Appendices

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Padé approximants of random Stieltjes series
by
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Appendix A. Proof of proposition 1.2

As mentioned in the introduction, the theory of products of random matrices is a convenient means of deducing the uniqueness of the invariant measure, as well as the positivity of the real part of the complex Lyapunov exponent. For this purpose, we shall have to deal with products of 2×2 matrices with real or complex entries; so, in the following, $\mathbb{K}$ will stand for either $\mathbb{R}$ or $\mathbb{C}$. Set

$$\overline{\mathbb{K}} := \mathbb{K} \cup \{\infty\}\,.$$

We define an equivalence relation in the set of nonzero vectors in $\mathbb{K}^2$ via

$$\begin{pmatrix} u \\ v \end{pmatrix} \sim \begin{pmatrix} u' \\ v' \end{pmatrix} \quad \text{if } \exists\, w \in \mathbb{K}\backslash\{0\} \text{ such that } \begin{pmatrix} u \\ v \end{pmatrix} = w \begin{pmatrix} u' \\ v' \end{pmatrix}\,.$$

The set of the equivalence classes is called the *projective space* $P(\mathbb{K}^2)$. Let

$$\left[\begin{pmatrix} u \\ v \end{pmatrix}\right] \in P\left(\mathbb{K}^2\right)\,.$$

We shall identify this equivalence class with

$$z = \mathcal{P}\left(\left[\begin{pmatrix} u \\ v \end{pmatrix}\right]\right) := \begin{cases} u/v & \text{if } v \neq 0 \\ \infty & \text{otherwise} \end{cases} \in \overline{\mathbb{K}}\,.$$

The results of Furstenberg & Kesten (1960) and Furstenberg (1963) concern the typical asymptotic behaviour of the product of independent, identically distributed random elements of some group acting on a compact topological space. In our particular context, the relevant group is the subgroup of $\mathrm{GL}\left(2, \mathbb{K}^2\right)$ consisting of 2×2 matrices with determinant ± 1, and the topological space is $P\left(\mathbb{K}^2\right)$. The invertible matrices

$$\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \tag{A 1}$$

are drawn at random from a distribution which we shall denote by $\tilde{\mu}$. The action of the matrix $\mathcal{A}$ on the projective space can be expressed as

$$\mathcal{A} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} \mathcal{F}(z) \\ 1 \end{pmatrix}\,, \tag{A 2}$$

where $\mathcal{F} : \overline{\mathbb{K}} \to \overline{\mathbb{K}}$ is the linear fractional transformation defined by

$$\mathcal{F}(z) = \mathcal{P}\left(\mathcal{A}\begin{pmatrix} z \\ 1 \end{pmatrix}\right) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \mathbb{K} \text{ and } cz+d \neq 0 \\ \infty & \text{if } c \neq 0 \text{ and } z = -d/c \\ a/c & \text{if } z = \infty \text{ and } c \neq 0 \\ \infty & \text{if } c = 0 \text{ and } z = \infty \end{cases} . \tag{A 3}$$

Thus, we have an obvious connection between products

$$\mathcal{U}_n := \mathcal{A}_n \cdots \mathcal{A}_2 \mathcal{A}_1$$

of the $\mathcal{A}_n$ and the Markov chain $\hat{\mathbf{Z}}$ such that

$$\hat{Z}_0 = z, \quad \hat{Z}_{n+1} = \mathcal{F}_n\left(\hat{Z}_n\right) . \tag{A 4}$$

Indeed,

$$\hat{Z}_n = \mathcal{U}_n \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} . \tag{A 5}$$

We are interested in the rate of growth of this product as $n \to \infty$; this is quantified by the number

$$\gamma_{\tilde{\mu}} := \lim_{n \to \infty} \frac{1}{n} \mathbb{E}\left(\ln |\mathcal{U}_n|\right) , \tag{A 6}$$

where $|\cdot|$ denotes some matrix norm. This limit exists whenever

$$\mathbb{E}(\log^+ |\mathcal{A}|) < \infty . \tag{A 7}$$

In the literature on products of random matrices, the name "Lyapunov exponent" is usually reserved for $\gamma_{\tilde{\mu}}$; as we shall see, it is in fact the real part of the complex Lyapunov exponent introduced earlier.

The following specialisation of Furstenberg's theorem will be the most useful for our purpose:

Theorem 5.1. *Let the $\mathcal{A}_n$ be independent 2×2 random matrices with determinant ± 1 and a common distribution $\tilde{\mu}$. Suppose that there is no measure on $P\left(\mathbb{K}^2\right)$ that is invariant under the action of the smallest subgroup of $\mathrm{GL}(2, \mathbb{K}^2)$ generated by the support of $\tilde{\mu}$. Then*

1. *there exists a unique probability measure $\tilde{\nu}_{\tilde{\mu}}$ on $P\left(\mathbb{K}^2\right)$ that is invariant under the action of matrices in the support of $\tilde{\mu}$.*
2. *Let $z \in \mathbb{K}$. Then, for almost every realisation of the sequence $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$,*
$$\lim_{n \to \infty} \frac{1}{n} \ln \left| \mathcal{U}_n \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} \right| = \gamma_{\tilde{\mu}} .$$
3. *If condition (A 7) holds, then the number $\gamma_{\tilde{\mu}}$ is strictly positive and is given by the formula*
$$\gamma_{\tilde{\mu}} = \int_{P(\mathbb{K}^2)} \int_{\mathrm{GL}(2,\mathbb{K})} \ln \frac{\left| \mathcal{A}\begin{pmatrix} z \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} z \\ 1 \end{pmatrix} \right|} \, \tilde{\mu}\left(\mathrm{d}\mathcal{A}\right) \, \tilde{\nu}_{\tilde{\mu}}\left(\mathrm{d}\begin{pmatrix} z \\ 1 \end{pmatrix}\right) .$$

Proof. See Bougerol & Lacroix (1985), Part A, theorem 4.4 on p. 32. The last formula appears in Part A, theorem 3.6 on p. 27 of the same reference. □

Corollary 5.2. *Let $t \in \mathbb{C}\backslash\mathbb{R}_-$. Let the s_n be independent random variables in $\mathbb{R}_+$ with a common distribution μ that has at least two points of increase. Set*

$$\mathcal{A}_n = \begin{pmatrix} 0 & 1 \\ 1 & \frac{s_n}{\sqrt{t}} \end{pmatrix} .$$

Then μ induces a probability distribution $\tilde{\mu}$ on $\mathrm{GL}(2,\mathbb{C})$ such that the hypothesis of theorem 5.1 is fulfilled, and the image ν_μ of the invariant measure $\tilde{\nu}_{\tilde{\mu}}$ under the map $\mathcal{P}$ is the unique stationary distribution for the Markov chain $\hat{\mathbf{Z}}$. Furthermore, if we also assume that

$$\int_{\mathbb{R}_+} s^\varepsilon \, \mu(\mathrm{d}s) < \infty$$

for some $\varepsilon > 0$, then we have the formula

$$\gamma_{\tilde{\mu}} = -\int_{\mathbb{C}} \ln|z| \, \nu_\mu(\mathrm{d}z) .$$

Proof. Same as that of Carmona & Lacroix (1990), corollary IV.4.26. □

Next, we elaborate the consequences of these results for the asymptotic behaviour of the sequence defined by the recurrence relation

$$u_{n+1} - u_{n-1} = \frac{s_{n+1}}{\sqrt{t}} \, u_n \, , \quad n = 0\,,1\,,2\,,\ldots \tag{A 8}$$

with $u_0 \neq 0$.

Corollary 5.3. *Under the same hypothesis on μ, for every $t \in \mathbb{C}\backslash\mathbb{R}_-$, for almost every realisation of the sequence* $\mathbf{s}$*, we have*

$$\lim_{n\to\infty} \frac{\ln u_n}{n} = \Lambda_\mu(t)$$

independently of the starting values $u_0 \neq 0$ and u_{-1}.

Proof. This follows from the equality

$$\frac{\ln u_n}{n} = \frac{\ln u_0}{n} - \frac{1}{n} \sum_{j=1}^{n} \ln \hat{Z}_j \, ,$$

the definition of Λ_μ, and the uniqueness of the stationary measure for the Markov chain $\hat{\mathbf{Z}}$. □

When discussing the nature of the spectrum of the measure σ, we also need to consider the case where

$$t = -x + \mathrm{i}0\pm \, , \quad x > 0 \, .$$

In this case,

$$\mathcal{A}_n = \begin{pmatrix} 0 & 1 \\ 1 & \mp\mathrm{i}\frac{s_n}{\sqrt{x}} \end{pmatrix}$$

and so the recurrence defining the Markov chain $\hat{\mathbf{Z}}$ is

$$\hat{Z}_{n+1} = \frac{1}{\hat{Z}_n + \mp\mathrm{i}\frac{s_{n+1}}{\sqrt{x}}}\,.$$

Once again, theorem 5.1 implies the existence of a unique stationary measure μ_ν. We remark that this stationary measure is concentrated on the imaginary axis. Indeed, define $\hat{Y}_n$ via

$$\hat{Z}_n = \mathrm{i}\hat{Y}_n\,.$$

Then

$$\hat{Y}_{n+1} = \frac{-1}{\hat{Y}_n + \mp\frac{s_{n+1}}{\sqrt{x}}}\,.$$

The resulting Markov chain $\hat{\mathbf{Y}} = (\hat{Y}_0\,, \hat{Y}_1\,, \dots)$ corresponds to the product of the real (Schrödinger) matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & \frac{s_n}{\sqrt{x}} \end{pmatrix}\,.$$

By applying theorem 5.1 with $\mathbb{K} = \mathbb{R}$, we deduce that $\hat{\mathbf{Y}}$ has a unique stationary measure supported on $\mathbb{R}$. The fact that the measure μ_ν must be concentrated on the imaginary axis then follows by uniqueness.

Corollary 5.4. *Let u_n be the sequence defined by the recurrence relation (A 8) with $u_0 \neq 0$ and $t = -x + \mathrm{i}0\pm$. Under the same hypothesis on μ, for every $x \in \mathbb{R}_+$, for almost every realisation of the sequence* $\mathbf{s}$*, we have*

$$\lim_{n\to\infty} \frac{\ln u_n}{n} = \Lambda_\mu(-x + \mathrm{i}0\pm)$$

independently of the starting values $u_0 \neq 0$ and u_{-1}.

Finally, the positivity of $\gamma_{\tilde{\mu}}$, asserted in theorem 5.1, leads to the

Corollary 5.5. *Under the same hypothesis on μ, for every $t \in \mathbb{C}\backslash\mathbb{R}_-$,*

$$\mathrm{Re}\,[\Lambda_\mu(t)] > 0\,.$$

Also, for every $x \in \mathbb{R}_+$,

$$\mathrm{Re}\,[\Lambda_\mu(-x + \mathrm{i}0\pm)] > 0\,.$$

Proof. Take the real part in equation (1.20) and use corollary 5.2 . □

The statements made in corollaries 5.2-5.5 are of the form:

Let t be fixed, then for almost every realisation of the sequence $\mathbf{s}$*, etc.*

But the proof of proposition 1.2 follows easily from them by a well-known argument based on the use of Fubini's theorem; see, for instance, Ishii (1973).

Electronic Supplementary Material

Appendix B. The complex Lyapunov exponent for the gamma distribution

In this appendix, we derive the formula (1.26) for the complex Lyapunov exponent when μ is the gamma distribution. For this purpose, it is convenient to adopt the notation used in Marklof *et al.* (2005); so we set

$$a_n := \frac{s_n}{\sqrt{|t|}} \quad \text{and} \quad \alpha := -\frac{\arg t}{2} \in [-\pi/2,\, \pi/2]\,.$$

Then

$$Z = \cfrac{1}{a_1 \mathrm{e}^{\mathrm{i}\alpha} + \cfrac{1}{a_2 \mathrm{e}^{\mathrm{i}\alpha} + \cfrac{1}{a_3 \mathrm{e}^{\mathrm{i}\alpha} + \cdots}}}\,.$$

The random variable Z takes values in the set

$$S_\alpha := \{ z \in \mathbb{C} \,:\, |\arg z| \le |\alpha| \}\,.$$

In Marklof *et al.* (2005), we showed that, if the a_n are gamma-distributed with parameters p and s, then the probability density function of Z is given explicitly by

$$f_\alpha(z) = \frac{\sin(2|\alpha|)}{\left|2K_p\left(2\mathrm{e}^{\mathrm{i}\alpha}/s\right)\right|^2} \frac{1}{r^2 \sin^2(\alpha+\theta)} \left[\frac{\sin(\alpha-\theta)}{\sin(\alpha+\theta)}\right]^{p-1} \times \exp\left\{ -\frac{\sin(2\alpha)}{s} \left[\frac{1}{r\sin(\alpha-\theta)} + \frac{r}{\sin(\alpha+\theta)} \right] \right\}\,, \tag{B 1}$$

where $r = |z|$ and $\theta = \arg z$. This is valid for $|\alpha| < \pi/2$— which is all we need for the purpose of calculating the complex Lyapunov exponent, since the cases $\alpha = \pm\pi/2$ may be obtained by letting α tend to the appropriate limit. In Marklof *et al.* (2005), we derived the formula

$$-\int_{S_\alpha} \ln|z| f_\alpha(z)\,\mathrm{d}z = \mathrm{Re}\left[\frac{\partial_p K_p\left(\frac{2}{s}\mathrm{e}^{-\mathrm{i}\alpha}\right)}{K_p\left(\frac{2}{s}\mathrm{e}^{-\mathrm{i}\alpha}\right)} \right]\,.$$

So there only remains to show that

$$-\int_{S_\alpha} \ln\left(\mathrm{e}^{\mathrm{i}\arg z}\right) f_\alpha(z)\,\mathrm{d}z = \mathrm{i}\,\mathrm{Im}\left[\frac{\partial_p K_p\left(\frac{2}{s}\mathrm{e}^{-\mathrm{i}\alpha}\right)}{K_p\left(\frac{2}{s}\mathrm{e}^{-\mathrm{i}\alpha}\right)} \right]\,. \tag{B 2}$$

We shall need the

Lemma 5.6. *Let w and W be two complex numbers with positive real part. Then*

$$K_p(w)\,\partial_p K_p(W) = -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-2pu} \int_0^\infty \left(u + \ln\frac{2v}{w\mathrm{e}^u + W\mathrm{e}^{-u}} \right) \times \exp\left\{ -v - \frac{w^2 + 2wW\cosh(2u) + W}{4v} \right\} \frac{\mathrm{d}v}{v}\,\mathrm{d}u\,.$$

Proof. The Bessel function of the second kind has the integral representation (see Watson 1966, §6.22)

$$K_p(z) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-z\cosh x - px}\,\mathrm{d}x\,.$$

Hence

$$K_p(W)\partial_p K_p(w) = -\frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-W\cosh y - py - w\cosh x - px\right\} x\,\mathrm{d}x\mathrm{d}y\,.$$

We make the change of variables

$$x = X + Y \quad \text{and} \quad y = X - Y\,.$$

Then, after some re-arrangement,

$$K_p(W)\,\partial_p K_p(w) = -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-2pX}$$
$$\times\int_{-\infty}^{\infty} \exp\left\{-\mathrm{e}^{Y}\frac{W\mathrm{e}^{-X}+w\mathrm{e}^{X}}{2} - \mathrm{e}^{-Y}\frac{W\mathrm{e}^{X}+w\mathrm{e}^{-X}}{2}\right\}(X+Y)\mathrm{d}Y\mathrm{d}X\,.$$

Make the substitution

$$v = \mathrm{e}^{Y}\frac{W\mathrm{e}^{-X}+w\mathrm{e}^{X}}{2}$$

in the second integral. Then

$$K_p(W)\,\partial_p K_p(w) = -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-2pX}$$
$$\times\int_{-\infty}^{\infty} \exp\left\{-v - \frac{W^2+2wW\cosh(2X)+w^2}{4v}\right\}\left(X + \ln\frac{2v}{W\mathrm{e}^{-X}+w\mathrm{e}^{X}}\right)\frac{\mathrm{d}v}{v}\mathrm{d}X\,.$$

The desired result follows after we set $X = u$ and take the imaginary part. □

Returning to the proof of equation (B 2), let us write

$$\mathrm{Im}\int_{S_\alpha} \ln\left(\mathrm{e}^{-\mathrm{i}\arg z}\right)\, f_\alpha(z)\,\mathrm{d}z = \frac{1}{4\left|K_p\left(\frac{2}{s}\mathrm{e}^{-\mathrm{i}\alpha}\right)\right|^2}\,\mathcal{I}\,, \qquad \text{(B 3)}$$

where

$$\mathcal{I} := \mathrm{Im}\int_{-\alpha}^{\alpha}\int_0^{\infty} \ln\left(\mathrm{e}^{-\mathrm{i}\theta}\right)\frac{\sin(2|\alpha|)}{r^2\sin^2(\alpha+\theta)}\left[\frac{\sin(\alpha-\theta)}{\sin(\alpha+\theta)}\right]^{p-1}$$
$$\times\exp\left\{-\frac{\sin(2\alpha)}{s}\left[\frac{1}{r\sin(\alpha-\theta)} + \frac{r}{\sin(\alpha+\theta)}\right]\right\}\mathrm{d}r\mathrm{d}\theta\,.$$

Replace the variable r by

$$v = \frac{r\sin(2\alpha)}{s\,\sin(\alpha+\theta)}$$

and then the variable θ by

$$t = \frac{\sin(\alpha - \theta)}{\sin(\alpha + \theta)}\,.$$

A straightforward calculation leads to

$$\mathcal{I} = \operatorname{Im} \int_0^\infty \ln\left(\frac{\sqrt{t\varphi(t)}}{\mathrm{e}^{\mathrm{i}\alpha} + t\mathrm{e}^{-\mathrm{i}\alpha}}\right) t^{p-1} \int_0^\infty \exp\left\{-v - \frac{4}{s^2}\frac{\varphi(t)}{4v}\right\} \frac{\mathrm{d}v}{v}\mathrm{d}t\,, \tag{B 4}$$

where

$$\varphi(t) = t + \frac{1}{t} + 2\cos\alpha\,.$$

Set

$$t = \mathrm{e}^{-2u}\,.$$

Then equation (B 4) becomes

$$\mathcal{I} = -2\operatorname{Im}\int_{-\infty}^\infty \ln\left(\mathrm{e}^{u+\mathrm{i}\alpha} + \mathrm{e}^{-u-\mathrm{i}\alpha}\right)\,\mathrm{e}^{-2pu} \times \int_0^\infty \exp\left\{-v - \frac{4}{s^2}\frac{\mathrm{e}^{-\mathrm{i}2\alpha} + 2\cosh(2u) + \mathrm{e}^{\mathrm{i}2\alpha}}{4v}\right\}\frac{\mathrm{d}v}{v}\mathrm{d}u\,. \tag{B 5}$$

We use lemma 5.6 with $W = \overline{w} = 2\mathrm{e}^{-\mathrm{i}\alpha}/s$ to obtain

$$\mathcal{I} = -4\,\operatorname{Im}\left[K_p\left(\frac{2}{s}\mathrm{e}^{-\mathrm{i}\alpha}\right)\,\partial_p K_p\left(\frac{2}{s}\mathrm{e}^{\mathrm{i}\alpha}\right)\right] = 4\,\operatorname{Im}\left[K_p\left(\frac{2}{s}\mathrm{e}^{\mathrm{i}\alpha}\right)\,\partial_p K_p\left(\frac{2}{s}\mathrm{e}^{-\mathrm{i}\alpha}\right)\right]\,. \tag{B 6}$$

The result is then an immediate consequence of equation (B 3).

Appendix C. A result of Goldsheid & Khoruzhenko (2005)

In order to deduce the existence of the integrated density of states from proposition 2.2, we shall need some technical results contained in Goldsheid & Khoruzhenko (2005).

Let $\{\mathscr{A}_n\}_{n\in\mathbb{Z}_+}$ be a deterministic sequence of square matrices of increasing dimension n, and let

$$p_n(z) = \frac{1}{n}\ln\left|\det\left(\mathscr{A}_n - z\mathscr{I}_n\right)\right| = \int_{\mathbb{C}} \ln|w - z|\,\kappa_n(\mathrm{d}w)\,,$$

where $\mathscr{I}_n$ is the $n\times n$ identity matrix and

$$\kappa_n = \frac{1}{2\pi}\Delta p_n$$

is the normalised eigenvalue counting measure of $\mathscr{A}_n$. Define

$$\tau_R := \overline{\lim_{n\to\infty}}\int_{|w|\geq R} \ln|w|\,\kappa_n(\mathrm{d}w)\,,\quad R\geq 1\,. \tag{C 1}$$

Proposition 5.7. *Assume that there is a function $p : \mathbb{C} \to [-\infty, \infty)$ such that*

$$p_n(z) \xrightarrow[n\to\infty]{} p(z) \quad \text{for Lebesgue-almost every } z \in \mathbb{C}\,.$$

If $\tau_1 < \infty$, then

1. *p is locally integrable.*
2. *The measure*
$$\kappa := \frac{1}{2\pi}\Delta p$$
is a probability measure.
3. *We have*
$$\int_{|w|\ge 1} \ln|w|\,\kappa(\mathrm{d}w) \le \tau_1 < \infty$$
and the sequence $\{\kappa_n\}_{n\in\mathbb{Z}_+}$ converges weakly to κ.

Proof. See Goldsheid & Khoruzhenko (2005), proposition 1.3. □

To apply this result in our context, we set $\mathscr{A}_n = \mathscr{J}_n$. Proposition 2.2 then takes care of the first assumption. There remains to show the finiteness of τ_1. Goldsheid & Khoruzhenko remark that the following inequalities hold:

$$\tau_1 \le \overline{\lim_{n\to\infty}} \frac{1}{2n} \operatorname{tr}\ln\left(\mathscr{I}_n + \mathscr{J}_n\mathscr{J}_n^*\right)$$

and

$$\frac{1}{n}\operatorname{tr}\ln\left(\mathscr{I}_n + \mathscr{J}_n\mathscr{J}_n^*\right) \le \frac{\alpha}{n}\sum_{j=0}^{n-1} \ln\left(1+\beta|\mathbf{r}_j|^2\right)\,, \tag{C 2}$$

for some positive constants α and β independent of n and $\mathbf{r}_j$, the jth row of $\mathscr{J}_n$. Now,

$$\begin{aligned}|\mathbf{r}_n|^2 &= h_{n-1}^2 + v_n^2 + h_n^2 \\ &= \frac{1}{s_{2n+2}^2 s_{2n+1} s_{2n+3}} + \frac{1}{s_{2n+1}^2}\left(\frac{1}{s_{2n+2}} + \frac{1}{s_{2n+3}}\right)^2 + \frac{1}{s_{2n}^2 s_{2n-1} s_{2n+1}}\end{aligned}$$

and, so by repeated use of the elementary inequality

$$ab \le \frac{1}{2}\left(a^2+b^2\right)\,,$$

we obtain readily

$$|\mathbf{r}_n|^2 \le \frac{5}{2}\left(\frac{1}{s_{2n-1}^4} + \cdots + \frac{1}{s_{2n+3}^4}\right)\,.$$

It follows that

$$\begin{aligned}\ln\left(1+\beta|\mathbf{r}_n|^2\right) &\le \ln\left(1+\frac{25}{2}\beta\max\left\{s_{2n-1}^{-4},\,\ldots,\,s_{2n+3}^{-4}\right\}\right) \\ &\le \ln\left(1+\frac{25}{2}\beta s_{2n-1}^{-4}\right) + \cdots + \ln\left(1+\frac{25}{2}\beta s_{2n+3}^{-4}\right)\,.\end{aligned}$$

We deduce from the ergodic theorem that the right-hand side of equation (C 2) is bounded independently of n provided that

$$\int_{\mathbb{R}_+} \ln\left(1 + \frac{25}{2}\frac{\beta}{s^4}\right) \mu(\mathrm{d}s) < \infty\,.$$

This last inequality follows easily from our hypothesis that

$$\int_{\mathbb{R}_+} |\ln s| \; \mu(\mathrm{d}s) < \infty \quad \text{and} \quad \int_{\mathbb{R}_+} s^{\varepsilon} \, \mu(\mathrm{d}s) < \infty\,.$$

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