}_j - \vec{v}_i| \geq |\vec{v}_k|,$$

which proves the result.  $\square$

As a corollary of Proposition 16, we also deduce the following result.

**Proposition 17.** *If  $\mathcal{G}_{\mathbb{S}_1^1}(M) = K$  then, for all  $1 \leq i < j \leq K - 1$ , the angle between  $\vec{v}_i$  and  $\vec{v}_j$  must be greater than  $\pi/3$ . Also, for all  $1 \leq i < j < k \leq K - 1$ , the vector  $\vec{v}_i$  does not lie in the positive cone determined by  $\vec{v}_j$  and  $\vec{v}_k$ .*

*Proof.* For the first part of the proposition, if  $1 \leq i < j \leq K - 1$ , then the fact that the angle between  $\vec{v}_i$  and  $\vec{v}_j$  must be greater than  $\pi/3$  follows (5.2) together with Proposition 2.

For the second part, suppose by way of contradiction that  $1 \leq i < j < k \leq K - 1$ , and that the vector  $\vec{v}_i$  does lie in the positive cone determined by  $\vec{v}_j$  and  $\vec{v}_k$ . Note that  $|\vec{v}_i| < |\vec{v}_j| < |\vec{v}_k|$  and that, by the first part of the proposition, the angle between  $\vec{v}_j$  and  $\vec{v}_k$  is greater than  $2\pi/3$ . With these observations in mind, consider Figure 7. The figure is rotated so that the vector  $\vec{v}_k$  is aligned with the positive  $x$ -axis. Depending on the orientation of the vectors involved, it may also be reflected about the  $x$ -axis. The vector  $-\vec{v}_j - \vec{v}_k$  must lie in the sector indicated in red. Once  $\vec{v}_j$  is chosen, the vector  $\vec{v}_i - \vec{v}_j - \vec{v}_k$  must lie in a sector of the circle of radius  $|\vec{v}_i|$  centered at  $-\vec{v}_j - \vec{v}_k$ , as indicated by the blue region in the figure. However, no matter what choice is made for  $\vec{v}_j$ , this sector will lie completely within the open disc of radius  $|\vec{v}_k|$  centered at the origin.

Since this contradicts (5.3), we conclude that  $\vec{v}_i$  can not lie in the positive cone determined by  $\vec{v}_j$  and  $\vec{v}_k$ .  $\square$

*Proof of Theorem 8.* Suppose, contrary to the statement of the theorem, that  $\mathcal{G}_{\mathbb{S}_1^1}(M) \geq 6$ . Consider the collection of vectors  $\vec{v}_i$ , with  $1 \leq i \leq 5$ . We will say that two vectors from this collection are *consecutive* if there is no other vector from the collection which lies in their positive cone. Every vector in the collection is consecutive to two others.

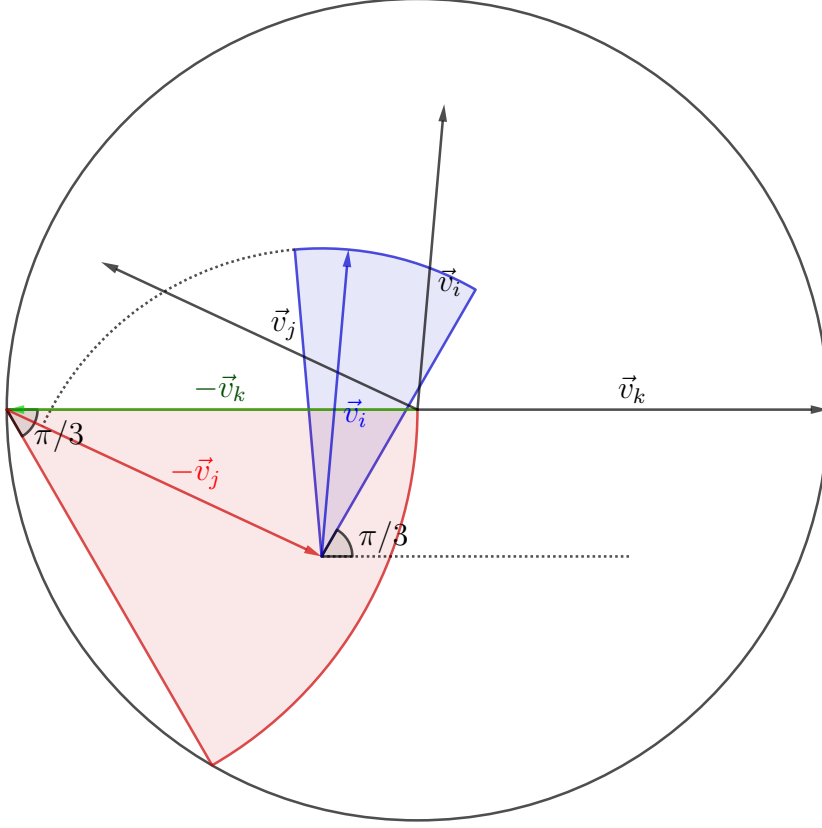


FIGURE 7. Illustration corresponding to the contradictory hypothesis that  $\vec{v}_i$  lies in the positive cone determined by  $\vec{v}_j$  and  $\vec{v}_k$ .

By Proposition 17, the angle between any pair of consecutive vectors is greater than  $\pi/3$ . Since there are five vectors in the collection, it follows that if  $i, j$ , and  $k$  are distinct indices with  $1 \leq i, j, k \leq 5$  and if  $\vec{v}_i$  is consecutive to both  $\vec{v}_j$  and  $\vec{v}_k$ , then the angle between  $\vec{v}_j$  and  $\vec{v}_k$  is less than  $\pi$ . In other words, in the situation just described, the vector  $\vec{v}_i$  lies in the positive cone determined by  $\vec{v}_j$  and  $\vec{v}_k$ . Therefore, by Proposition 17, if  $i, j$ , and  $k$  are distinct and if  $\vec{v}_i$  is consecutive to both  $\vec{v}_j$  and  $\vec{v}_k$ , then it must be the case that  $i > \min\{j, k\}$ . However, the vector  $\vec{v}_1$  is consecutive to two vectors  $\vec{v}_j$  and  $\vec{v}_k$ , with  $1 < j < k$ , and this gives a contradiction. Therefore we conclude that  $\mathcal{G}_{\mathbb{S}_1^1}(M) \leq 5$ .  $\square$

## 6. EXPLICIT UPPER BOUNDS IN DIMENSION $d > 2$

Let  $G = \mathrm{SL}(d+1, \mathbb{R})$  and  $\Gamma = \mathrm{SL}(d+1, \mathbb{Z})$ . As in the proof of Theorem 7, for each  $M \in \Gamma \backslash G$  we suppose that  $K \in \mathbb{N}$  and  $\{(u_i, \vec{v}_i)\}_{i=1}^K \subseteq M$  are chosen so that conditions (V1)-(V3) hold. In this section we will prove the following statement, which by (2.14) implies Theorem 1 in dimension  $d \geq 3$ .

**Theorem 9.** *Let  $d \geq 3$  and  $\mathcal{D} = \mathbb{S}_1^{d-1}$ . Then for any  $M \in G$  we have that*

$$(6.1) \quad \mathcal{G}_{\mathcal{D}}(M) \leq \sigma_d + 1.$$

We will use the following analogues of Propositions 3 and 9.



**Proposition 18.** *Let  $d \geq 3$  and  $\mathcal{D} = \mathbb{S}_1^{d-1}$ . If  $(u, \vec{v}) \in \mathcal{Q}_{\mathcal{D}}(M)$  with  $u \in (-1/2, 1/2)$ , then  $|\vec{v}_K| \leq |\vec{v}|$ . It follows that if, for some integer  $1 \leq i \leq K$  we have that  $u_i \in (-1/2, 1/2)$ , then we must have that  $i = K$ .*

*Proof.* Suppose first that  $u \in (-1/2, 0]$ . Then  $-u \in [0, 1/2)$  and  $(-u, -\vec{v}) \in \mathcal{Q}_{\mathcal{D}}(M)$ . Therefore, for any  $t \in (0, 1)$ , any shortest vector in  $\mathcal{Q}_{\mathcal{D}}(M, t)$  can have length at most  $|\vec{v}|$ . If  $u \in [0, 1/2)$  the argument is symmetric, so this verifies the first claim of the proposition.

For the second claim, apply the first with  $(u, \vec{v}) = (u_i, \vec{v}_i)$ . Then it follows from properties (V1)-(V3) that  $i = K$ .  $\square$

**Proposition 19.** *Let  $d \geq 3$  and  $\mathcal{D} = \mathbb{S}_1^{d-1}$ . If  $1 \leq i, j \leq K$  and*

$$(6.2) \quad -1 < u_i < u_j \leq -1/2 \quad \text{or} \quad 1/2 \leq u_j < u_i < 1,$$

*then the angle between  $\vec{v}_i$  and  $\vec{v}_j$  is greater than  $\pi/3$ .*

*Proof.* Suppose the hypotheses of the proposition are satisfied and that  $u_i, u_j \in (-1, -1/2]$  with  $u_i < u_j$ . Then conditions (V1) and (V3) imply that  $i < j$  and that  $|\vec{v}_i| < |\vec{v}_j|$ . If the angle between  $\vec{v}_i$  and  $\vec{v}_j$  were less than or equal to  $\pi/3$  then by Proposition 2 we would have that

$$(6.3) \quad |\vec{v}_i - \vec{v}_j| < |\vec{v}_j|.$$

Since  $(u_i - u_j, \vec{v}_i - \vec{v}_j) \in \mathcal{Q}_{\mathcal{D}}(M)$  satisfies  $u_i - u_j \in (-1/2, 1/2)$ , we could then deduce from Proposition 18 that

$$(6.4) \quad |\vec{v}_K| \leq |\vec{v}_i - \vec{v}_j| < |\vec{v}_j|,$$

which contradicts (V1). Therefore the angle between  $\vec{v}_i$  and  $\vec{v}_j$  is greater than  $\pi/3$ . The argument is symmetric if  $u_i, u_j \in [1/2, 1)$  with  $u_i > u_j$ .  $\square$

*Proof of Theorem 9.* By Proposition 18, there is at most one value of  $1 \leq i \leq K$  with  $u_i \in (-1/2, 1/2)$ . If such a value exists then it must be  $K$ .

For each  $1 \leq i \leq K - 1$  let

$$(6.5) \quad \vec{x}_i = \frac{\vec{v}_i}{|\vec{v}_i|} \in \mathbb{S}_1^{d-1}.$$

By Proposition 19 any points  $\vec{x}_i$  and  $\vec{x}_j$  with  $i \neq j$  are separated by an angle greater than  $\pi/3$ . Therefore the collection of spheres of radius  $1/2$  centered at the points  $\vec{x}_i$ , for  $1 \leq i \leq K - 1$ , do not overlap, and they are all tangent to the sphere of radius  $1/2$  centered at the origin. It follows that

$$(6.6) \quad K - 1 \leq \sigma_d,$$

and this completes the proof of Theorem 9.  $\square$

## 7. CONTINUITY AND LOCAL UPPER/LOWER BOUNDS

We now turn to the case of general  $\mathcal{D} \subseteq \mathbb{S}_1^{d-1}$ . We will show in this section that there are choices of  $\mathcal{D}$  for which  $F_{\mathcal{D}}(M, t)$  is unbounded. First we establish local upper bounds (i.e. upper bounds for  $M$  on compacta) that hold for general  $\mathcal{D}$ .

**Proposition 20.** *Suppose that  $\mathcal{D} \subseteq \mathbb{S}_1^{d-1}$  has non-empty interior and that  $\mathcal{C} \subset \Gamma \backslash G \times (0, 1)$  is compact. Then the following must hold:*

- (i) *There exists a number  $\kappa(\mathcal{C}) > 0$  such that  $F_{\mathcal{D}}(M, t) < \kappa(\mathcal{C})$  if  $(\Gamma M, t) \in \mathcal{C}$ , and*
- (ii)  *$F$  is continuous at every point  $(\Gamma M, t) \in \mathcal{C}$  with*

$$(7.1) \quad (\mathbb{Z}^{d+1}M \setminus \{0\}) \cap \partial((-t, 1-t) \times (0, \kappa(\mathcal{C}])\mathcal{D}) = \emptyset.$$

We emphasise that in (ii), relation (7.1) needs to be verified only for one specific but arbitrary  $\kappa(\mathcal{C}) > F_{\mathcal{D}}(M, t)$ .

*Proof.* This is analogous to the proof of [26, Prop. 2]: The existence of a constant  $\kappa(\mathcal{C})$  in (i) is guaranteed by the lattice point constructed via Minkowski's theorem in the proof of Proposition 1, where uniformity over  $\mathcal{C}$  follows from Mahler's compactness criterion. Claim (ii) follows from the uniform discreteness of  $\mathbb{Z}^{d+1}M$ , and in particular from the fact that the packing radii of the sets  $\mathbb{Z}^{d+1}M$  are uniformly bounded away from 0 whenever  $M$  is restricted to lie in a bounded subregion of  $\Gamma \backslash G$  (again, a consequence of Mahler's criterion).  $\square$

Let us now extend the above uniform upper bound to all  $t \in (0, 1)$ , with  $M$  in a compact set. Given a bounded subset  $\mathcal{A} \subset \mathbb{R}^{d+1}$  with non-empty interior, and  $M \in G$ , we define the *covering radius* (also called *inhomogeneous minimum*)

$$(7.2) \quad \rho(M, \mathcal{A}) = \inf\{\theta > 0 \mid \theta\mathcal{A} + \mathbb{Z}^{d+1}M = \mathbb{R}^{d+1}\}.$$

Because  $\mathcal{A}$  has non-empty interior,  $\rho(M, \mathcal{A}) < \infty$ . We will in the following take  $\mathcal{A}$  to be of the form  $\mathcal{A} = (0, 1) \times (0, r]\mathcal{D}$ . Then  $\mathcal{A} \subset \lambda\mathcal{A}$  for any  $\lambda > 1$ , and thus  $\theta\mathcal{A} + \mathbb{Z}^{d+1}M = \mathbb{R}^{d+1}$  for every  $\theta > \rho(M, \mathcal{A})$ . Therefore, for such  $\theta$ , the set  $\theta\mathcal{A} + \vec{x}$  intersects  $\mathbb{Z}^{d+1}M$  in at least one point, for every  $\vec{x} \in \mathbb{R}^{d+1}$ .

For a given set  $\mathcal{C} \subset \Gamma \backslash G$ , we define

$$(7.3) \quad \bar{\rho}(\mathcal{C}, \mathcal{A}) = \sup_{\Gamma M \in \mathcal{C}} \rho(M, \mathcal{A}).$$

It is well known that  $\bar{\rho}(\mathcal{C}, \mathcal{A}) < \infty$  for every compact  $\mathcal{C} \subset \Gamma \backslash G$ . For  $\theta > 0$ , set

$$(7.4) \quad D(\theta) = \begin{pmatrix} \theta^d & 0 \\ 0 & \theta^{-1}\mathbf{1}_d \end{pmatrix} \in G.$$

**Proposition 21.** *Let  $\mathcal{D} \subseteq \mathbb{S}_1^{d-1}$  with non-empty interior. Assume  $\mathcal{C} \subset \Gamma \backslash G$  is compact, and  $\theta > \bar{\rho}(\mathcal{C}, (0, 1) \times (0, 1]\mathcal{D})$ . Then*

$$(7.5) \quad F_{\mathcal{D}}(M, t) \leq \theta^{d+1}$$

for  $\Gamma M \in \mathcal{C}D(\theta)^{-1}$  and  $t \in (0, 1)$ .

*Proof.* This is almost identical to the proof of [26, Prop. 3]. The set  $\mathcal{A}_{t,\theta} = (-t, 1-t) \times (0, \theta^{d+1}]\mathcal{D} \subset \mathbb{R}^{d+1}$  has non-empty interior. The task is to show that  $\mathcal{A}_{t,\theta}$  intersects  $\mathbb{Z}^{d+1}M$  in at least one point, for every  $\Gamma M \in \mathcal{C}D(\theta)^{-1}$  and every  $t \in (0, 1)$ . Now  $\mathcal{A}_{t,\theta} \cap \mathbb{Z}^{d+1}M \neq \emptyset$  is equivalent to  $\theta^d\mathcal{A}_{t,1} \cap \mathbb{Z}^{d+1}MD(\theta) \neq \emptyset$ . The latter holds because the assumption that  $\theta > \bar{\rho}(\mathcal{C}, (0, 1) \times (0, 1]\mathcal{D})$  implies that  $\theta\mathcal{A}_{t,1} \cap \mathbb{Z}^{d+1}M' \neq \emptyset$  for every  $\Gamma M' \in \mathcal{C}$ , and  $\Gamma M' = \Gamma MD(\theta) \in \mathcal{C}$  by assumption.  $\square$

We have the following corollary.

**Proposition 22.** *Let  $\mathcal{D} \subseteq \mathbb{S}_1^{d-1}$  have non-empty interior. Assume  $\mathcal{C} \subset \Gamma \backslash G$  is compact, and  $\theta > \bar{\rho}(\mathcal{C}, (0, 1) \times (0, 1] \mathcal{D})$ . Then there is a constant  $C_{\theta, \mathcal{D}} < \infty$  such that*

$$(7.6) \quad \mathcal{G}_{\mathcal{D}}(M) \leq C_{\theta, \mathcal{D}}$$

for  $\Gamma M \in \mathcal{CD}(\theta)^{-1}$ .

*Proof.* This follows from an upper bound, uniform in  $\Gamma M \in \mathcal{CD}(\theta)^{-1}$ , on the number of lattice points in a given bounded set (via Mahler's criterion); see the proof of [26, Prop. 4].  $\square$

The key point is now that, in addition to the above upper bounds, we can find open sets  $\mathcal{U} \subset \Gamma \backslash G$ , on which  $\mathcal{G}_{\mathcal{D}, \mathcal{N}}(M)$  can exceed any given value. This requires however that  $\mathcal{D}$  is contained in a hemisphere. Let  $\mathcal{H} \subset \mathbb{S}_1^{d-1}$  be an (arbitrary) open hemisphere, and  $\mathcal{D} \subset \mathbb{S}_1^{d-1}$  with non-empty interior so that  $\mathcal{D}^{\text{cl}} \subset \mathcal{H}$ . Choose  $d + 1$  row vectors  $\vec{e}_0, \vec{e}_1, \dots, \vec{e}_d \in \mathbb{S}_1^{d-1}$  with the properties

- (i)  $\vec{e}_1 \in \mathcal{D}^\circ$  and  $\vec{e}_0 \in \mathcal{H} \setminus \mathcal{D}$  such that  $\vec{e}_0 \cdot \vec{e}_1 > 0$ ;
- (ii)  $\vec{e}_2, \dots, \vec{e}_d \notin \mathcal{D}$  such that  $(\vec{e}_1, \dots, \vec{e}_d)$  forms an orthonormal basis of  $\mathbb{R}^d$  with

$$\det \begin{pmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_d \end{pmatrix} = 1.$$

Given  $\epsilon > 0$ , define the matrix

$$(7.7) \quad M_\epsilon = \begin{pmatrix} \epsilon & -\epsilon \vec{e}_0 \\ 0 & \vec{e}_1 \\ 0 & \epsilon^{-1/(d-1)} \vec{e}_2 \\ \vdots & \vdots \\ 0 & \epsilon^{-1/(d-1)} \vec{e}_d \end{pmatrix}.$$

Note that  $\det M_\epsilon = 1$ , and thus the vectors

$$\vec{b}_0 = (\epsilon, -\epsilon \vec{e}_0), \vec{b}_1 = (0, \vec{e}_1), \vec{b}_2 = (0, \epsilon^{-1/(d-1)} \vec{e}_2), \dots, \vec{b}_d = (0, \epsilon^{-1/(d-1)} \vec{e}_d)$$

form a basis of the unimodular lattice  $\mathcal{L}_\epsilon = \mathbb{Z}^{d+1} M_\epsilon$ .

Let  $L(x)$  denote the largest integer strictly less than  $x$ . That is, in terms of the floor function  $L(x) = \lfloor x \rfloor$  if  $x \notin \mathbb{Z}$  and  $L(x) = x - 1$  if  $x \in \mathbb{Z}$ .

**Proposition 23.** *Let  $\mathcal{D} \subset \mathcal{H}$  and  $M_\epsilon$  be as above. Then there exist  $\lambda \in (0, 1)$  and  $\epsilon_0 > 0$  such that, for any  $\epsilon \in (0, \epsilon_0]$  and  $t \in (\lambda, 1)$ ,*

- (i)  $F_{\mathcal{D}}(M_\epsilon, t) = |\vec{e}_1 - \epsilon L(\epsilon^{-1}(1-t)) \vec{e}_0|$  and
- (ii)  $F_{\mathcal{D}}$  is continuous at  $(\Gamma M_\epsilon, t) \in \Gamma \backslash G \times (0, 1)$  if  $t \notin 1 + \epsilon \mathbb{Z}$ .

*Proof.* (i) Fix  $s_0 \in (0, \vec{e}_0 \cdot \vec{e}_1)$ , and denote by  $s_- \leq 0 \leq s_+$  the infimum and supremum of all  $s$  such that  $\vec{e}_1 - s \vec{e}_0 \in \mathbb{R}_{>0} \mathcal{D}$ ; since  $\vec{e}_1 \in \mathcal{D}^\circ$  we have  $s_- < 0 < s_+$ . Note also that

$$(7.8) \quad \frac{d}{ds} |\vec{e}_1 - s \vec{e}_0| = \frac{s - \vec{e}_0 \cdot \vec{e}_1}{|\vec{e}_1 - s \vec{e}_0|} < 0$$

if  $s \leq s_0$ .

We are interested in the lattice points from  $\mathbb{Z}^{d+1}M_\epsilon$  contributing to (2.4), i.e.,

$$(7.9) \quad \mathcal{Q}_{\mathcal{D}}(M_\epsilon, t) = \{(u, \vec{v}) \in \mathbb{Z}^{d+1}M_\epsilon \mid -t < u < 1-t, \vec{v} \in \mathbb{R}_{>0}\mathcal{D}\},$$

and in particular those with minimal  $|\vec{v}|$ . We begin with those elements of the form

$$(7.10) \quad (u, \vec{v}) = m_0\vec{b}_0 + m_1\vec{b}_1 = (\epsilon m_0, m_1\vec{e}_1 - \epsilon m_0\vec{e}_0), \quad m_0 \in \mathbb{Z}, m_1 \in \mathbb{Z}_{\geq 1}.$$

By construction, we have  $m_1\vec{e}_1 - \epsilon m_0\vec{e}_0 \in \mathbb{R}_{>0}\mathcal{D}^\circ$  if and only if  $s_- < \epsilon m_0/m_1 < s_+$ ; and  $m_1\vec{e}_1 - \epsilon m_0\vec{e}_0 \in \mathbb{R}_{>0}\mathcal{D}^{\text{cl}}$  if and only if  $s_- \leq \epsilon m_0/m_1 \leq s_+$ . In view of (7.8), the length  $|m_1\vec{e}_1 - \epsilon m_0\vec{e}_0|$  is strictly decreasing (as a function of  $m_0/m_1$ ) if  $\epsilon m_0/m_1 \leq s_0$ . Let us restrict our attention to those  $t$  for which  $1-t < \min\{s_0, s_+\}$ . Then the condition  $\epsilon m_0 < 1-t$  implies  $\epsilon m_0/m_1 < s_+$  and  $\epsilon m_0/m_1 \leq s_0$  for all  $m_1 \geq 1$ . Hence the smallest value of  $|m_1\vec{e}_1 - \epsilon m_0\vec{e}_0|$  is obtained for  $m_0 = L(\epsilon^{-1}(1-t))$  and  $m_1 = 1$ . In summary, we have shown thus far that for  $t > 1 - \min\{s_0, s_+\}$ ,

$$(7.11) \quad F_{\mathcal{D}}(M_\epsilon, t) \leq |\vec{e}_1 - \epsilon L(\epsilon^{-1}(1-t))\vec{e}_0|.$$

Note that

$$(7.12) \quad |\vec{e}_1 - s\vec{e}_0| = \sqrt{1 - 2s\vec{e}_0 \cdot \vec{e}_1 + s^2} < 1$$

for  $0 < s \leq s_0$  and  $\vec{e}_0 \cdot \vec{e}_1 > s_0$ , and hence  $F_{\mathcal{D}}(M_\epsilon, t) < 1$ .

What we need to establish now is that all other elements in  $\mathcal{Q}_{\mathcal{D}}(M_\epsilon, t)$  that are not of the form (7.10) have larger  $|\vec{v}|$ . Consider first the set of vectors

$$(7.13) \quad (u, \vec{v}) = m_0\vec{b}_0 + m_1\vec{b}_1 = (\epsilon m_0, m_1\vec{e}_1 - \epsilon m_0\vec{e}_0), \quad m_0 \in \mathbb{Z}, m_1 \in \mathbb{Z}_{\leq 0},$$

Since  $\vec{e}_0 \in \mathcal{H} \setminus \mathcal{D}$  and  $\vec{e}_1 \in \mathcal{D}^\circ$  we have  $m_1\vec{e}_1 - \epsilon m_0\vec{e}_0 \notin \mathbb{R}_{>0}\mathcal{D}$  for all  $m_1 \leq 0$ , and hence the corresponding vectors are not in  $\mathcal{Q}_{\mathcal{D}}(M_\epsilon, t)$ .

Next consider the remaining cases

$$(7.14) \quad (u, \vec{v}) = m_0\vec{b}_0 + \cdots + m_d\vec{b}_d, \quad m_0, m_1 \in \mathbb{Z}, (m_2, \dots, m_d) \in \mathbb{Z}^{d-1} \setminus \{0\}.$$

We need to understand whether any of these vectors can lie in  $\mathcal{Q}_{\mathcal{D}}(M_\epsilon, t)$  and satisfy  $|\vec{v}| < 1$ . We make the following observations:

- (a) The domain  $(-1, 1) \times (0, 1)\mathcal{D}$  is bounded.
- (b) The vector  $m_0\vec{b}_0$  has bounded length  $|m_0\vec{b}_0| = \sqrt{2}|\epsilon m_0| < \sqrt{2}$  (since  $-1 < \epsilon m_0 < 1$ ).
- (c) The vector  $m_1\vec{b}_1 + \cdots + m_d\vec{b}_d$  has length at least  $\epsilon^{-1/(d-1)}$  since  $(m_2, \dots, m_d) \in \mathbb{Z}^{d-1} \setminus \{0\}$ .

Therefore, as long as  $\epsilon$  is sufficiently small,  $(-1, 1) \times (0, 1)\mathcal{D}$  will not contain any vectors of the form (7.14). Since we have now considered all vectors for our restricted values of  $t$ , this establishes claim (i) with  $\lambda = 1 - \min\{s_0, s_+\}$ .

(ii) We need to establish that, for  $\lambda < t < 1$ ,  $t \notin 1 + \epsilon\mathbb{Z}$ , the function  $F_{\mathcal{D}}$  is continuous at  $(\Gamma M_\epsilon, t)$ . By Proposition 20 (with  $\mathcal{C} = \{(\Gamma M_\epsilon, t)\}$ ), it is sufficient to check that

$$(7.15) \quad (\mathbb{Z}^{d+1}M_\epsilon \setminus \{0\}) \cap \partial((-t, 1-t) \times (0, \kappa]\mathcal{D}) = \emptyset,$$

for any fixed choice of  $\kappa > F_{\mathcal{D}}(\Gamma M_\epsilon, t)$ . We fix  $\kappa$  so that  $F_{\mathcal{D}}(\Gamma M_\epsilon, t) < \kappa < 1$  and

$$(7.16) \quad \kappa \notin \{|m_1\vec{e}_1 - \epsilon m_0\vec{e}_0| \mid (m_0, m_1) \in \mathbb{Z}_{\geq 1}^2\}.$$

Since vectors with  $(m_2, \dots, m_d) \in \mathbb{Z}^{d-1} \setminus \{0\}$  are outside the bounded domain  $((-t, 1-t) \times (0, \kappa] \mathcal{D})^{\text{cl}}$  for  $\epsilon$  sufficiently small (by the argument in the proof of fact (i) above), what we are aiming to show is equivalent to

$$(7.17) \quad \{(\epsilon m_0, m_1 \vec{e}_1 - \epsilon m_0 \vec{e}_0) \mid (m_0, m_1) \in (\mathbb{Z}^2 \setminus \{0\}) \cap \partial((-t, 1-t) \times (0, \kappa] \mathcal{D})\} = \emptyset.$$

By the same argument as above, for  $m_1 \leq 0$  we have  $m_1 \vec{e}_1 - \epsilon m_0 \vec{e}_0 \notin \mathbb{R}_{\geq 0} \mathcal{D}^{\text{cl}}$  (unless  $(m_0, m_1) = 0$ , which is excluded). So we can assume  $m_1 \geq 1$  from now on. Then, for  $m_0 \leq 0$ , we have by the monotonicity (7.8) that  $|m_1 \vec{e}_1 - \epsilon m_0 \vec{e}_0| \geq m_1 |\vec{e}_1| \geq |\vec{e}_1| = 1$ . Therefore  $m_1 \vec{e}_1 - \epsilon m_0 \vec{e}_0 \notin [0, \kappa] \mathcal{D}^{\text{cl}}$ . What remains is to check that

$$(7.18) \quad \{(\epsilon m_0, m_1 \vec{e}_1 - \epsilon m_0 \vec{e}_0) \mid (m_0, m_1) \in \mathbb{Z}_{\geq 1}^2 \cap \partial((-t, 1-t) \times (0, \kappa] \mathcal{D})\} = \emptyset.$$

The truth of relation (7.18) is equivalent to the truth of both

$$(7.19) \quad \{(\epsilon m_0, m_1 \vec{e}_1 - \epsilon m_0 \vec{e}_0) \mid (m_0, m_1) \in \mathbb{Z}_{\geq 1}^2 \cap (\{-t, 1-t\} \times [0, \kappa] \mathcal{D}^{\text{cl}})\} = \emptyset$$

and

$$(7.20) \quad \{(\epsilon m_0, m_1 \vec{e}_1 - \epsilon m_0 \vec{e}_0) \mid (m_0, m_1) \in \mathbb{Z}_{\geq 1}^2 \cap ([-t, 1-t] \times \partial((0, \kappa] \mathcal{D}))\} = \emptyset.$$

The first relation (7.19) is automatically satisfied since (a) by assumption  $\epsilon m_0 \neq 1-t$  for any integer  $m_0$ , and (b)  $\epsilon m_0 \neq -t$  because  $t > 0$  and  $m_0 > 0$ .

As to the second relation (7.20), the statement  $m_1 \vec{e}_1 - \epsilon m_0 \vec{e}_0 \in \partial((0, \kappa] \mathcal{D})$  implies  $s_- = \epsilon m_0 / m_1$  or  $s_+ = \epsilon m_0 / m_1$  or  $|m_1 \vec{e}_1 - \epsilon m_0 \vec{e}_0| = \kappa$ . The first option is not possible since  $s_- < 0$ , and third is excluded by assumption. If the second option holds, then, since  $\epsilon m_0 \in [-t, 1-t]$ , we have  $m_1 s_+ \leq 1-t$  and thus  $s_+ \leq (1-t)/m_1$ . But this contradicts our assumption  $1-t < s_+$ . Hence (7.20) holds and the proof is complete.  $\square$

The following lower bound on the number of distinct values of  $t \mapsto F_{\mathcal{D}}(M, t)$  is a corollary of the previous proposition.

**Proposition 24.** *Let  $\mathcal{H} \subset \mathbb{S}_1^{d-1}$  be an arbitrary open hemisphere, and suppose that  $\mathcal{D} \subset \mathbb{S}_1^{d-1}$  has non-empty interior and satisfies  $\mathcal{D}^{\text{cl}} \subset \mathcal{H}$ . Then there is a constant  $c_{\mathcal{D}} > 0$  such that, for any  $\epsilon > 0$ , there exists an open subset  $\mathcal{U}_{\epsilon} \subset \Gamma \backslash G$  and integer  $N_{\epsilon}$  with the property that, for all  $\Gamma M \in \mathcal{U}_{\epsilon}$  and  $N \geq N_{\epsilon}$ ,*

$$(7.21) \quad \mathcal{G}_{\mathcal{D}, N}(M) \geq c_{\mathcal{D}} \epsilon^{-1}.$$

*Proof.* Proposition 23 (i) shows that  $\mathcal{G}_{\mathcal{D}}(M_{\epsilon}) \geq c_{\mathcal{D}} \epsilon^{-1}$  for a sufficiently small  $c_{\mathcal{D}} > 0$ . Denote the distinct elements of the set  $\{F_{\mathcal{D}}(M_{\epsilon}, t) \mid t \in (\lambda, 1)\}$  by  $0 < \varphi_1 < \dots < \varphi_{\mathcal{G}_{\mathcal{D}}(M_{\epsilon})} < 1$ . Let  $\delta = \max_i(\varphi_{i+1} - \varphi_i)$ . Then by the continuity established in Proposition 23 (ii), there exists a neighbourhood  $\mathcal{U}_{\epsilon} \subset \Gamma \backslash G$  of  $\Gamma M_{\epsilon}$  and an  $\eta_{\epsilon} > 0$  such that for  $t \notin 1 + \epsilon \mathbb{Z}$ , we have that

$$(7.22) \quad |F_{\mathcal{D}}(M, t') - F_{\mathcal{D}}(M_{\epsilon}, t)| < \delta/2,$$

whenever  $\Gamma M \in \mathcal{U}_{\epsilon}$  and  $t' \in (t - \eta_{\epsilon}, t + \eta_{\epsilon})$ . For  $N \geq \eta_{\epsilon}^{-1}$  we can find an integer  $n$  so that  $\frac{n}{N} \in (t - \eta_{\epsilon}, t + \eta_{\epsilon})$ . This implies  $\mathcal{G}_{\mathcal{D}, N}(M) \geq \mathcal{G}_{\mathcal{D}}(M_{\epsilon}) \geq c_{\mathcal{D}} \epsilon^{-1}$  for all  $\Gamma M \in \mathcal{U}_{\epsilon}$  and  $N \geq \eta_{\epsilon}^{-1}$ .  $\square$

*Proof of Theorem 5.* The proof of Theorem 5 now follows from the same argument as the proof of [26, Theorem 1]. For  $\vec{\alpha} \in P$ , we have that the set  $\{\Gamma A_{N_i + \frac{1}{2}}(\vec{\alpha}) \mid i \in \mathbb{N}\}$  is dense in  $\Gamma \backslash G$ ; see Section 2 of [26] for details. The claim on the limit inferior then

follows from Proposition 21, since by density we have infinite returns to a given compact subset. The claim on the limit superior follows from Proposition 24, since by density the above set intersects any given open neighbourhood  $\mathcal{U}_\epsilon$ .  $\square$

*Proof of Theorem 6.* This is analogous to the proof of Theorem 2 in [26]. Note that for  $\mathbb{Z}^d M_0 = \mathbb{Z}^d M_0$  and  $\vec{\alpha} = \vec{\alpha}_0 M_0$

$$(7.23) \quad A_N(\vec{\alpha}, \mathcal{L}) = \begin{pmatrix} 1 & \vec{\alpha}_0 \\ 0 & \mathbf{1}_d \end{pmatrix} \begin{pmatrix} N^{-1} & 0 \\ 0 & N^{1/d} \mathbf{1}_d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & M_0 \end{pmatrix}.$$

By assumption  $\vec{\alpha}$  is badly approximable by  $\mathbb{Q}\mathcal{L}$ , i.e.,  $\vec{\alpha}_0$  is badly approximable by  $\mathbb{Q}^d$ . Hence, in view of Dani's correspondence, the set  $\{\Gamma A_{N_+}(\vec{\alpha}_0 M_0, \mathbb{Z}^d M_0) \mid N \in \mathbb{N}\}$  is contained in a compact subset of  $\Gamma \backslash G$ . The claim then follows from Proposition 21.  $\square$

We now return to the special case  $\mathcal{D} = \mathbb{S}_1^{d-1}$ , and provide the remaining ingredients for the proof of Theorem 2.

**Proposition 25.** *Let  $\mathcal{D} = \mathbb{S}_1^{d-1}$ , and fix a matrix  $A_{N_+}(\vec{\alpha})$  as in (2.2) with  $N_+ := N + \frac{1}{2}$ ,  $N \in \mathbb{N}$ . Then the following hold.*

- (i) *For given  $n = 1, \dots, N$ , the function  $t \mapsto F_{\mathcal{D}}(A_{N_+}(\vec{\alpha}), t)$  is constant on the interval  $I_n = (N_+^{-1}(n - \frac{1}{2}), N_+^{-1}n]$ .*
- (ii)  *$F_{\mathcal{D}}$  is continuous at  $(\Gamma A_{N_+}(\vec{\alpha}), t) \in \Gamma \backslash G \times (0, 1)$  if  $t, 1 - t \notin N_+^{-1}\mathbb{Z}$ .*

*Proof.* Throughout this proof set  $\mathcal{D} = \mathbb{S}_1^{d-1}$ . Then

$$(7.24) \quad F_{\mathcal{D}}(A_{N_+}(\vec{\alpha}), t) = N_+^{1/d} \min \{ |k\vec{\alpha} + \vec{\ell}| > 0 \mid -N_+t < k < N_+(1-t), k \in \mathbb{Z}, \vec{\ell} \in \mathbb{Z}^d M_0 \}.$$

The set  $(-N_+t, N_+(1-t)) \cap \mathbb{Z}$  is independent of the choice of  $t \in I_n$ ; this proves (i). In view of Proposition 20, claim (ii) holds if

$$(7.25) \quad (\mathbb{Z}^{d+1} A_{N_+}(\vec{\alpha}) \setminus \{0\}) \cap \partial((-t, 1-t) \times (0, \kappa] \mathcal{D}) = \emptyset,$$

for a fixed choice of  $\kappa > \sup_{t \in (0,1)} F_{\mathcal{D}}(A_{N_+}(\vec{\alpha}), t)$ . The set

$$(7.26) \quad \tilde{\mathcal{M}}(M) = \{ |\vec{v}| > 0 \mid (u, \vec{v}) \in \mathbb{Z}^{d+1} M, |u| \leq 1 \}$$

is locally finite for every fixed  $M \in G$  (cf. (2.11)), and so we can choose  $\kappa \notin \tilde{\mathcal{M}}(A_{N_+}(\vec{\alpha}))$ . This means that the lattice  $\mathbb{Z}^{d+1} A_{N_+}(\vec{\alpha})$  does not intersect the set  $[-1, 1] \times \kappa \mathcal{D}$ . Furthermore, by the assumption  $t, 1 - t \notin N_+^{-1}\mathbb{Z}$ , we have

$$(7.27) \quad (\mathbb{Z}^{d+1} A_{N_+}(\vec{\alpha}) \setminus \{0\}) \cap (\{-t, 1-t\} \times (0, \kappa] \mathcal{D}) = \emptyset,$$

which establishes (7.25), and hence completes the proof of claim (ii).  $\square$

*Proof of Theorem 2.* In the following  $\mathcal{D} = \mathbb{S}_1^{d-1}$ . We fix  $N \in \mathbb{N}$  until the last step of the proof. By (2.6) and Proposition 25 (i) we have

$$(7.28) \quad \delta_{n,N}(\mathcal{D}) = N_+^{-1/d} F_{\mathcal{D}}(A_{N_+}(\vec{\alpha}_0, \mathcal{L}_0), t_n),$$

for any  $t_n \in I_n$ , and hence

$$(7.29) \quad g_N(\vec{\alpha}_0, \mathcal{L}_0) = |\{F_{\mathcal{D}}(A_{N_+}(\vec{\alpha}_0, \mathcal{L}_0), t_n) \mid n = 1, \dots, N\}|.$$

Choose  $\delta > 0$  sufficiently small so that the elements of the set  $\{F_{\mathcal{D}}(A_{N_+}(\vec{\alpha}_0, \mathcal{L}_0), t_n) \mid n = 1, \dots, N\}$  are separated by at least  $\delta$ . Fix any  $t_n \in I_n$  such that  $t_n, 1 - t_n \notin N_+^{-1}\mathbb{Z}$ . Proposition 25 (ii) implies that  $F_{\mathcal{D}}$  is continuous at  $(\Gamma A_{N_+}(\vec{\alpha}_0, \mathcal{L}_0), t_n)$ , for  $n = 1, \dots, N$ . That is, there exists a neighbourhood  $\mathcal{U} \subset \Gamma \backslash G$  of the point  $\Gamma A_{N_+}(\vec{\alpha}_0, \mathcal{L}_0)$  and an  $\eta > 0$  such that

$$(7.30) \quad |F_{\mathcal{D}}(M, t') - F_{\mathcal{D}}(A_{N_+}(\vec{\alpha}_0, \mathcal{L}_0), t_n)| < \delta/2,$$

whenever  $\Gamma M \in \mathcal{U}$  and  $t' \in (t_n - \eta, t_n + \eta)$ . For every integer  $\tilde{N} \geq \eta^{-1}$  and  $n = 1, \dots, N$  we can find a positive integer  $m_n \leq \tilde{N}$  so that  $\frac{m_n}{N_+} \in (t_n - \eta, t_n + \eta)$ . This implies that  $\mathcal{G}_{\mathcal{D}, \tilde{N}}(M) \geq \mathcal{G}_{\mathcal{D}}(A_{N_+}(\vec{\alpha}_0, \mathcal{L}_0)) = g_N(\vec{\alpha}_0, \mathcal{L}_0)$  for all  $\Gamma M \in \mathcal{U}$  and  $\tilde{N} \geq \eta^{-1}$ .

We can now conclude the proof as for Theorem 5. Let  $\vec{\alpha} \in P$ . The density of the orbit  $\{\Gamma A_{N_i + \frac{1}{2}}(\vec{\alpha}, \mathcal{L}) \mid i \in \mathbb{N}\}$  implies

$$(7.31) \quad \limsup_{i \rightarrow \infty} g_{N_i}(\vec{\alpha}, \mathcal{L}) \geq g_N(\vec{\alpha}_0, \mathcal{L}_0).$$

Theorem 2 now follows by taking the supremum over  $N \in \mathbb{N}$ . □

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