Nikolai Chernov

10 November 1956 – 7 August 2014

Chernov Memorial Lectures

The Lorentz gas – kinetic theory via dynamical renormalisation

Jens Marklof
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Semi-dispersing billiards

Функциональный анализ и его приложения,
1982, т. 16, вып. 4, 35—46.

Structure of transversal leaves in multidimensional semidispersing billiards

УДК 517.919.2 Construction of transverse fiberings in multidimensional semidispersed billiards

ПОСТРОЕНИЕ ТРАНСВЕРСАЛЬНЫХ СЛОЕВ В МНОГОМЕРНЫХ ПОЛУРАССЕИВАЮЩИХ БИЛЬЯРДАХ

Н. И. Чернов
N.I. Chernov

§ 1. Введение

Одним из важных классов динамических систем являются системы бильярдного типа (бильярды), которые порождаются движением частицы ... евклидова пространства или d-мерного евклидова ... подмножество U, причем гра-...

для которого (14) будет выполнено при всех $t > 0$. Отсюда

$$\chi^+(x_0, u_0) = \overline{\lim}\, \frac{1}{t} \ln \| dT^t u_0 \| \leqslant -\rho \ln D < 0,$$

где χ^+ — характеристический показатель Ляпунова (см. [3] и [4]). Лемма 9 доказана.

Аналогичные рассуждения показывают, что пространство векторов, удовлетворяющих (14), j_0-мерно и его проекция на $\mathcal{T}_{q_0}Q$ совпадает с $J_+(x_0)$. Поскольку $dv_T = B_T dq_T$ при всех T для компонент вектора $dT^T u$, то для компонент вектора $u_0 = (dq_0, dv_0)$ выполнено соотношение $dv_0 = Bdq_0$.

Автор выражает благодарность Я. Г. Синаю за постановку задачи и полезные обсуждения, Я. Б. Песину и А. Крамли за ряд ценных замечаний, а также Н. В. Щербине за полезные обсуждения.

ЦИТИРОВАННАЯ ЛИТЕРАТУРА

1. *Синай Я. Г.* Динамические системы с упругими отражениями. Эргодические свойства рассеивающих бильярдов.— УМН, 1970, т. 25, вып. 2, с. 141—191.

Uspekhi Mat. Nauk **45**:3 (1990), 97–134 Russian Math. Surveys **45**:3 (1990), 105–152

Markov partitions for two-dimensional hyperbolic billiards

L.A. Bunimovich, Ya.G. Sinai, and N.I. Chernov

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seen from the interior of Q), *focussing* if $\varkappa(q) < 0$ for all $q \in \Gamma_i$ (that is, Γ_i is convex as viewed from the outside), and *neutral* if $\varkappa(q) \equiv 0$ on Γ_i (in this case Γ_i is a segment of a straight line).

Definition. A billiard in Q is called *scattering* if all components of ∂Q are scattering (Fig. 1).

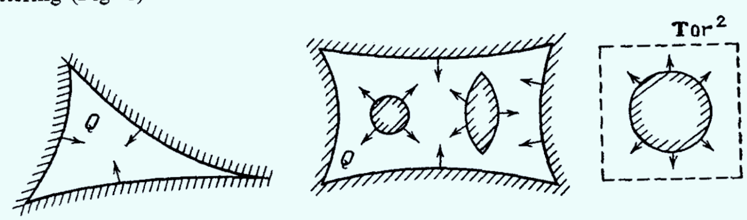


Fig. 1

The stochastic properties of two-dimensional scattering billiards have been studied rather exhaustively. They are ergodic, are K-systems (see [10]), and in

Statistical properties of hyperbolic billiards

NEW PROOF OF SINAI'S FORMULA FOR THE ENTROPY OF HYPERBOLIC BILLIARD SYSTEMS. APPLICATION TO LORENTZ GASES AND BUNIMOVICH STADIUMS

N. I. Chernov

§1. INTRODUCTION

A *billiard system* is a dynamical system generated by the free motion of material points i with elastic reflections from the walls. Billiards provide convenient models in a number of ar example, the survey [3]). They have recently been used for research on quantum chaos (see [3

If the walls of the container are concave inward the billiard system is said to be a d concavity is not strict, i.e., there are flat sections, the billiard system is called semidispersing Well- known statistical models for solid spheres, Lorentz gases, and Rayleigh gases reduce to dis systems. These billiard systems have strong stochastic properties and have a structure that negatively curved surfaces. Namely, they are characterized by exponential instability of trajector been proved to be ergodic, to be mixing, to have the K-property [9, 13, 24, 26], and to have billiard systems [2, 18] are similar to them. By analogy to geodesic flows, we will call all of tl

Russian Math. Surveys **64**:4 651–699

Uspekhi Mat. Nauk **64**:4 73–124
DOI 10.1070/RM2009v064n04ABEH004630

Anomalous current in periodic Lorentz gases with infinite horizon

D. I. Dolgopyat and N. I. Chernov

Abstract. Electric current is studied in a two-dimensional periodic Lorentz gas in the presence of a weak homogeneous electric field. When the horizon is finite, that is, free flights between collisions are bounded, the resulting current $\mathbf{J}$ is proportional to the voltage difference $\mathbf{E}$, that is, $\mathbf{J} = \frac{1}{2}\mathbf{D}^*\mathbf{E} +$

Journal of Statistical Physics, Vol. 74, Nos. 1/2, 1994

Statistical Properties of the Periodic Lorentz Gas. Multidimensional Case

N. I. Chernov[1]

Received November 3, 1992; final July 30, 1993

In 1981 Bunimovich and Sinai established the statistical properties of the planar periodic Lorentz gas with finite horizon. Our aim is to extend their theory to the multidimensional Lorentz gas. In that case the Markov partitions of the Bunimovich–Sinai type, the main tool of their theory, are not available. We use a crude approximation to such partitions, which we call Markov sieves. Their

CONVERGENCE OF MOMENTS FOR DISPERSING BILLIARDS WITH CUSPS

P. BÁLINT, N. CHERNOV, D. DOLGOPYAT

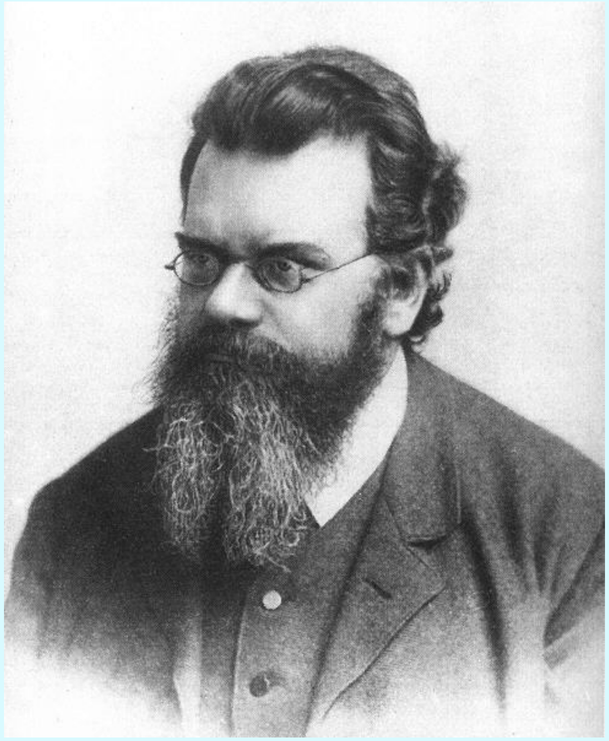
ABSTRACT. Dispersing billiards with cusps are deterministic dynamical systems with a mild degree of chaos, exhibiting "intermittent" behavior that alternates between regular and chaotic patterns. They are characterized by decay of correlations of order $1/n$ and a central limit theorem with a non-classical scaling factor of $\sqrt{n \log n}$. We show that the pth moments of the so normalized ergodic sums converge to the moments of the limit normal distribution only for $p < 2$ and diverge for $p > 2$. The critical second moments converge, but their limit is double the second moment of the normal distribution.

Part 1

Kinetic theory for the Lorentz gas
(general scatterer configurations)

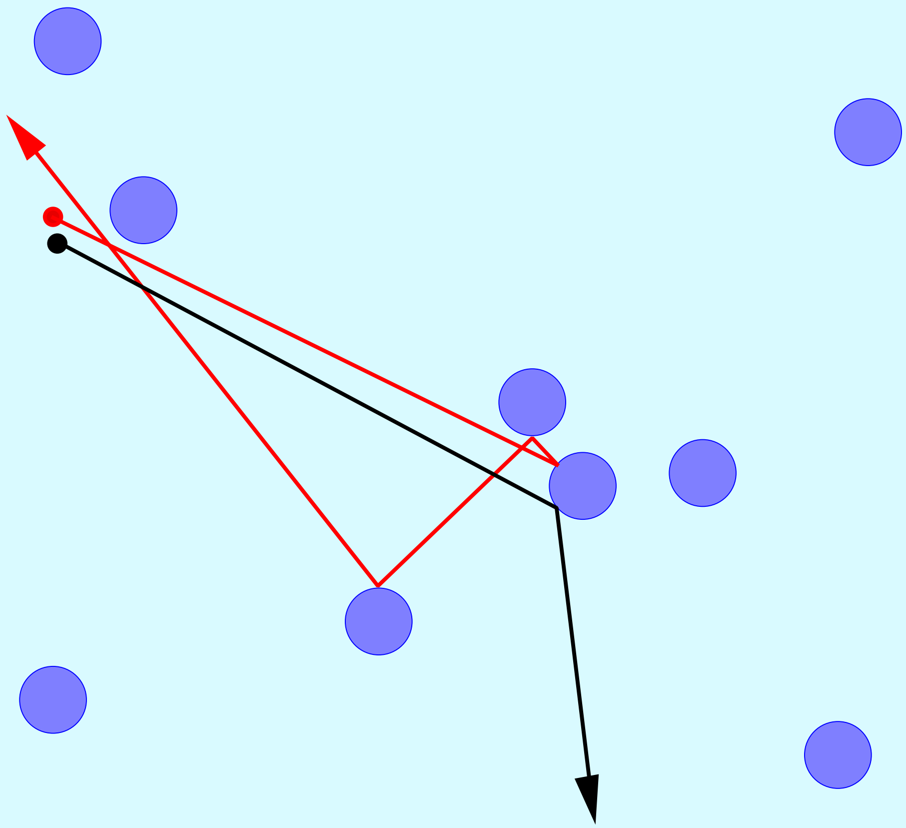
Maxwell and Boltzmann

James Clerk Maxwell (1831-1879)

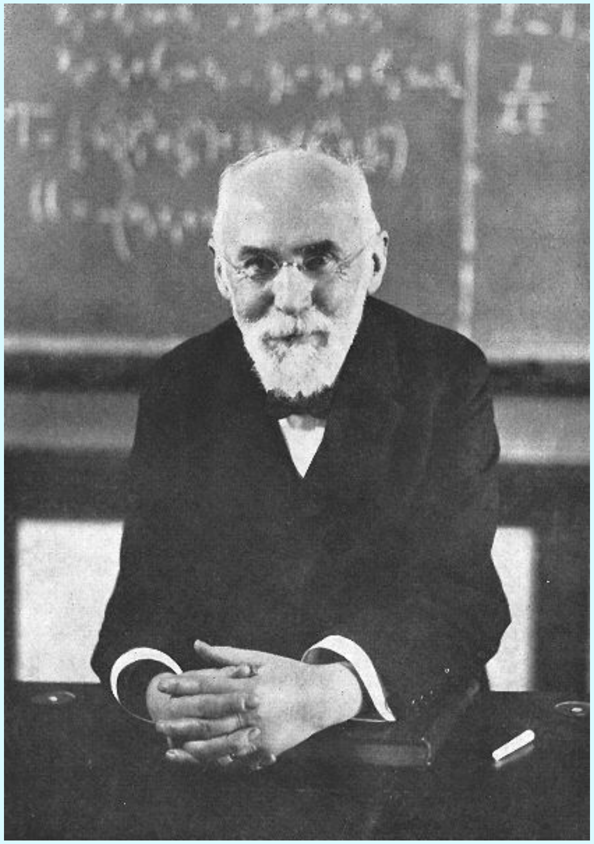


Ludwig Boltzmann (1844-1906)

The Lorentz gas

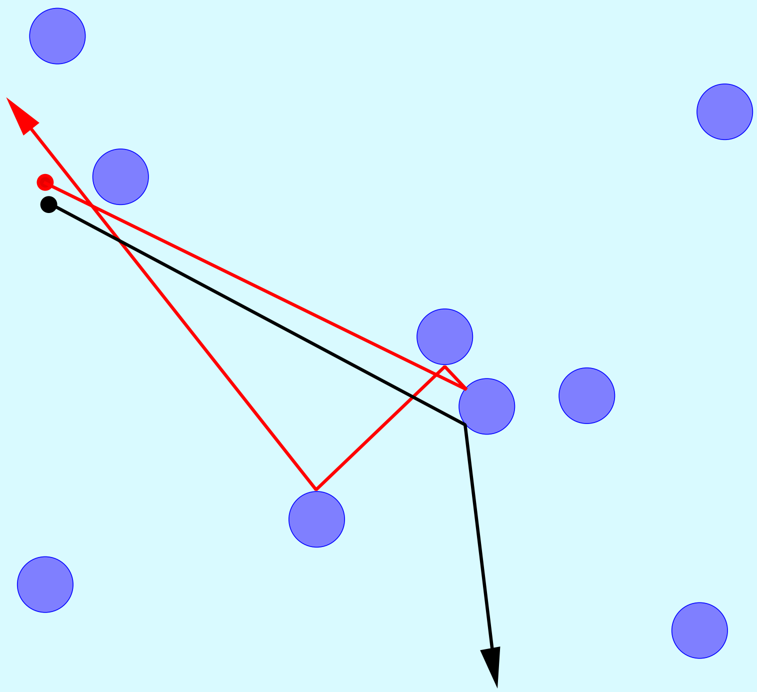


Arch. Neerl. (1905)



Hendrik Lorentz (1853-1928)

The Lorentz gas



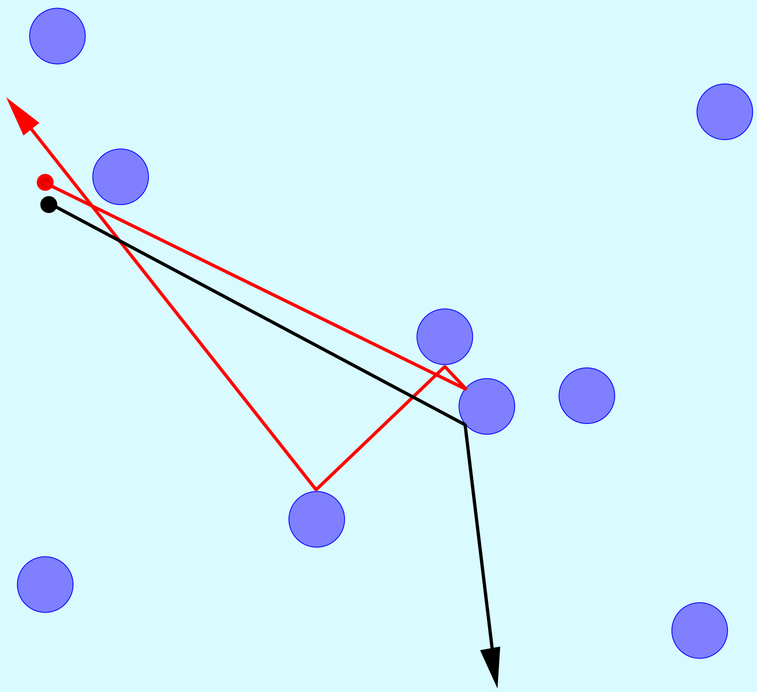
- $\mathcal{P}$ locally finite subset of $\mathbb{R}^d$ with density one, i.e.,

$$\lim_{R\to\infty} \frac{\#(\mathcal{P} \cap R\mathcal{D})}{R^d} = \mathrm{vol}\,\mathcal{D}$$

  for all bounded sets $\mathcal{D} \subset \mathbb{R}^d$ with $\mathrm{vol}\,\partial\mathcal{D} = 0$

- scatterers are fixed open balls of radius ρ centered at the points in $\mathcal{P}$

The Lorentz gas



- the particles are assumed to be non-interacting
- each test particle moves with constant velocity $\boldsymbol{v}(t)$ between collisions
- the scattering is specular reflection; **we can also treat scattering by compactly supported, spherically symmetric potentials**
- we assume w.l.o.g. $\|\boldsymbol{v}(t)\| = 1$

Diffusion in the classical periodic Lorentz gas (dimension two)

In the case of fixed scattering radius ρ, proofs of CLT for the Lorentz gas are currently restricted to the 2-dim periodic setting.

Finite horizon:

- Bunimovich & Sinai (Comm Math Phys 1980): Standard CLT for finite horizon
- Melbourne & Nicol (Annals Prob 2009): More general invariance principles

Infinite horizon:

- Bleher (J Stat Phys 1992): Heuristics for CLT with $t \log t$ mean square displacement
- Szász & Varjú (J Stat Phys 2007): Proof of CLT for billiard map
- Dolgopyat & Chernov (Russ Math Surveys, 2009): Proof of CLT & invariance principle in continuous time

Diffusion in the classical periodic Lorentz gas (higher dimension)

The problem in higher dimensions is control of complexity of singularities

- Chernov (J Stat Phys 1994)
- Balint & Tóth (AHP 2008, Nonlinearity 2012)

and in the case of infinite horizon the subtle geometry of free flight channels

- Dettmann (J Stat Phys 2012)
- Nadori, Szasz & Varju (CMP 2014)

As we will see, the problem becomes tractable if we consider the small scatterer (Boltzmann-Grad) limit $\rho \to 0$. In particular (taking first $\rho \to 0$ then $t \to \infty$)

- JM & Balint Tóth (CMP 2017): CLT with $t \log t$ mean square displacement in any dimension (with time t measured in units of the mean collision time); builds on JM & Strömbergsson (Annals Math 2010 & 2011, GAFA 2011)

Diffusion in the classical aperiodic/random Lorentz gas

For fixed ρ, still a major open problem–no CLT established so far.

- Dolgopyat, Szasz & Varju (Duke 2009): finite local perturbations
- Lenci (ETDS 2003/06); Christadoro, degli Esposti, Lenci & Seri (Chaos 2010, J Stat Phys 2011); Lenci & Troubetzkoy (Phys D 2011): recurrence properties

$\vdots$

- Demers & Liverani (CMP 2023): construction of Birkhoff cones that contract under transfer operator

What can be said in the Boltzmann-Grad limit $\rho \to 0$?

The Boltzmann-Grad (=low-density) limit

- Consider the dynamics in the limit of small scatterer radius ρ
- $\big(\boldsymbol{q}(t), \boldsymbol{v}(t)\big)$ = "microscopic" phase space coordinate at time t
- A volume argument shows that for $\rho \to 0$ the mean free path length (i.e., the average time between consecutive collisions) is asymptotic to

$$\frac{1}{\text{total scattering cross section}} = \frac{1}{\rho^{d-1} \,\text{vol}\, B_1^{d-1}}$$

- We thus measure position and time in the "macroscopic" coordinates

$$\big(\boldsymbol{Q}(t), \boldsymbol{V}(t)\big) = \big(\rho^{d-1}\boldsymbol{q}(\rho^{1-d}t), \boldsymbol{v}(\rho^{1-d}t)\big)$$

- Time evolution of initial data $(\boldsymbol{Q}_0, \boldsymbol{V}_0)$:

$$\big(\boldsymbol{Q}(t), \boldsymbol{V}(t)\big) = \Phi_\rho^t(\boldsymbol{Q}_0, \boldsymbol{V}_0)$$

The linear Boltzmann equation

- Time evolution of a particle cloud with initial density $f \in \mathrm{L}^1$:

$$f_t^{(\rho)}(\boldsymbol{Q},\boldsymbol{V}) := f\Big(\Phi_\rho^{-t}(\boldsymbol{Q},\boldsymbol{V})\Big)$$

In his 1905 paper Lorentz suggested that $f_t^{(\rho)}$ is governed, as $\rho \to 0$, by the linear Boltzmann equation:

$$\left[\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla_{\boldsymbol{Q}}\right] f_t(\boldsymbol{Q},\boldsymbol{V}) = \int_{\mathrm{S}_1^{d-1}} \Big[f_t(\boldsymbol{Q},\boldsymbol{V}') - f_t(\boldsymbol{Q},\boldsymbol{V})\Big]\sigma(\boldsymbol{V},\boldsymbol{V}')d\boldsymbol{V}'$$

where $\sigma(\boldsymbol{V},\boldsymbol{V}')$ is the differential cross section of the individual scatterer.
E.g.: $\sigma(\boldsymbol{V},\boldsymbol{V}') = \frac{1}{4}\|\boldsymbol{V}-\boldsymbol{V}'\|^{3-d}$ for specular reflection at a hard sphere

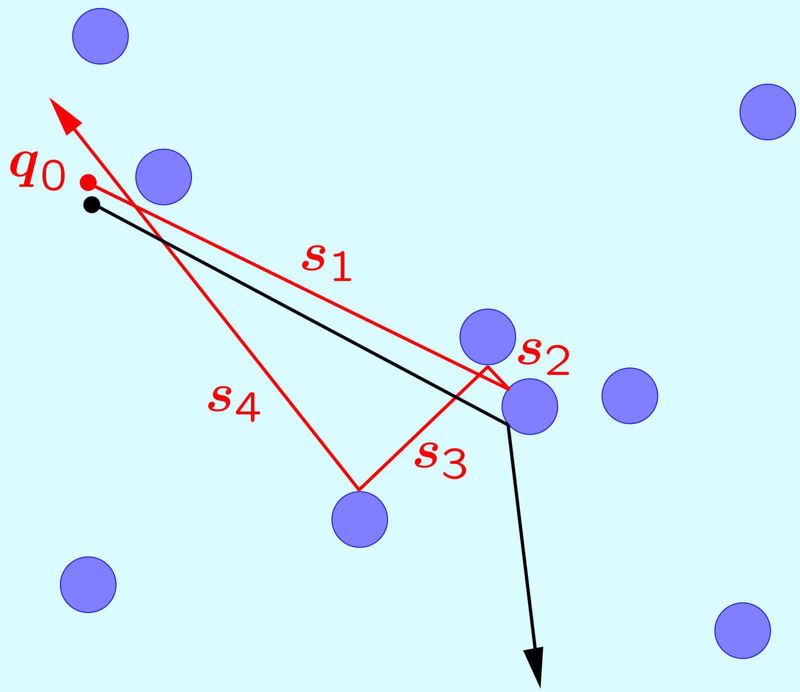
Applications: Neutron transport, radiative transfer, . . .

Main questions:

- What are the random flight processes that emerge in the Boltzmann-Grad limit?

- What are the associated kinetic transport equations?

Key microscopic quantities

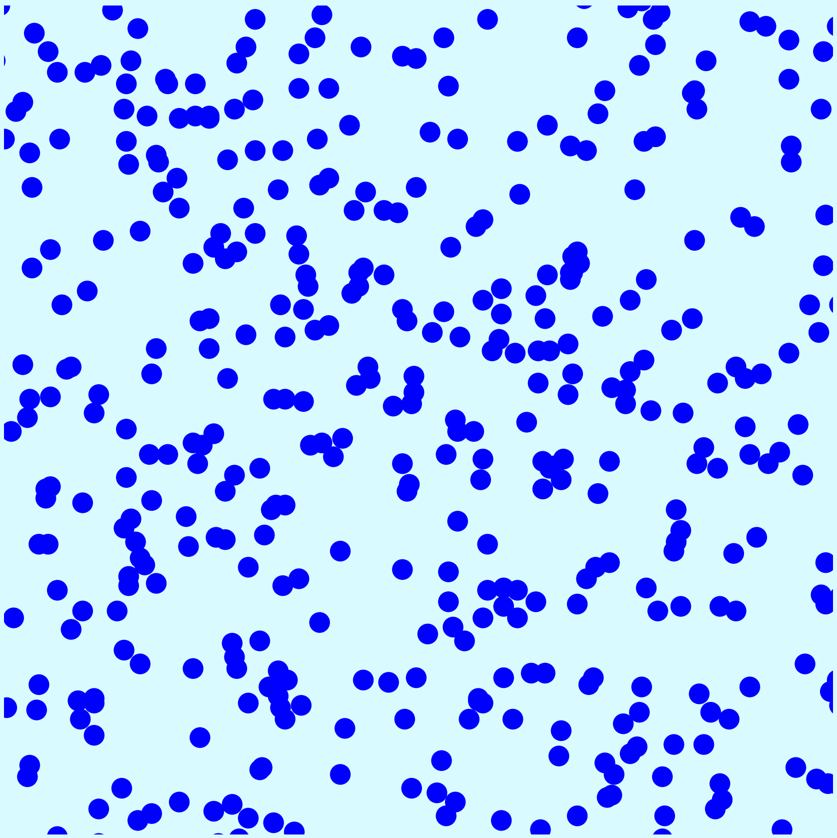


- $\boldsymbol{q}_0, \boldsymbol{v}_0$ initial particle position and velocity ($\|\boldsymbol{v}_0\| = 1$)
- $\tau_1 = \tau_1(\boldsymbol{q}_0, \boldsymbol{v}_0)$ first hitting time

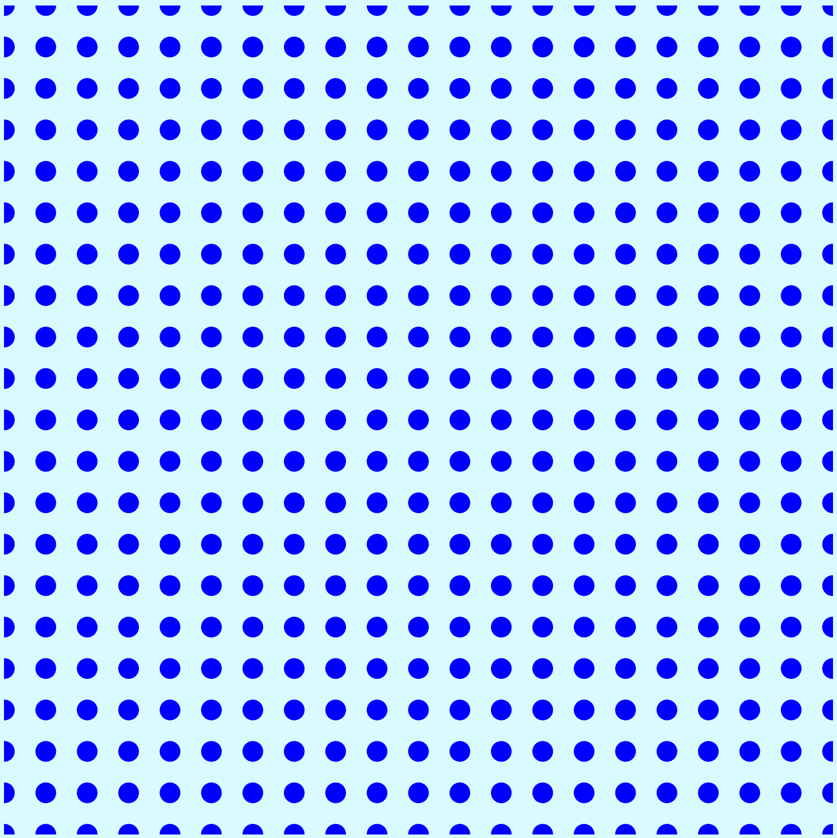
- $\boldsymbol{v}_n = \boldsymbol{v}_n(\boldsymbol{q}_0, \boldsymbol{v}_0)$ velocity after nth collision
- $\tau_{n+1} = \tau_{n+1}(\boldsymbol{q}_0, \boldsymbol{v}_0)$ free path lengths after nth collision
- $\boldsymbol{s}_n = \tau_n \boldsymbol{v}_{n-1}$ travel intinerary

- mean free path $\sim \frac{1}{\rho^{d-1} \operatorname{vol} B_1^{d-1}}$

The Boltzmann-Grad limit



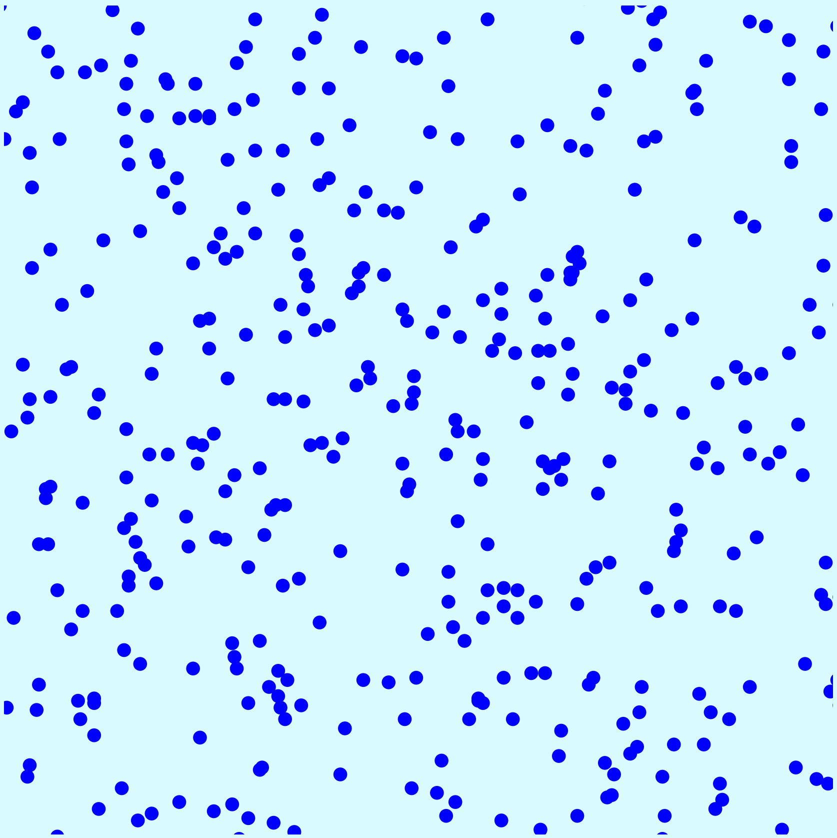
Fixed random scatterer configuration



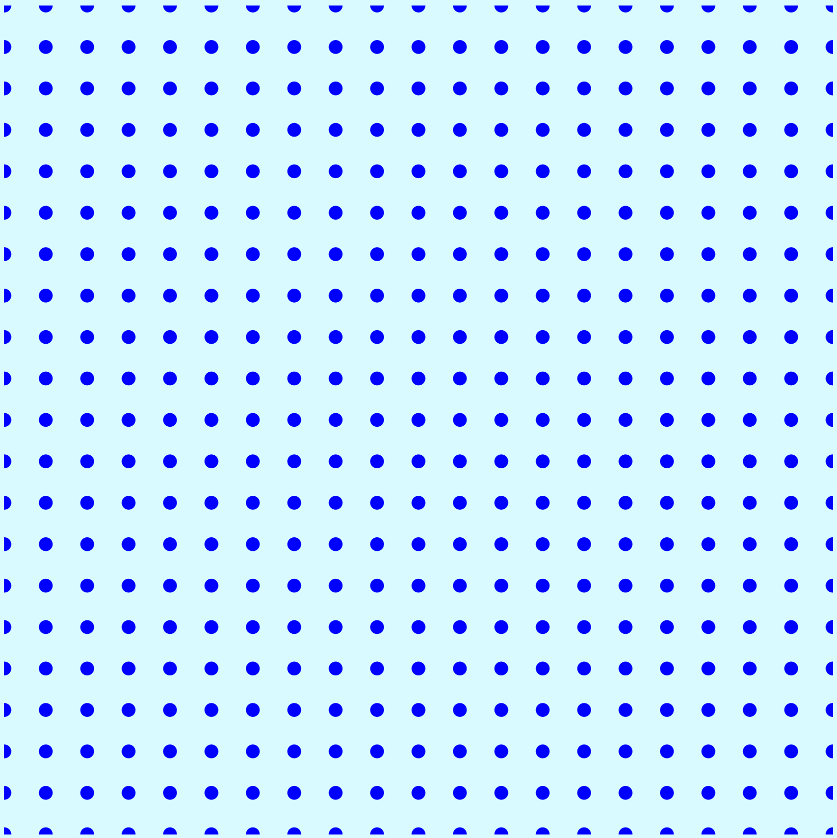
Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $\rho = 1/4$, mean free path$= \frac{1}{2\rho} = 2$

The Boltzmann-Grad limit



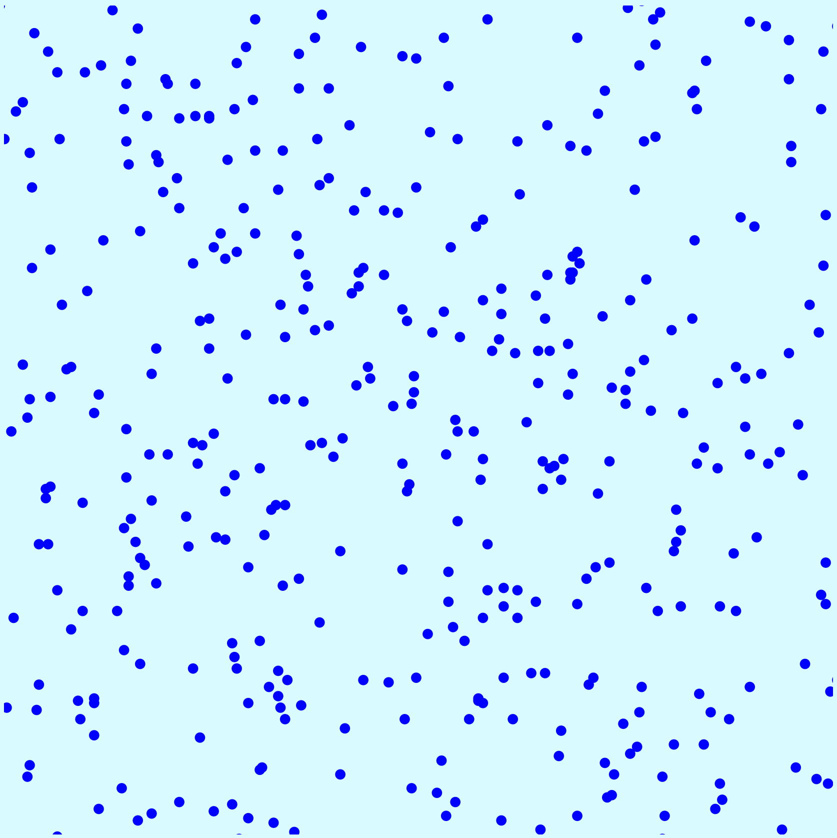
Fixed random scatterer configuration



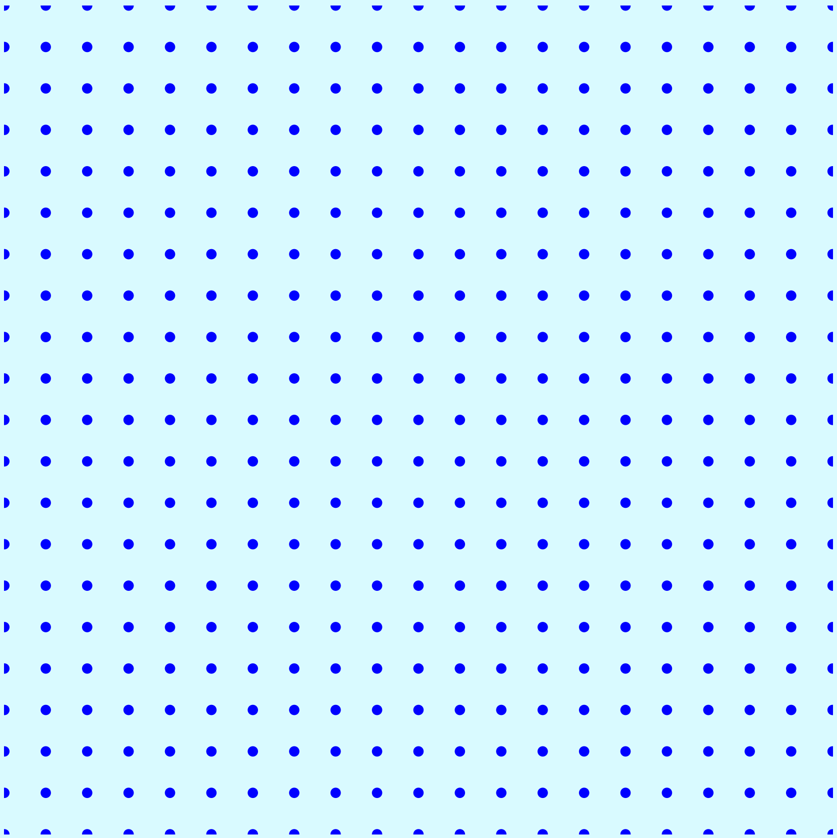
Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $\rho = 1/6$, mean free path$= \frac{1}{2\rho} = 3$

The Boltzmann-Grad limit



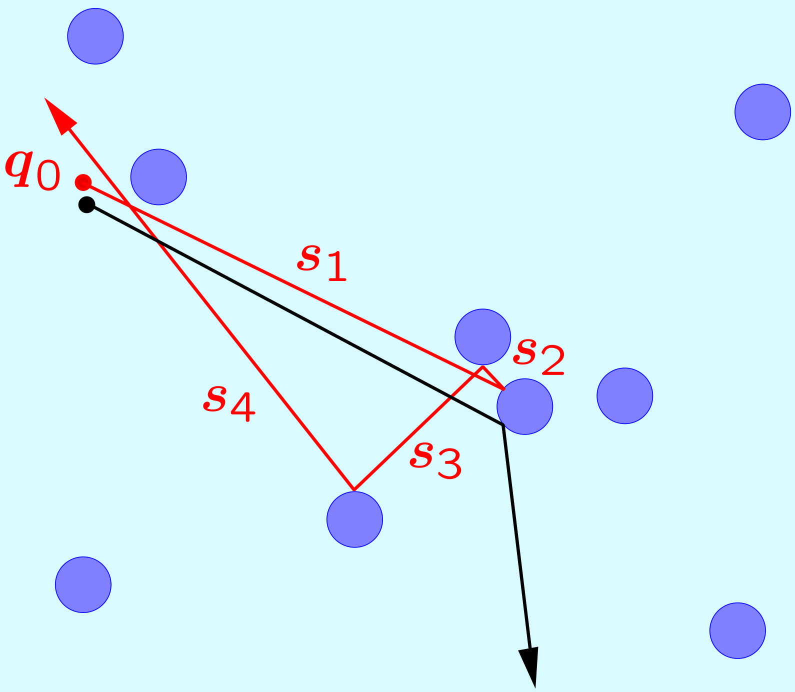
Fixed random scatterer configuration



Periodic scatterer configuration $\mathbb{Z}^2$

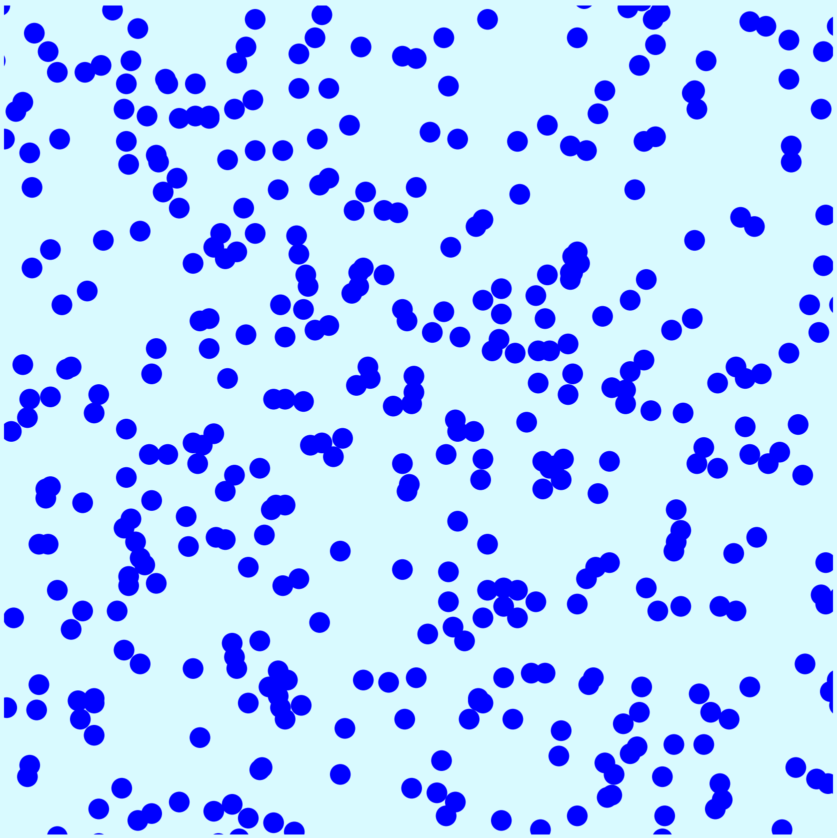
Scattering radius $\rho = 1/8$, mean free path$= \frac{1}{2\rho} = 4$

Key macroscopic quantities

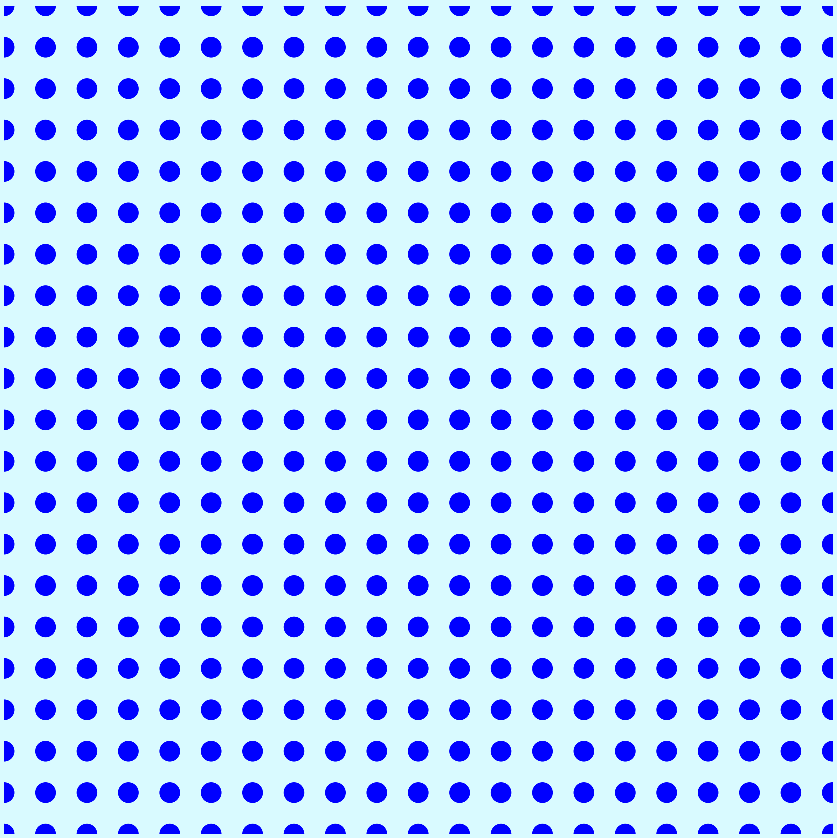


- $\boldsymbol{Q}_0 = \rho^{d-1}\boldsymbol{q}_0, \boldsymbol{V}_0 = \boldsymbol{v}_0$
- $\mathcal{T}_1 = \rho^{d-1}\tau_1(\rho^{1-d}\boldsymbol{Q}_0, \boldsymbol{V}_0)$
- $\boldsymbol{V}_n = \boldsymbol{v}_n(\rho^{1-d}\boldsymbol{Q}_0, \boldsymbol{V}_0)$
- $\mathcal{T}_{n+1} = \rho^{d-1}\tau_{n+1}(\rho^{1-d}\boldsymbol{Q}_0, \boldsymbol{V}_0)$
- $\boldsymbol{S}_n = \mathcal{T}_n\boldsymbol{V}_{n-1} = \rho^{d-1}\boldsymbol{s}_n$
- (macro) mean free path $\frac{1}{\mathrm{vol}\, B_1^{d-1}}$

The Boltzmann-Grad limit



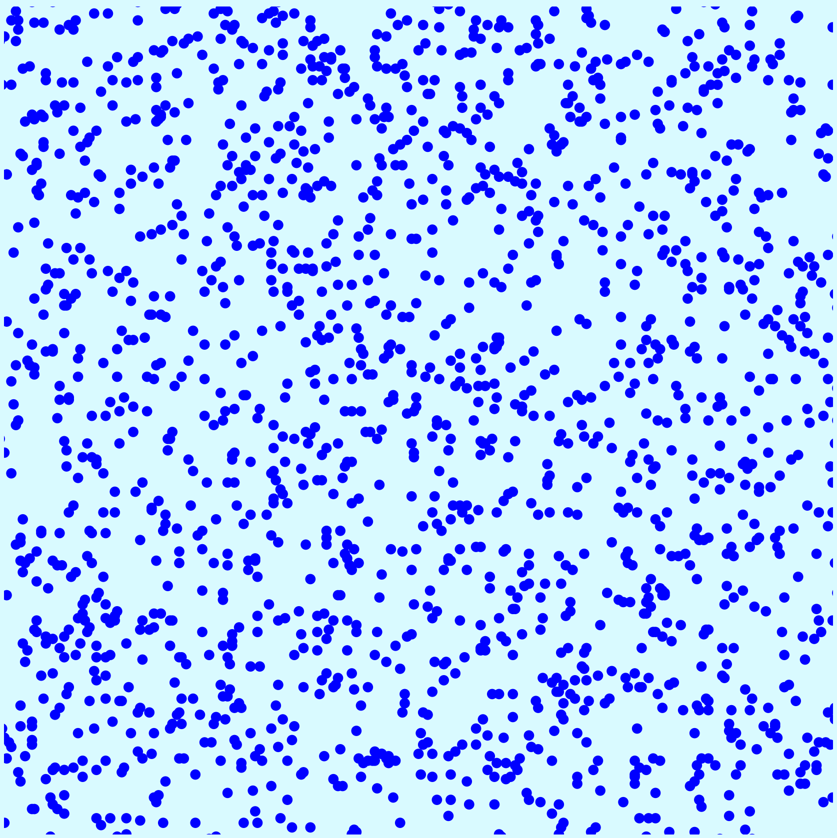
Fixed random scatterer configuration



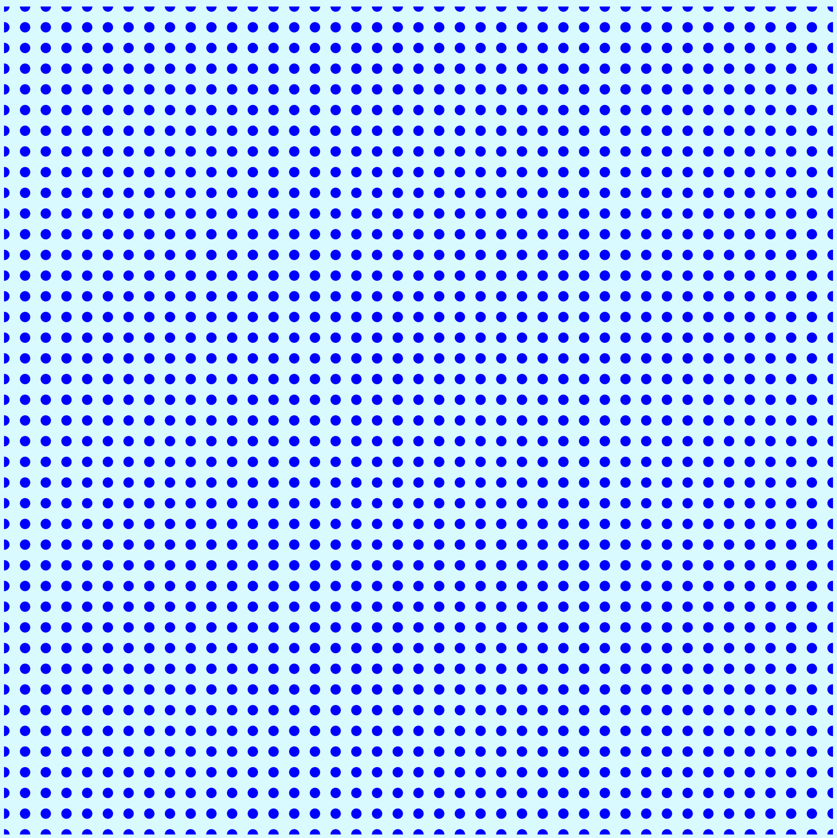
Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $\rho = 1/4$, mean free path$= \frac{1}{2\rho} = 2$

The Boltzmann-Grad limit



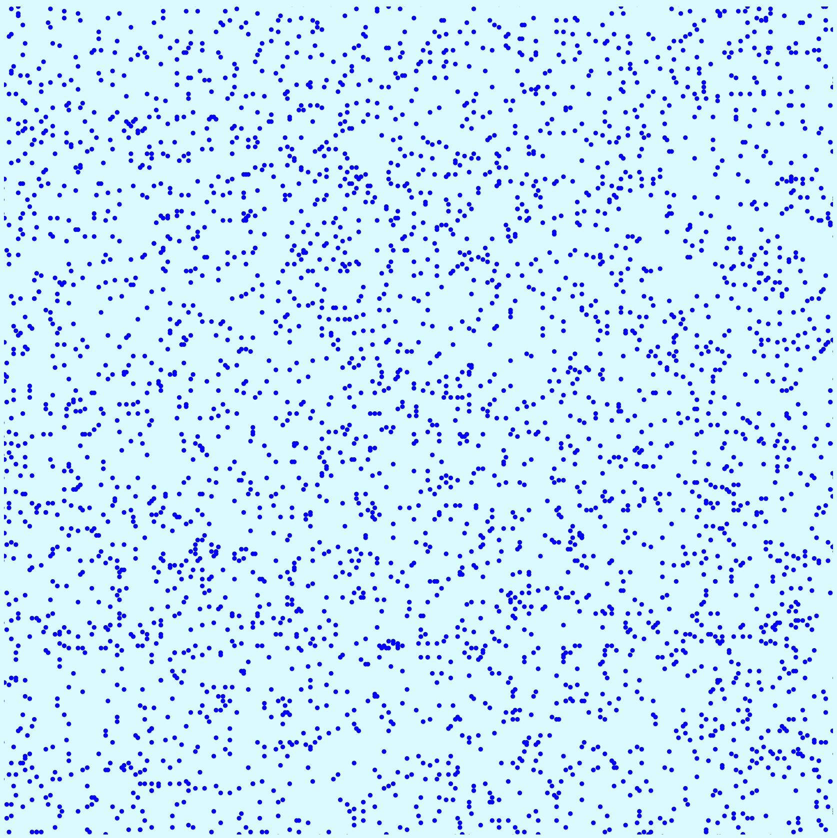
Fixed random scatterer configuration



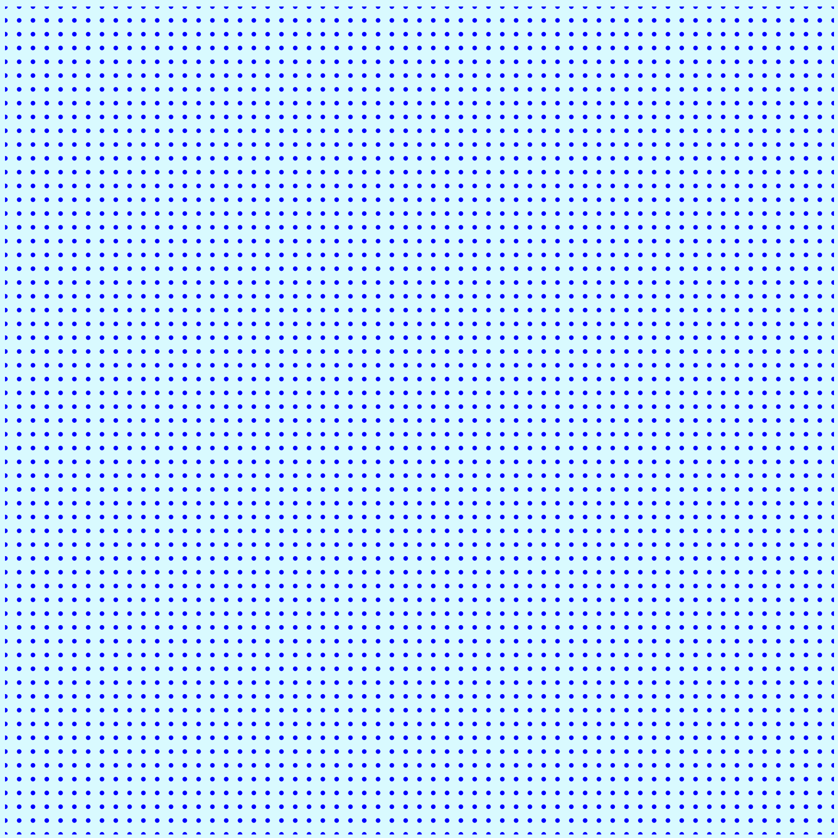
Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $\rho = 1/4$; $1/2$-zoom: macroscopic mean free path=1

The Boltzmann-Grad limit



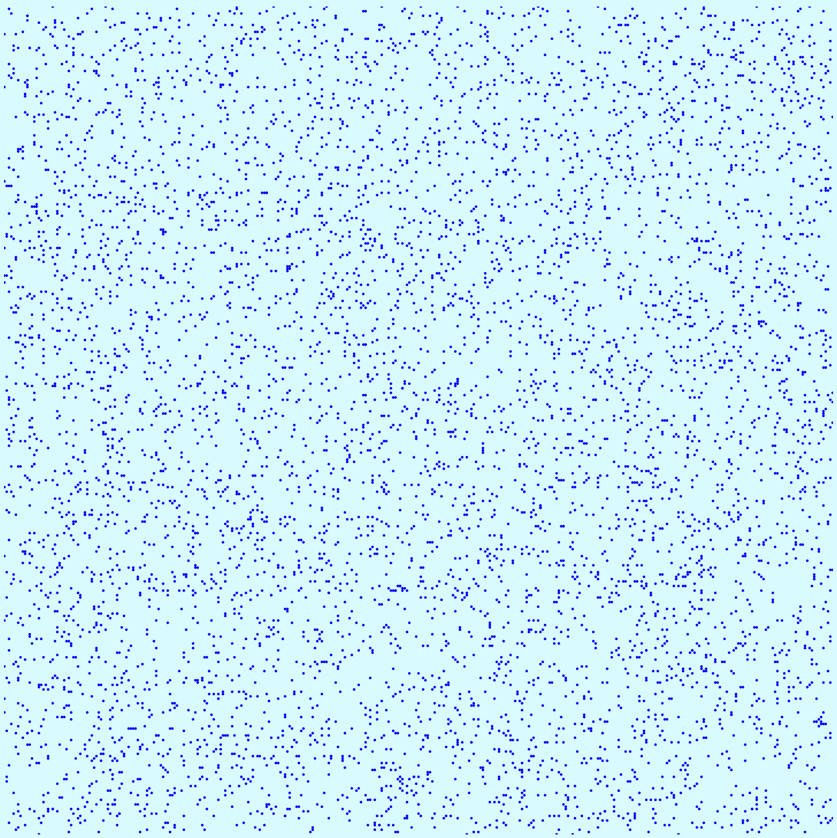
Fixed random scatterer configuration



Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $\rho = 1/6$; $1/3$-zoom: macroscopic mean free path=1

The Boltzmann-Grad limit



Fixed random scatterer configuration

Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $\rho = 1/8$; $1/4$-zoom: macroscopic mean free path=1

The main result

- $n_t = n_t(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ the number of collisions within time t, i.e.,

$$n_t = \max\left\{n \in \mathbb{Z}_{\geq 0} : T_n \leq t\right\}, \qquad T_n := \sum_{j=1}^{n} \mathcal{T}_j.$$

- For $(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ random w.r.t. $\Lambda \in P_{\mathrm{ac}}(\mathrm{T}^1(\mathbb{R}^d))$,

$$\Theta^{(\rho)} : t \mapsto \Theta^{(\rho)}(t) = \left(\boldsymbol{Q}_0 + \sum_{j=1}^{n_t} \mathcal{T}_j \boldsymbol{V}_{j-1} + (t - T_{n_t})\boldsymbol{V}_{n_t}, \boldsymbol{V}_{n_t}\right)$$

  defines a **random flight process**.*

Theorem A (JM & Strömbersson, Memoirs AMS 2024)
Let $\mathcal{P}$ be admissible (see below). Then, for any $\Lambda \in P_{\mathrm{ac}}(\mathrm{T}^1(\mathbb{R}^d))$, there is a random flight process Θ such that $\Theta^{(\rho)}$ converges to Θ in distribution, as $\rho \to 0$.

*Instead of hard sphere scatterers, we can even allow smooth, compactly supported, radially symmetric potentials so that the scattering is dispersing (e.g. Muffin-tin Coulomb potentials).

Outline of proof

The key is to establish the following discrete time analogue of Theorem A.

Theorem B (JM & Strömbersson, Memoirs AMS 2024)
Let $\mathcal{P}$ be admissible.
Then, for random initial data $(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ with distribution $\Lambda \in P_{\mathrm{ac}}(\mathrm{T}^1(\mathbb{R}^d))$,

$$\Big\langle \mathcal{T}_j(\boldsymbol{Q}_0, \boldsymbol{V}_0), \boldsymbol{V}_j(\boldsymbol{Q}_0, \boldsymbol{V}_0) \Big\rangle_{j=1}^{\infty}$$

converges in distribution to the random sequence

$$\Big\langle \xi_j, \boldsymbol{v}_j \Big\rangle_{j=1}^{\infty}$$

(which in general does not form a Markov chain).

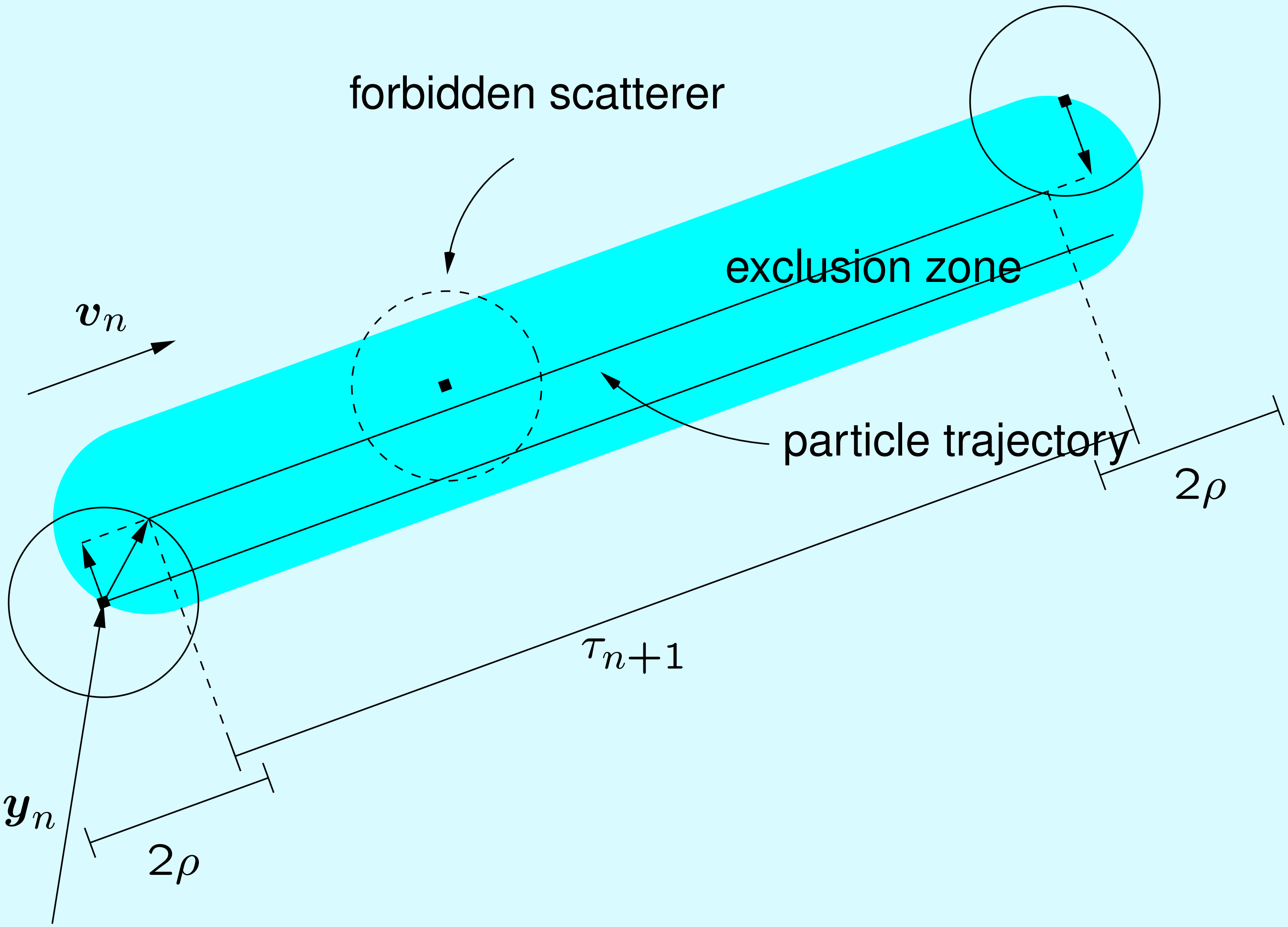
There are three steps:

1. **Rescaling** and spherical equidistribution for each individual inter-collision flight
2. Markovianisation of the limit process through introduction of a **marking** of $\mathcal{P}$
3. **Induction** on the number of inter-collision flights

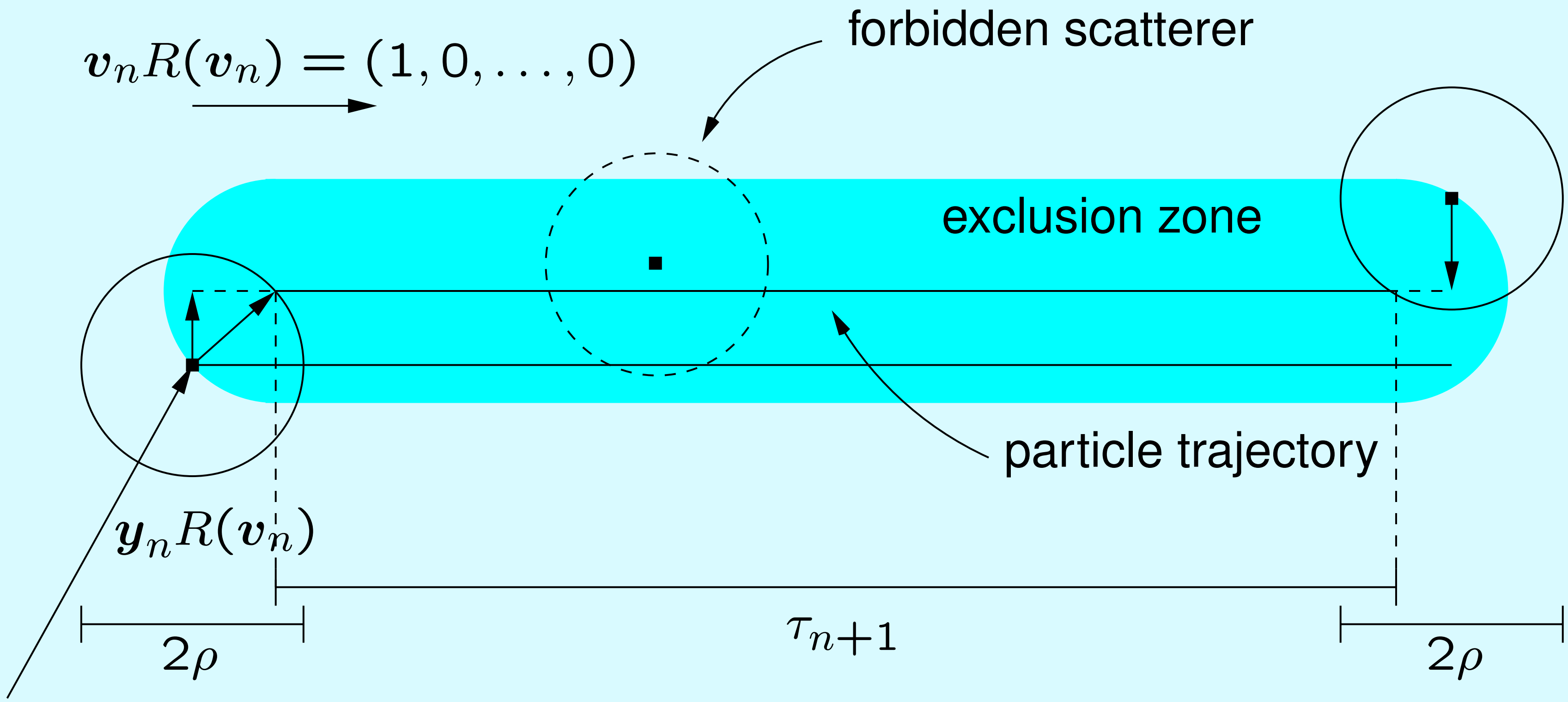
Step 1: Rescaling

Define $R(\boldsymbol{v}) : \mathsf{S}_1^{d-1} \to \mathsf{SO}(d)$ such that $\boldsymbol{v}R(\boldsymbol{v}) = \boldsymbol{e}_1 = (1, 0, \ldots, 0)$ and

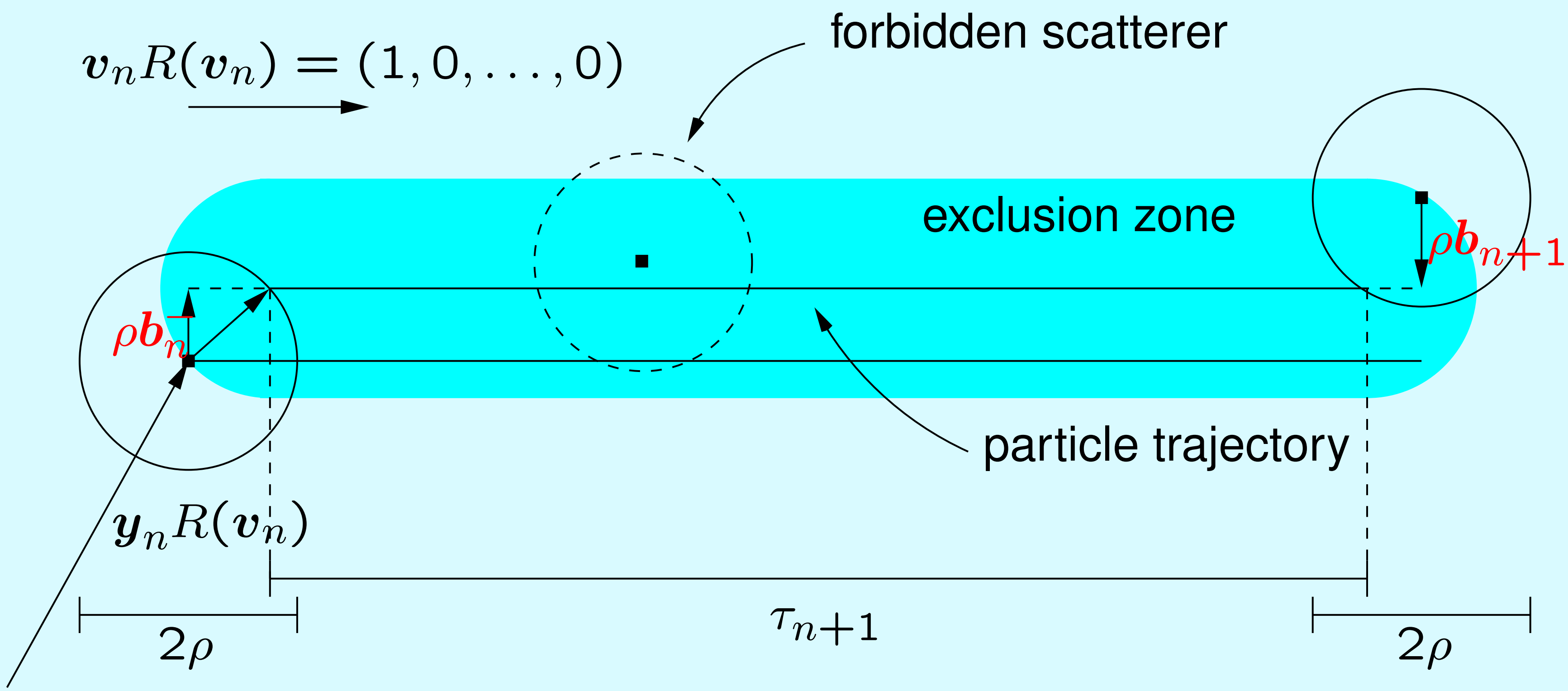
$$D_\rho = \begin{pmatrix} \rho^{d-1} & \mathbf{0} \\ {}^{\mathrm{t}}\mathbf{0} & \rho^{-1} 1_{d-1} \end{pmatrix} \in \mathsf{SL}(d, \mathbb{R}).$$



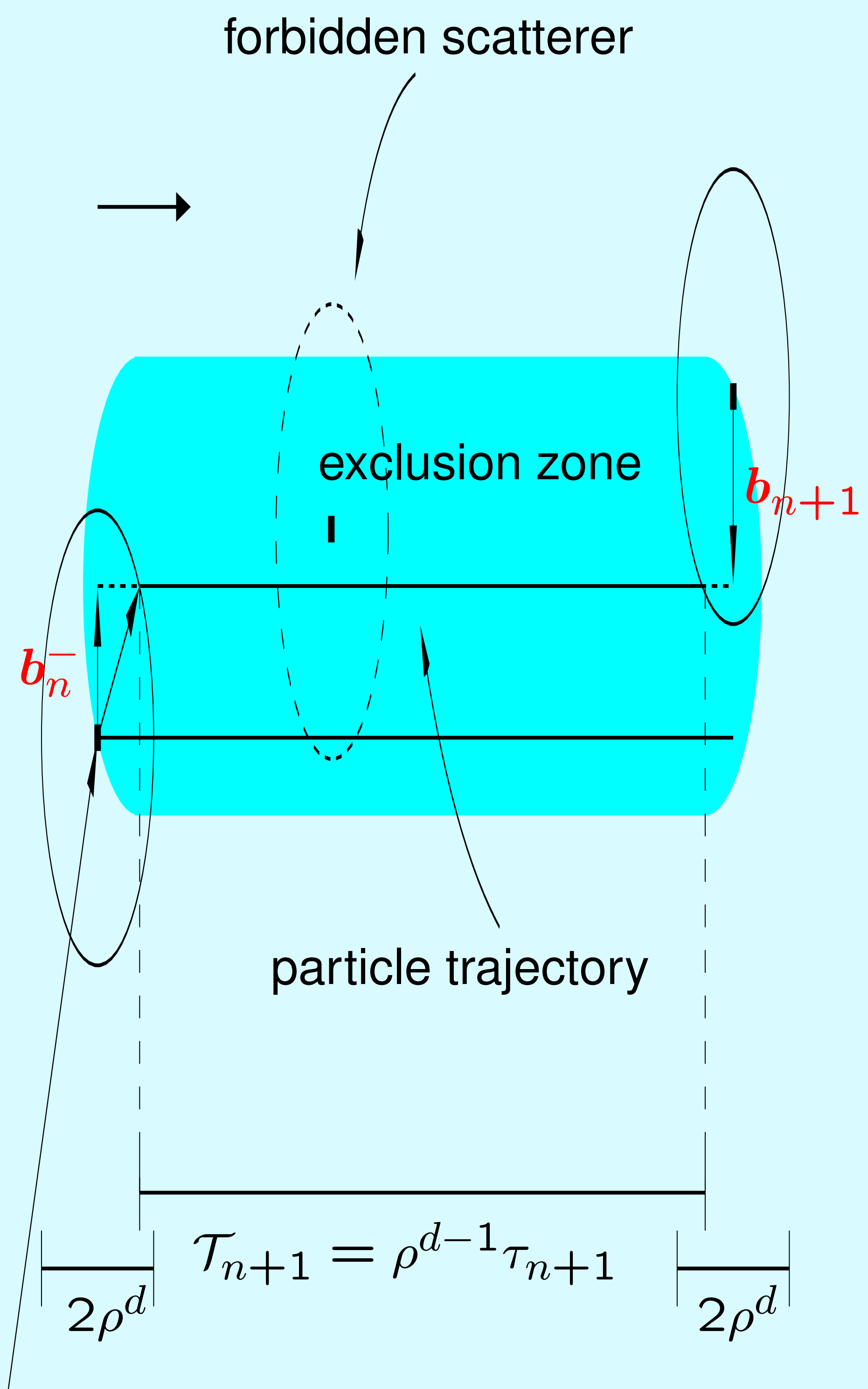
Applying $R(\boldsymbol{v}_n)D_\rho$ to this cylinder orients it along the e_1-axis and makes it well proportioned. First apply $R(\boldsymbol{v}_n)$.



It is important to keep track of the exit parameters $\boldsymbol{b}_n^-$ and impact parameters $\boldsymbol{b}_n$.



Now apply D_ρ.


forbidden scatterer
exclusion zone
b_{n+1}
b_n^-
particle trajectory
$\mathcal{T}_{n+1} = \rho^{d-1}\tau_{n+1}$
$2\rho^d$
$2\rho^d$

Step 2: Marking

- Under the above rescaling the cylinder converges to a $(\rho, \boldsymbol{v}_n)$-independent cyclinder (with flat caps).

- The point set $\mathcal{P}$ has been replaced by the random point set $(\mathcal{P} - \boldsymbol{y}_n) R(\boldsymbol{v}_n) D_\rho$.

- For $\boldsymbol{y}$ fixed and $\boldsymbol{v}$ random, limit distribution of $(\mathcal{P} - \boldsymbol{y}) R(\boldsymbol{v}) D_\rho$ can in general depend on $\boldsymbol{y} \in \mathcal{P}$. In order to keep track of this, we assign a **mark** to each $\boldsymbol{y}$; we want the space of marks to be "nice".

Assumptions on the scatterer configuration $\mathcal{P}$

We say $\mathcal{P}$ is **admissible** if there exists a compact metric space Σ with Borel probability measure m, and map $\varsigma : \mathcal{P} \to \Sigma$ (the marking) such that for

$$\mathcal{X} = \mathbb{R}^d \times \Sigma, \quad \mu_{\mathcal{X}} = \mathrm{vol} \times \mathsf{m}$$
$$\tilde{\mathcal{P}} = \{(\boldsymbol{y}, \varsigma(\boldsymbol{y})) : \boldsymbol{y} \in \mathcal{P})\} \subset \mathcal{X} \qquad \text{(the marked point set)}$$

we have

- **Assumption 1** (density)

$$\lim_{R\to\infty} \frac{\#(\tilde{\mathcal{P}} \cap R\mathcal{D})}{R^d} = \mu_{\mathcal{X}}(\mathcal{D})$$

  for all bounded sets $\mathcal{D} \subset \mathcal{X}$ with $\mu_{\mathcal{X}}(\partial\mathcal{D}) = 0$
- **Assumption 2** (spherical equidistribution) For $\boldsymbol{v}$ random according to λ a.c. w.r.t. vol measure on S_1^{d-1}

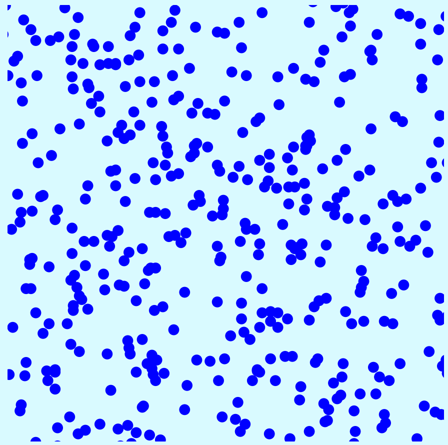
$$\tilde{\Xi}_{\rho,\boldsymbol{y}} = (\tilde{\mathcal{P}} - \boldsymbol{y}) R(\boldsymbol{v}) D_\rho \xrightarrow{\mathrm{d}} \tilde{\Xi}_{\varsigma(\boldsymbol{y})} \qquad (\rho \to 0)^*$$

  uniformly for all $\boldsymbol{y} \in \mathcal{P}$ in balls of radius $\asymp \rho^{1-d}$, where $\tilde{\Xi}_\varsigma$ **depends only on** $\varsigma \in \Sigma$
- ...and more

*for $M \in \mathrm{SL}(d, \mathbb{R})$ set $(\boldsymbol{y}, \varsigma(\boldsymbol{y}))M = (\boldsymbol{y}M, \varsigma(\boldsymbol{y}))$

Examples for admissible $\mathcal{P}$

Example 1: $\mathcal{P} =$ a **fixed** realization of the Poisson process in $\mathbb{R}^d$ with intensity 1, and $\Sigma = \{1\}$; proof that our assumptions are satisfied **a.s.** is non-trivial, follows ideas of Boldrighini, Bunimovich and Sinai (J Stat Phys 1983) $\rightarrow$ classical linear Boltzmann equation

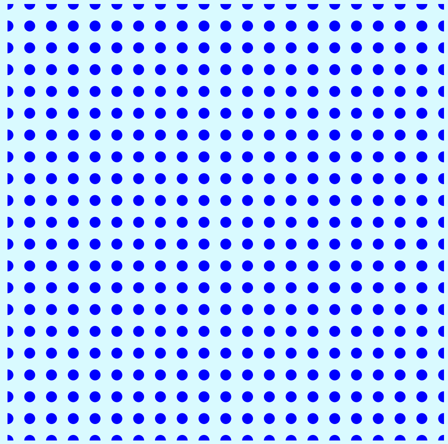


Previous results:

- Galavotti (Phys Rev 1969 & report 1972): Poisson distributed hard-sphere scatterer configuration $\mathcal{P}$
- Spohn (Comm Math Phys 1978): extension to more general random scatterer configurations $\mathcal{P}$ and potentials
- Boldrighini, Bunimovich and Sinai (J Stat Phys 1983): prove convergence for almost every FIXED scatterer configuration $\mathcal{P}$ (w.r.t. the Poisson random measure), for hard sphere scatterers

- Spohn (Comm Math Phys 1978): Implies CLT for limit process (standard CLT for Markovian random flight process)
- Lutsko & Tóth (CMP 2020): Intermediate joint Boltzmann-Grad/diffusive scaling limits

Examples for admissible $\mathcal{P}$



Example 2: The periodic Lorentz gas, $\mathcal{P} = \mathbb{Z}^d$ (or any other Euclidean lattice of co-volume 1) and $\Sigma = \{1\}$; proof uses spherical equidistribution on space of lattices

JM & Strömbergsson (Annals of Math 2010)

. . . more on this in the next lecture. . .

Previous results:

- Caglioti and Golse (Comptes Rendus 2008, J Stat Phys 2010)
- JM & Strömbergsson (Nonlinearity 2008, Annals of Math 2010/2011, GAFA 2011)

- Polya (Arch Math Phys 1918): "Visibility in a forest" $(d = 2)$
- Dahlquist (Nonlinearity 1997); Boca, Cobeli, Zaharescu (CMP 2000); Caglioti, Golse (CMP 2003); Boca, Gologan, Zaharescu (CMP 2003); Boca, Zaharescu (CMP 2007): Limit distributions for the free path lengths for various sets of initial data $(d = 2)$
- Dumas, Dumas, Golse (J Stat Phys 1997): Asymptotics of mean free path lengths $(d \geq 2)$
- Bourgain, Golse, Wennberg (CMP 1998); Golse, Wennberg (CMP 2000): bounds on possible weak limits $(d \geq 2)$

- Boca & Gologan (Annales I Fourier 2009), Boca (NY J Math 2010): honeycomb lattice

Examples for admissible $\mathcal{P}$

Example 3: $\mathcal{P} = \bigcup_{i=1}^{m} (\mathcal{L} + \boldsymbol{\alpha}_i)$ locally finite periodic point set (e.g. the honeycomb/hexagonal lattice), with $\mathcal{L}$ Euclidean lattice of covolume m; $\Sigma = \{1, 2, \ldots, m\}$. Admissible follows from spherical equidistribution, which here is a consequence of Ratner's measure classification theorem for $\mathrm{SL}(d,\mathbb{Z}) \ltimes (\mathbb{Z}^d)^m \backslash \mathrm{SL}(d,\mathbb{R}) \ltimes (\mathbb{R}^d)^m$.

JM & Strömbergsson (CMP 2014, Memoirs AMS 2024)

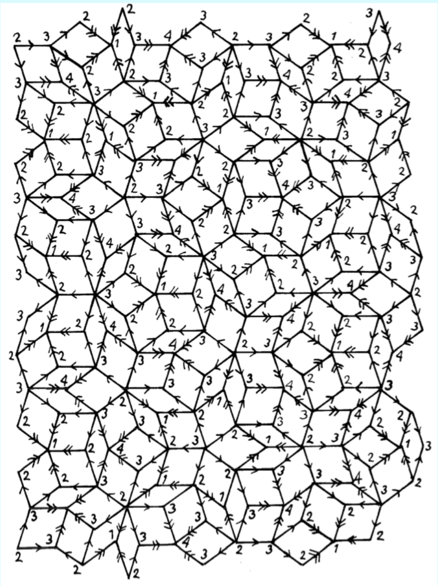
Previous results on free path length:

Boca & Gologan (Annales I Fourier 2009), Boca (NY J Math 2010)

Example 4: $\mathcal{P} =$ Euclidean cut-and-project set (e.g. the vertex set of a Penrose tiling) and $\Sigma \subset \mathbb{R}^k$ (the internal space in the c&p construction); proof of assumptions uses uses equidistribution of lower dimensional spheres in space of lattices, which is again a consequence of Ratner's measure classification theorem

JM & Strömbergsson (CMP 2014, Memoirs AMS 2024)

cf. also Ruhr, Smilansky & Weiss (JEMS 2023)



Examples for admissible $\mathcal{P}$

Example 5: $\mathcal{P} = \bigcup_{i=1}^{m} \mathcal{L}_i$ finite union of Euclidean lattices with $\mathcal{L}_i$ of positive covolume m_i; $\Sigma = \{1, 2, \ldots, m\}$. Admissible follows from spherical equidistribution, which here is a consequence of Ratner's measure classification theorem for $(\mathrm{SL}(d,\mathbb{Z})\backslash\,\mathrm{SL}(d,\mathbb{R}))^m$.

JM & Strömbergsson (J Stat Phys 2014): first collision only

Palmer & Strömbergsson (CMP 2024)

Example 6: $\mathcal{P}$ given by periodic Lorentz gas with random thinning or pertubation.

JM & Vinogradov (J Stat Phys 2014): first collision only

Extension to non-identical scatterers:

Avelin, preprint 2024

Step 3: Induction $\longrightarrow$ Main Theorem C

(Theorem C $\Rightarrow$ Theorem B $\Rightarrow$ Theorem A)

Theorem A
Let $\mathcal{P}$ be admissible. Then, for any $\Lambda \in P_{\mathrm{ac}}(\mathrm{T}^1(\mathbb{R}^d))$, there is a random flight process Θ such that $\Theta^{(\rho)}$ converges to Θ in distribution, as $\rho \to 0$.

Theorem B
Let $\mathcal{P}$ be admissible.
Then, for random initial data $(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ with law $\Lambda \in P_{\mathrm{ac}}(\mathrm{T}^1(\mathbb{R}^d))$,

$$\Big\langle \mathcal{T}_j(\boldsymbol{Q}_0, \boldsymbol{V}_0), \boldsymbol{V}_j(\boldsymbol{Q}_0, \boldsymbol{V}_0) \Big\rangle_{j=1}^{\infty}$$

converges in distribution to the random sequence

$$\Big\langle \xi_j, \boldsymbol{v}_j \Big\rangle_{j=1}^{\infty}$$

(which in general does not form a Markov chain).

Theorem C (JM & Strömbersson, Memoirs AMS 2024) Let $\mathcal{P}$ be admissible. Then, for random $(\boldsymbol{q}_0, \boldsymbol{v}_0) \sim \Lambda \in P_{\mathrm{ac}}(\mathrm{T}^1(\mathbb{R}^d))$, the random process

$$\mathbb{N} \to (\mathbb{R}_{>0} \cup \{\infty\}) \times \boldsymbol{\Sigma} \times \mathrm{S}_1^{d-1}$$
$$j \mapsto \Big(\mathcal{T}_j(\boldsymbol{q}_0, \boldsymbol{v}_0), \varsigma_j(\boldsymbol{q}_0, \boldsymbol{v}_0), \boldsymbol{V}_j(\boldsymbol{q}_0, \boldsymbol{v}_0)\Big)$$

converges in distribution to the second-order Markov process

$$j \mapsto \Big(\xi_j, \varsigma_j, \boldsymbol{v}_j\Big),$$

where for any Borel set $A \subset \mathbb{R}_{\geq 0} \times \boldsymbol{\Sigma} \times \mathrm{S}_1^{d-1}$,

$$\mathbb{P}\bigg((\xi_1, \varsigma_1, \boldsymbol{v}_1) \in A \,\bigg|\, (\boldsymbol{q}_0, \boldsymbol{v}_0)\bigg) = \int_A p(\boldsymbol{v}_0; \xi, \varsigma, \boldsymbol{v})\, d\xi\, d\mathrm{m}(\varsigma)\, d\boldsymbol{v},$$

and for $j \geq 2$,

$$\mathbb{P}\bigg((\xi_j, \varsigma_j, \boldsymbol{v}_j) \in A \,\bigg|\, (\boldsymbol{q}_0, \boldsymbol{v}_0),\ \Big\langle(\xi_i, \varsigma_i, \boldsymbol{v}_i)\Big\rangle_{i=1}^{j-1}\bigg)$$
$$= \int_A p_{\mathbf{0}}(\boldsymbol{v}_{j-2}, \varsigma_{j-1}, \boldsymbol{v}_{j-1}; \xi, \varsigma, \boldsymbol{v})\, d\xi\, d\mathrm{m}(\varsigma)\, d\boldsymbol{v}.$$

The functions $p, p_{\mathbf{0}}$ depend on $\mathcal{P}$ but are independent of Λ, and for any fixed $\boldsymbol{v}_0, \varsigma, \boldsymbol{v}$ both $p(\boldsymbol{v}_0\,;\,\cdot\,)$ and $p_{\mathbf{0}}(\boldsymbol{v}_0, \varsigma, \boldsymbol{v}\,;\,\cdot\,)$ are probability densities on $\mathbb{R}_{\geq 0} \times \boldsymbol{\Sigma} \times \mathrm{S}_1^{d-1}$. In particular $\mathbb{P}(\xi_j = \infty) = 0$ for all j.

(Theorem A $\Rightarrow$) Evolution of densities

Recall: a cloud of particles with initial density $f(\boldsymbol{Q},\boldsymbol{V})$ evolves in time t to

$$[L_\rho^t f](\boldsymbol{Q},\boldsymbol{V}) = f\Big(\Phi_\rho^{-t}(\boldsymbol{Q},\boldsymbol{V})\Big).$$

Theorem D (JM & Strömbersson, Memoirs AMS 2024)
Let $\mathcal{P}$ be admissible. Then for every $t>0$ there exists a linear operator

$$L^t : \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d)) \to \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$$

such that for every $f \in \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$ and any set $\mathcal{A} \subset \mathrm{T}^1(\mathbb{R}^d)$ with boundary of Liouville measure zero,

$$\lim_{\rho\to 0} \int_{\mathcal{A}} [L_\rho^t f](\boldsymbol{Q},\boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V} = \int_{\mathcal{A}} [L^t f](\boldsymbol{Q},\boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V}.$$

The operator L^t thus describes the macroscopic diffusion of the Lorentz gas in the Boltzmann-Grad limit $\rho \to 0$. (We in fact prove convergence of the Lorentz process to a random flight process.)

Note: The family $\{L^t\}_{t\geq 0}$ does in general *not* form a semigroup.

(Theorem C $\Rightarrow$) **A generalized linear Boltzmann equation**

Consider extended phase space coordinates $(\boldsymbol{Q}, \boldsymbol{V}, \varsigma, \xi, \boldsymbol{V}_+)$:

$(\boldsymbol{Q}, \boldsymbol{V}) \in \mathrm{T}^1(\mathbb{R}^d)$ — usual position and momentum
$\varsigma \in \Sigma$ — the mark of current scatterer location
$\xi \in \mathbb{R}_+$ — flight time until the next scatterer
$\boldsymbol{V}_+ \in \mathrm{S}_1^{d-1}$ — velocity after the next hit

$$\left[\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{Q}} - \frac{\partial}{\partial \xi}\right] f_t(\boldsymbol{Q}, \boldsymbol{V}, \varsigma, \xi, \boldsymbol{V}_+)$$
$$= \int_\Sigma \int_{\mathrm{S}_1^{d-1}} f_t(\boldsymbol{Q}, \boldsymbol{V}', \varsigma', 0, \boldsymbol{V})\, p_{\boldsymbol{0}}(\boldsymbol{V}', \varsigma', \boldsymbol{V}, \varsigma, \xi, \boldsymbol{V}_+) d\boldsymbol{V}' d\mathrm{m}(\varsigma').$$

with a collision kernel $p_{\boldsymbol{0}}(\boldsymbol{V}', \varsigma', \boldsymbol{V}, \varsigma, \xi, \boldsymbol{V}_+)$, which can be expressed as a product of the scattering cross section of an individual scatterer and a certain transition probability for hitting a given point on the next scatterer with mark ς after time ξ, given the present scatterer has mark ς'.

Summary

- Under natural assumptions on the scatterer configuration $\mathcal{P}$, we can prove convergence to a limiting random flight process as $\rho \to 0$ (Boltzmann-Grad limit).

- The limiting process depends on the scatterer configuration, and requires to keep track of some microscopic data (next velocity and flight time, marking).

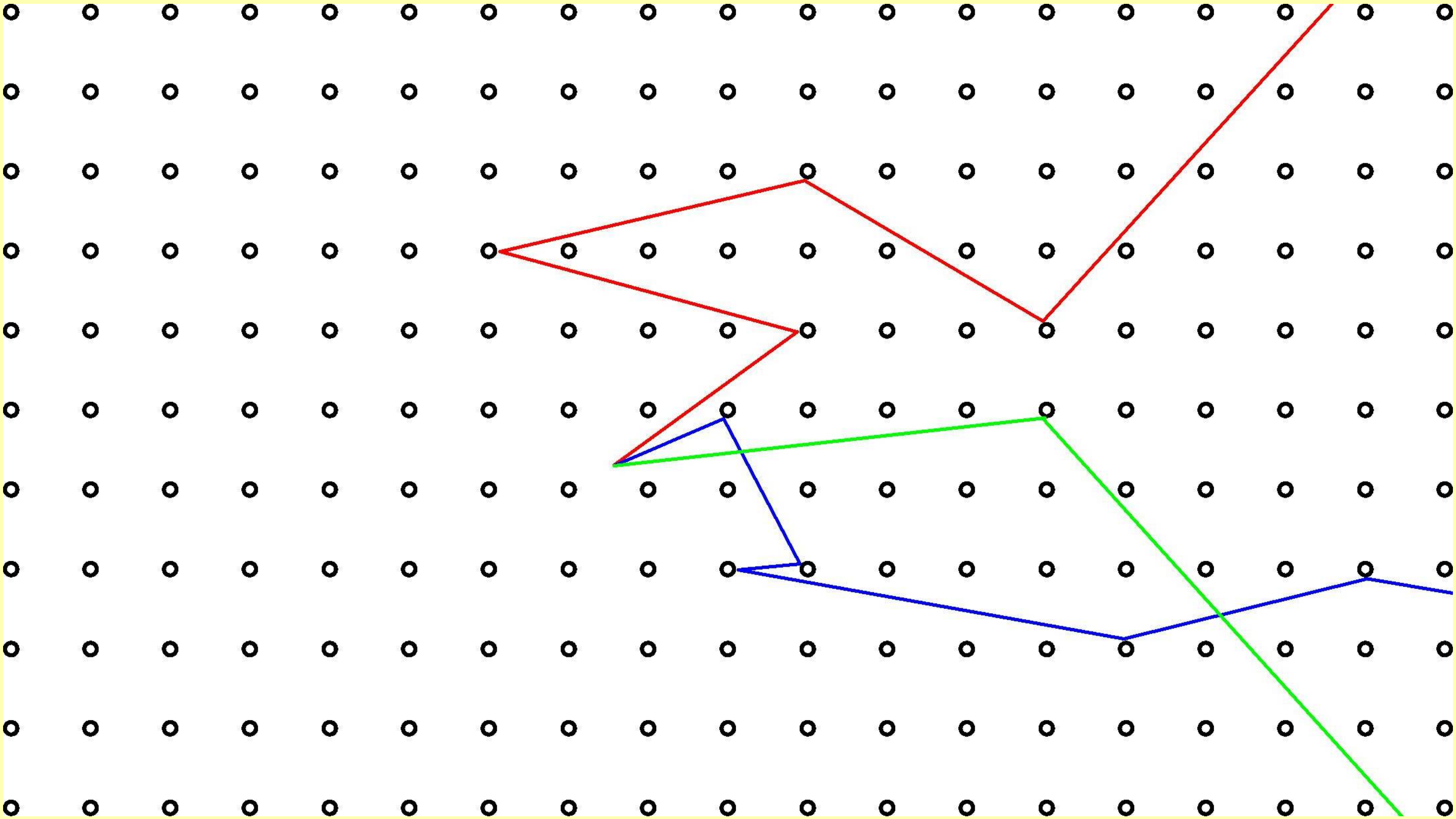
What’s next?

- Explicit limit distributions in the periodic setting, superdiffusion, entropy
- Connection with three gap theorem, higher-dimensional analogues
- Directional statistics, dynamics on point sets
- Open questions, future challenges

Part 2

The periodic Lorentz gas: limit distributions, superdiffusion, entropy, and higher dimensional three gap theorems

The periodic Lorentz gas



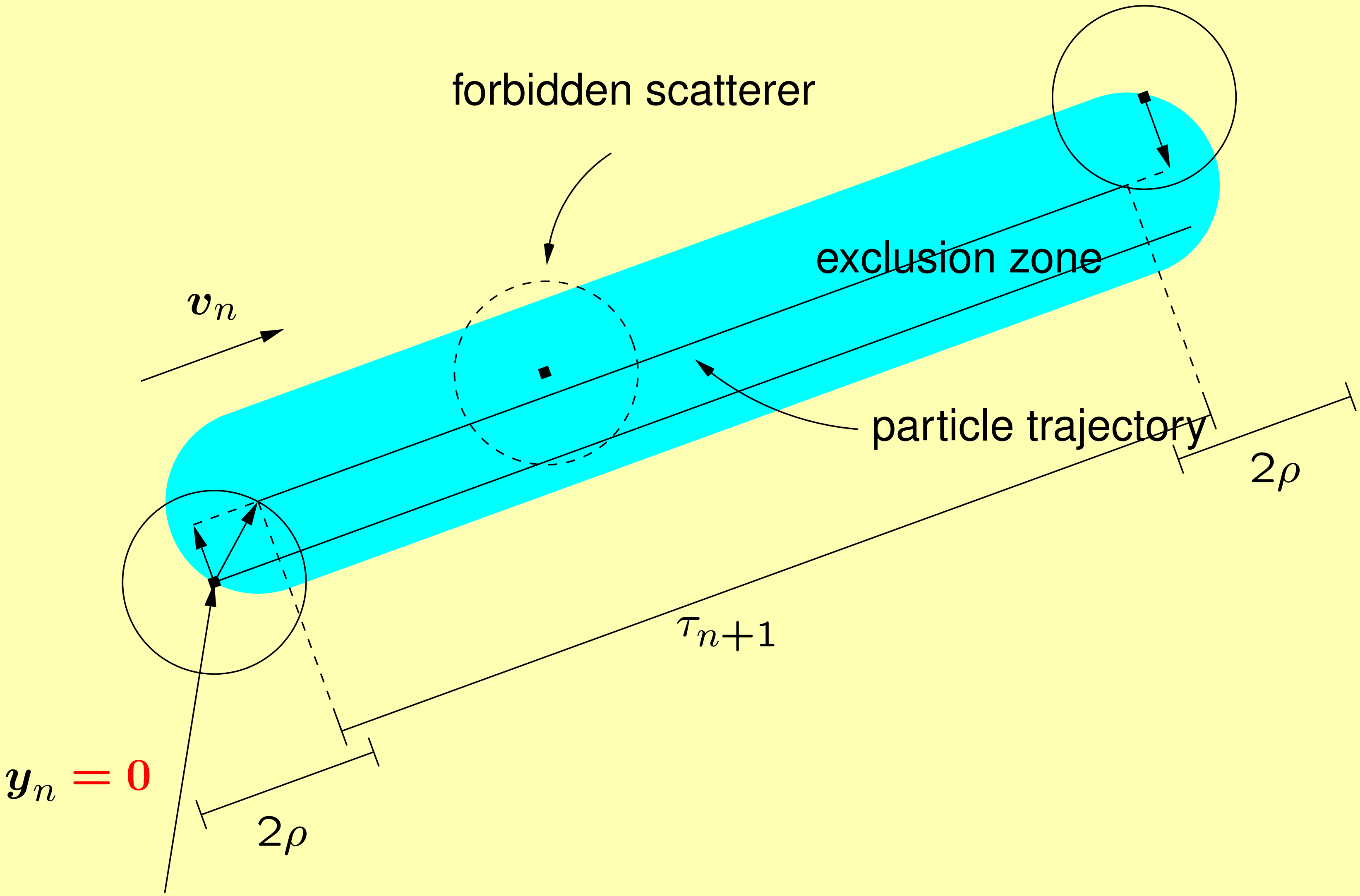
Recall the three steps, now for special case $\mathcal{P} = \mathbb{Z}^d$ (or general Euclidean lattice $\mathcal{L}$):

1. **Rescaling** and spherical equidistribution for each individual inter-collision flight
2. Markovianisation of the limit process ~~through introduction of a **marking** of $\mathcal{P}$~~
3. **Induction** on the number of inter-collision flights

Step 1: Rescaling

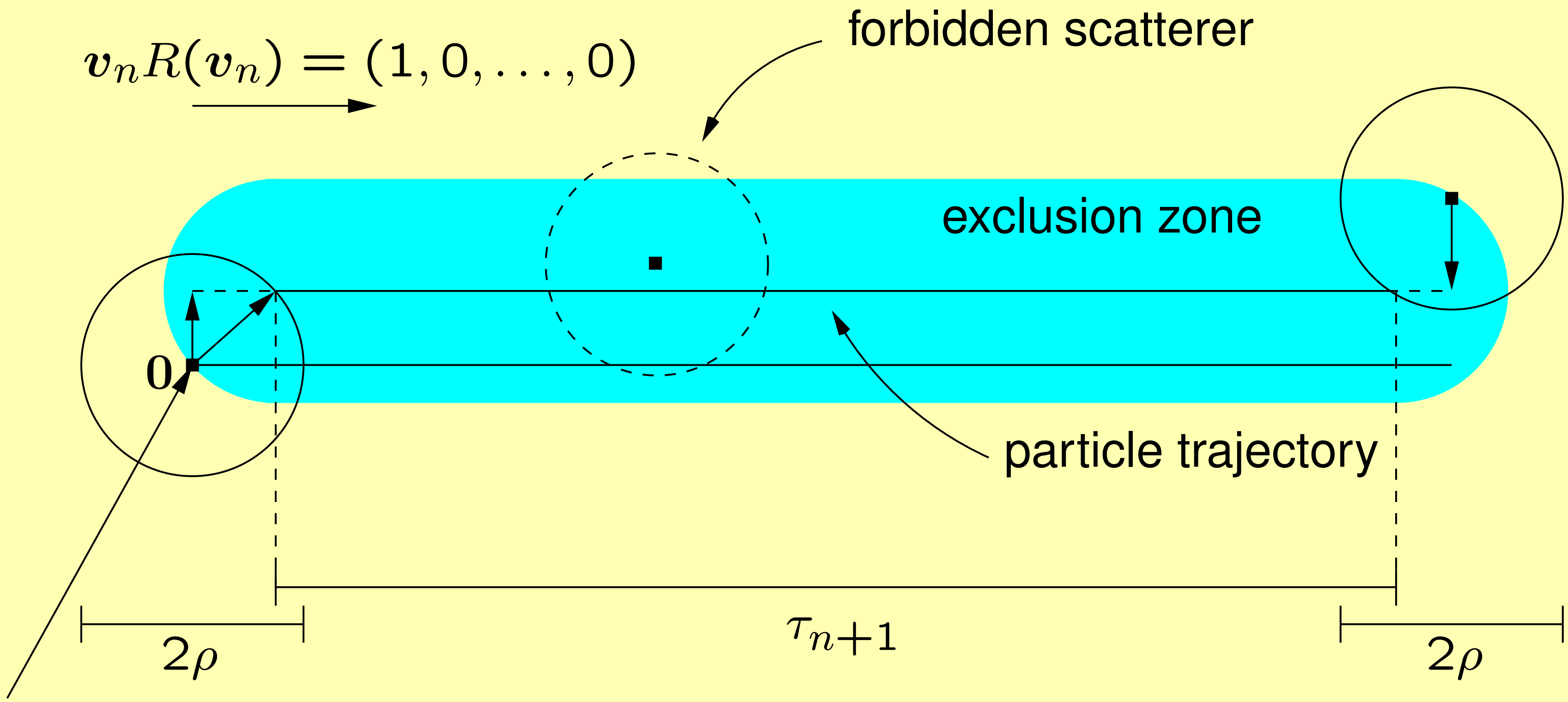
Define $R(\boldsymbol{v}) : \mathsf{S}_1^{d-1} \to \mathsf{SO}(d)$ such that $\boldsymbol{v}R(\boldsymbol{v}) = \boldsymbol{e}_1 = (1, 0, \ldots, 0)$ and

$$D_\rho = \begin{pmatrix} \rho^{d-1} & \mathbf{0} \\ {}^{\mathrm{t}}\mathbf{0} & \rho^{-1} 1_{d-1} \end{pmatrix} \in \mathsf{SL}(d, \mathbb{R}).$$

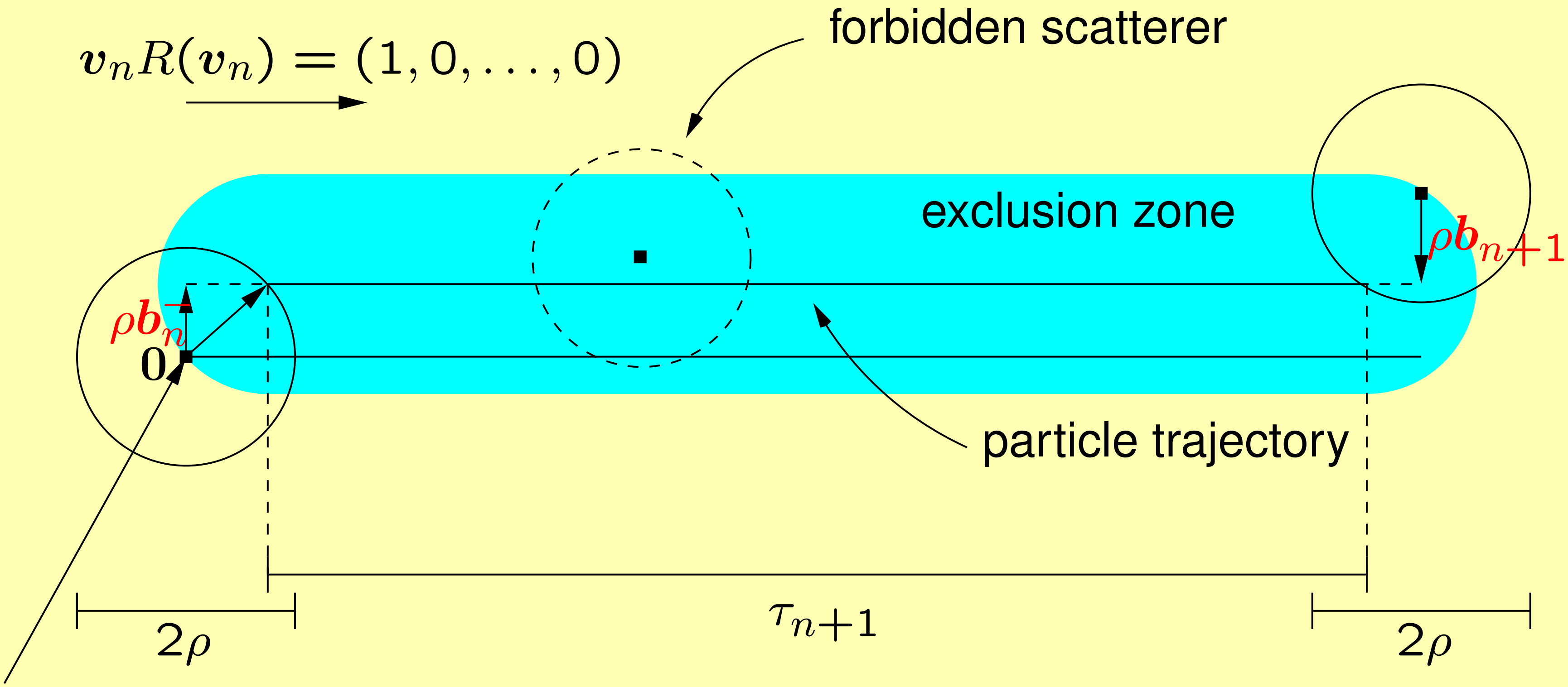


Applying $R(\boldsymbol{v}_n)D_\rho$ to this cylinder orients it along the e_1-axis and makes it well proportioned. First apply $R(\boldsymbol{v}_n)$.

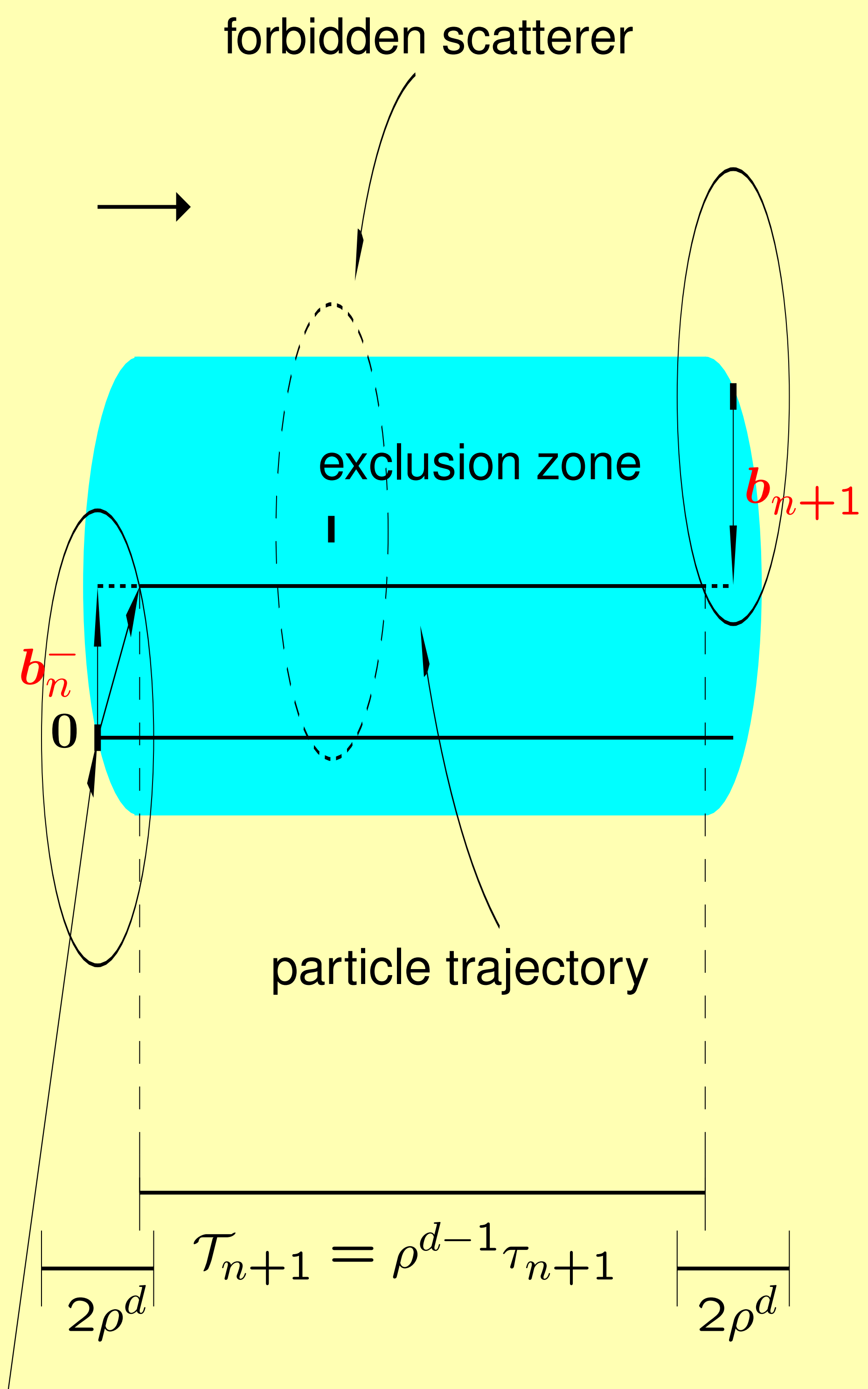
By translation-invariance, can assume w.l.o.g. scatterer is at origin, i.e. $\boldsymbol{y}_n = \boldsymbol{0}$.



It is important to keep track of the exit parameters $\boldsymbol{b}_n^-$ and impact parameters $\boldsymbol{b}_n$.



Now apply D_ρ.


forbidden scatterer
exclusion zone
b_{n+1}
b_n^-
0
particle trajectory
$\mathcal{T}_{n+1} = \rho^{d-1}\tau_{n+1}$
$2\rho^d$
$2\rho^d$

Spherical equidistribution in the space of lattices

- $\mathcal{L} \subset \mathbb{R}^d$—euclidean lattice of covolume one
- recall $\mathcal{L} = \mathbb{Z}^d M$ for some $M \in \mathsf{SL}(d,\mathbb{R})$, therefore the homogeneous space $X_1 = \mathsf{SL}(d,\mathbb{Z})\backslash \mathsf{SL}(d,\mathbb{R})$ parametrizes the space of lattices of covolume one
- μ_1—right-$\mathsf{SL}(d,\mathbb{R})$ invariant prob measure on X_1 (Haar)
- The following Theorem shows that in the limit $\rho \to 0$ the lattice $\mathbb{Z}^d K(\boldsymbol{v}) D_\rho$ behaves like a random lattice with respect to Haar measure μ_1

Theorem E. Let λ be an a.c. Borel probability measure on S_1^{d-1}. Then, for every bounded continuous function $f : X_1 \to \mathbb{R}$,

$$\lim_{\rho\to 0} \int_{\mathsf{S}_1^{d-1}} f(K(\boldsymbol{v})D_\rho)\, d\lambda(\boldsymbol{v}) = \int_{X_1} f(M)\, d\mu_1(M).$$

Theorem E is a direct consequence of the mixing property for the flow given by right multiplication by $D_{\exp(-t)}$.

$\Rightarrow$ Assumption 2 on spherical equidistribution of $\mathcal{P}$ is satisfied, and we hence we have established that the Lorentz converges in the Boltzmann-Grad limit:

Theorem C (for $\mathcal{P} = \mathcal{L}$) (JM & Strömbersson, Annals Math 2011)
For any $\Lambda \in P_{\mathrm{ac}}(\mathrm{T}^1(\mathbb{R}^d))$, the random process

$$\mathbb{N} \to (\mathbb{R}_{>0} \cup \{\infty\}) \times \mathrm{S}_1^{d-1}$$
$$j \mapsto \big(\mathcal{T}_j(\boldsymbol{q}_0, \boldsymbol{v}_0), \boldsymbol{v}_j(\boldsymbol{q}_0, \boldsymbol{v}_0)\big)$$

converges in distribution to the second-order Markov process $j \mapsto \big(\xi_j, \boldsymbol{v}_j\big)$ where for any Borel set $A \subset \mathbb{R}_{\geq 0} \times \mathrm{S}_1^{d-1}$,

$$\mathbb{P}\Big((\xi_1, \boldsymbol{v}_1) \in A \,\Big|\, (\boldsymbol{q}_0, \boldsymbol{v}_0)\Big) = \int_A p(\boldsymbol{v}_0; \xi, \boldsymbol{v})\, d\xi\, d\boldsymbol{v},$$

and for $j \geq 2$,

$$\mathbb{P}\Big((\xi_j, \boldsymbol{v}_j) \in A \,\Big|\, (\boldsymbol{q}_0, \boldsymbol{v}_0),\ \big\langle(\xi_i, \boldsymbol{v}_i)\big\rangle_{i=1}^{j-1}\Big) = \int_A p_0(\boldsymbol{v}_{j-2}, \boldsymbol{v}_{j-1}; \xi, \boldsymbol{v})\, d\xi\, d\boldsymbol{v}.$$

Note: If we condition on the velocities $\boldsymbol{v}_j$, the ξ_j form a sequence of independent (but not identically distributed) random variables. This is a key observation in the proof of. . .

Superdiffusive central limit theorem

Theorem F (JM & B. Tóth, CMP 2016)
Let $d \geq 2$ and fix a Euclidean lattice $\mathcal{L} \subset \mathbb{R}^d$ of covolume one. Assume $(\boldsymbol{q}_0, \boldsymbol{v}_0)$ random with distribution $\Lambda \in P_{\text{ac}}(\mathrm{T}^1(\mathbb{R}^d))$. Then, for any bounded continuous $f : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{t\to\infty} \lim_{\rho\to 0} \mathbb{E} f\left(\frac{\boldsymbol{q}(t) - \boldsymbol{q}_0}{\Sigma_d \sqrt{t \log t}}\right) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \mathrm{e}^{-\frac{1}{2}\|\boldsymbol{x}\|^2} d\boldsymbol{x},$$

with $\Sigma_d^2 := \frac{2^{1-d}\overline{\sigma}}{d^2(d+1)\zeta(d)}$.

For fixed ρ the analogous result is currently known only in dimension $d = 2$, see Szász & Varjú (J Stat Phys 2007), Chernov & Dolgopyat (Russ. Math Surveys 2009); Bálint, Bruin & Terhesiu (PTRF): small scatterers in intermediate scaling

Proof uses the Lindeberg CLT on the sums of independent random variables $\sum_{j=1}^n \xi_j \boldsymbol{v}_{j-1} \mathbb{1}(\xi_j^2 \leq j(\log j)^{1.99})$, conditioned on the $\boldsymbol{v}_j$, plus tail estimates on the collision times.

<u>Note:</u> $\lim_{t\to\infty} \lim_{\rho\to 0} \mathbb{E} \left\| \frac{\boldsymbol{q}(t) - \boldsymbol{q}_0}{\Sigma_d \sqrt{t \log t}} \right\|^2 = 2 \neq 1$ (cf. Balint, Chernov & Dolgopyat: Billiards with cusps)

Heavy tails of the path length distribution

For random initial data $(\boldsymbol{q}, \boldsymbol{v})$ leaving a scatterer (with a.c. distribution w.r.t. invariant measure of scattering map), consider the probability $F_\rho(\xi)$ of hitting the next scatterer at time $\tau_1(\boldsymbol{q}, \boldsymbol{v}; \rho) > \rho^{1-d}\xi$

- Boca, Zaharescu (CMP 2007): proof of convergence as $\rho \to 0$ and explicit formula in dimension $d = 2$

- Special case of Thm C: convergence

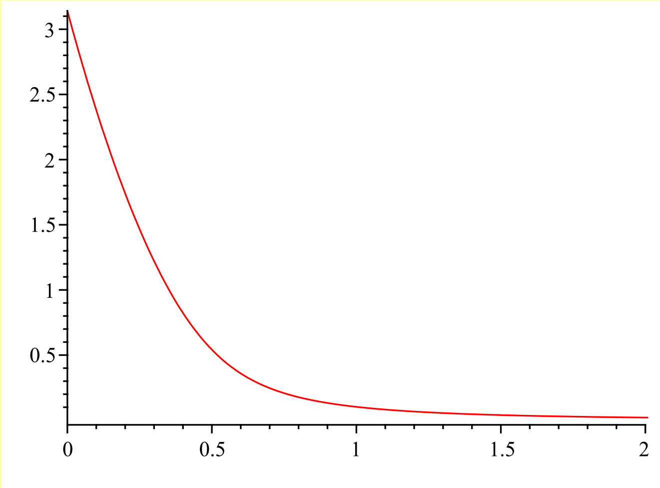
$$F_\rho(\xi) \to F(\xi) = \int_\xi^\infty \Psi_0(x)dx$$

in arbitrary dimension, with continuous limit density and tail ($\xi \to \infty$)

$$\Psi_0(\xi) \sim \frac{A_d}{\xi^3} \qquad A_d = \frac{2^{2-d}}{d(d+1)\zeta(d)} \qquad (*)$$

(JM & Strömbergsson, Annals 2010, GAFA 2011)

$\Rightarrow$ No second moment...superdiffusion!



$F(\xi)$ in dim. $d = 3$

Chernov's entropy asymptotics

NEW PROOF OF SINAI'S FORMULA FOR THE ENTROPY OF HYPERBOLIC BILLIARD SYSTEMS. APPLICATION TO LORENTZ GASES AND BUNIMOVICH STADIUMS

UDC 517.53+517.57

N. I. Chernov

§1. INTRODUCTION

4.1. One Spherical Scatterer. Assume that we have exactly one scatterer of radius r.

Proposition 4.1. If the radius r is sufficiently small, the entropy of the derived mapping T is

$$h(T) = -d(d-1)\ln r + O(1), \tag{4.1}$$

and the entropy of the phase flow is

We will now prove formula (4.1). By (2.7), the operator $\mathscr{B}(x)$ at the point $x = (q,v) \in M$ can be written in the form $\mathscr{B}(x) = \Theta_1 + (\tau_1 I + \ldots)^{-1} = \Theta_1 + \Delta_{\mathscr{B}}$, where Θ_1 is the operator of the form (2.4) associated with the mapping at the time t = 0. It is easy to see that Θ_1 has $2r^{-1}(v,n(q))^{-1}$ for an eigenvalue of multiplicity one, and $2r^{-1}(v,n(q))$ for an eigenvalue of multiplicity of two. We write $\tilde\Theta = r\Theta_1$ and, by (1.2), write

$$h(T) = \langle \ln\det(I + \tau r^{-1}\tilde\Theta_1 + \tau\Delta_{\mathscr{B}})\rangle = \langle \ln(\tau r^{-1})^{d-1}\rangle + \\ + \langle \ln\det(\tilde\Theta_1 + r\tau^{-1}I + r\Delta_{\mathscr{B}})\rangle = (d-1)[-\ln r + \langle \ln\tau\rangle] + \Delta_h.$$

It is clear that $\tau(x)$ is nonzero for sufficiently small r, so $\|\Delta_{\mathscr{B}}\| \leqslant$ const. It is not difficult to use this to show that

$$\lim_{r\to 0}\Delta_h = H(d) = \text{const}. \tag{4.4}$$

In addition, H(d) can be explicitly evaluated: H(2) = 1, H(3) = ln 4, and for d ≥ 4, $d \geqslant 4$ $H(d) = \ln 2^{d-1} - (d-3)\langle\ln(v, n(q))\rangle = \ln 2^{d-1} - (d-3)\,|\,S^{d-2}\,|\left(\int_0^1 t^{d-2}\ln\sqrt{1-t^2}\,dt\right)$.

Entropy and moments

Let $\nu_\rho(\boldsymbol{q}, \boldsymbol{v})$ the invariant measure for the scatterering map T of the Lorentz gas.

Proposition

For $-1 < \operatorname{Re}\alpha < 2$,

$$\lim_{\rho\to 0} \rho^{(d-1)\alpha} \int \tau_1(\boldsymbol{q}, \boldsymbol{v}; \rho)^\alpha \, d\nu_\rho(\boldsymbol{q}, \boldsymbol{v}) = \int_0^\infty \xi^\alpha \Psi_0(\xi) d\xi$$

Proof: Use escape of mass estimates (cf. last lecture).

Corollary: For $\rho \to 0$,

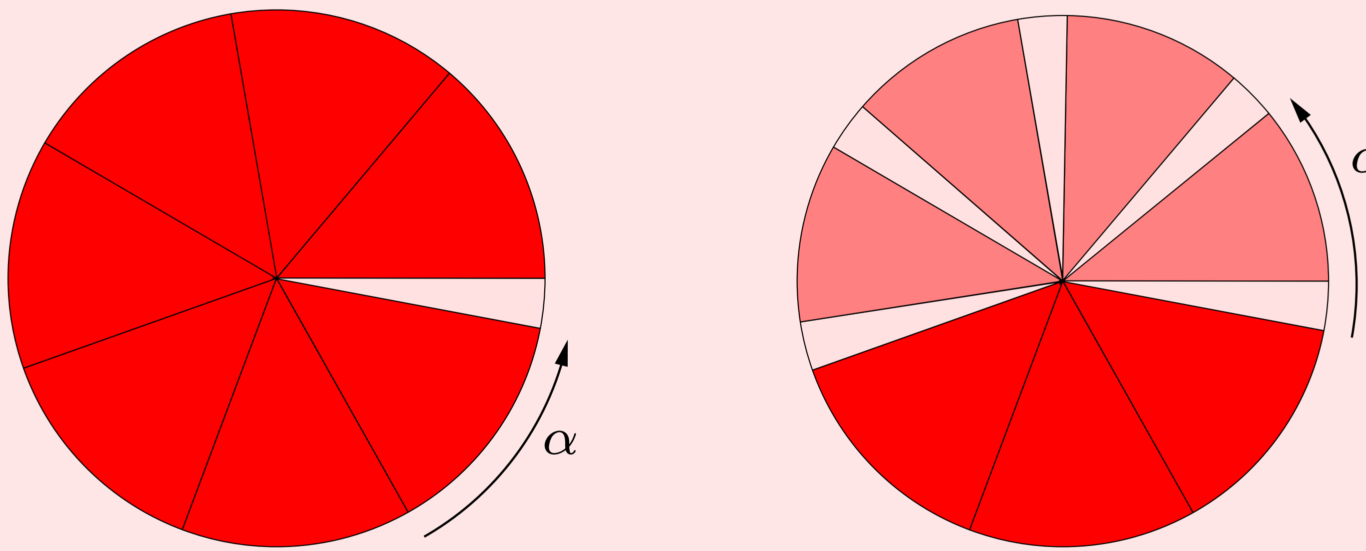
$$\int \log \tau_1(\boldsymbol{q}, \boldsymbol{v}; \rho) \, d\nu_\rho(\boldsymbol{q}, \boldsymbol{v}) = -(d-1) \log \rho + \int_0^\infty \log \xi \, \Psi_0(\xi) d\xi + o(1)$$

$$h(T) = -d(d-1) \log \rho + \int_0^\infty \log \xi \, \Psi_0(\xi) d\xi + H(d) + o(1)$$

[An interlude: The three gap theorem and higher dimensional versions]

The three gap theorem (Steinhaus conjecture)

"There are at most three distinct gap lengths in the fractional parts of the sequence $\alpha, 2\alpha, \ldots, N\alpha$, for any integer N and real number α."



Sós (1957), Surányi (1958), Świerczkowski (1959)

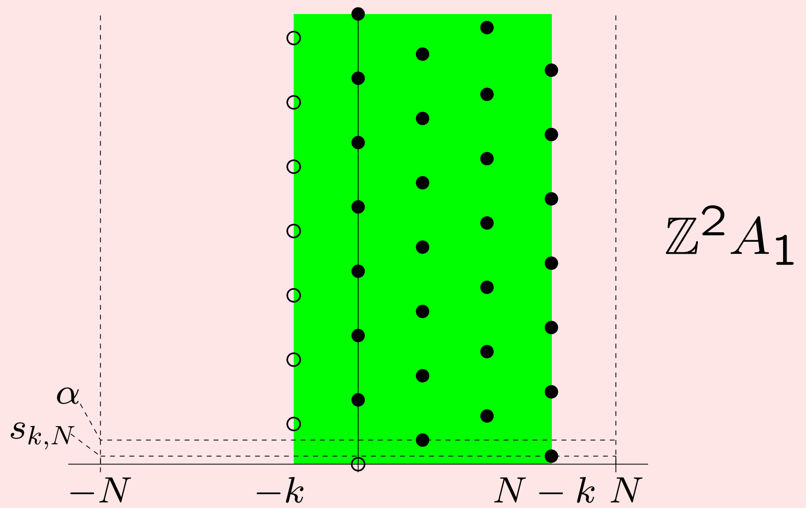
The three gap theorem and the space of lattices

JM & Strömbergsson (American Math. Monthly 2017)

The gap between $\xi_k = k\alpha \bmod 1$ and its *next* neighbour on $\mathbb{R}/\mathbb{Z}$ is given by

$$\begin{aligned} s_{k,N} &= \min\{(\ell-k)\alpha + n > 0 \mid (\ell,n)\in\mathbb{Z}^2,\ 0<\ell\leq N\} \\ &= \min\{m\alpha + n > 0 \mid (m,n)\in\mathbb{Z}^2,\ -k<m\leq N-k\} \\ &= \min\{y > 0 \mid (x,y)\in\mathbb{Z}^2 A_1,\ -k<x\leq N-k\}, \end{aligned}$$

with the matrix $A_1 = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$.



An $\mathsf{SL}(2,\mathbb{Z})$-invariant function

Set $G = \mathsf{SL}(2,\mathbb{R})$, $\Gamma = \mathsf{SL}(2,\mathbb{Z})$.

For $M \in G$, $0 < t \leq 1$, define

$$F(M,t) = \min \left\{ y > 0 \;\middle|\; (x,y) \in \mathbb{Z}^2 M,\ -t < x \leq 1 - t \right\}.$$

Key point:

$$\begin{aligned} s_{k,N} &= \frac{1}{N} \min \left\{ y > 0 \;\middle|\; (x,y) \in \mathbb{Z}^2 A_N,\ -\frac{k}{N} < x \leq 1 - \frac{k}{N} \right\} \\ &= \frac{1}{N} F\left(A_N, \frac{k}{N} \right) \end{aligned}$$

with $A_N = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N^{-1} & 0 \\ 0 & N \end{pmatrix} \in G$

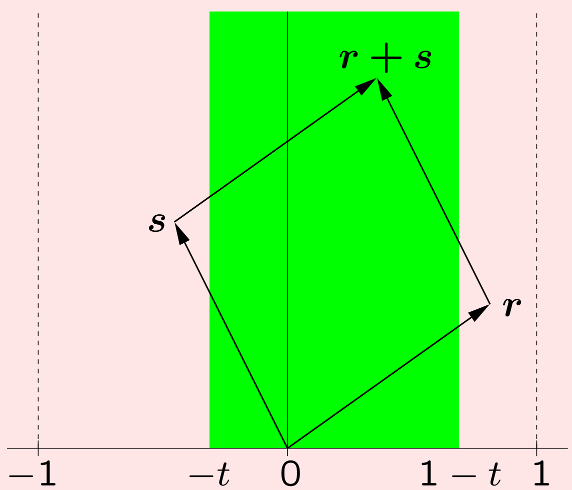
An $\mathsf{SL}(2,\mathbb{Z})$-invariant function

$$F(M,t) = \min\left\{y>0 \,\middle|\, (x,y)\in\mathbb{Z}^2M,\ -t<x\le 1-t\right\}.$$

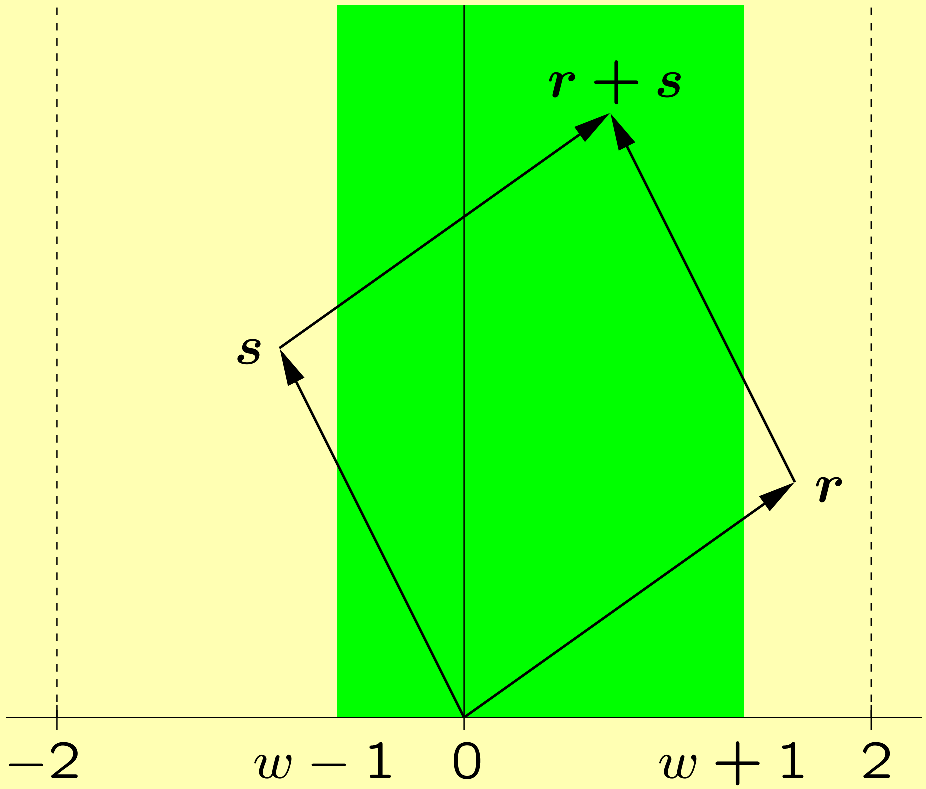
Proposition 1. F is well-defined as a function $\Gamma\backslash G\times(0,1]\to\mathbb{R}_{>0}$.

Proposition 2. For every given $M\in G$, the function $t\mapsto F(M,t)$ is piecewise constant and takes at most three distinct values. If there are three values, then the third is the sum of the first and second.

Proof:



Connection with the Lorentz gas?



The two linearly independent lattice vectors with lowest and second-lowest heights in the vertical strip between -2 and 2 form a basis. One can show that at any vertical strip of width one (in green) contains at least one of the three points, and hence the minimal height vector is either $\boldsymbol{r}$, $\boldsymbol{s}$ or $\boldsymbol{r}+\boldsymbol{s}$

JM & Strömbergsson, The three gap theorem and the space of lattices, American Math. Monthly 2017

- If $\mathbb{Z}^2 M$ is a Haar random lattice, then the minimal height vector $\boldsymbol{y} = (z, \xi)$ in the **green strip** $(w-1, w+1) \times \mathbb{R}_{>0}$ is distributed according the probability density

$$K_w(z,\xi) = \frac{6}{\pi^2}\, H\left(1 + \frac{\xi^{-1} - \max\big(|w|, |z-w|\big) - 1}{|z|}\right)$$

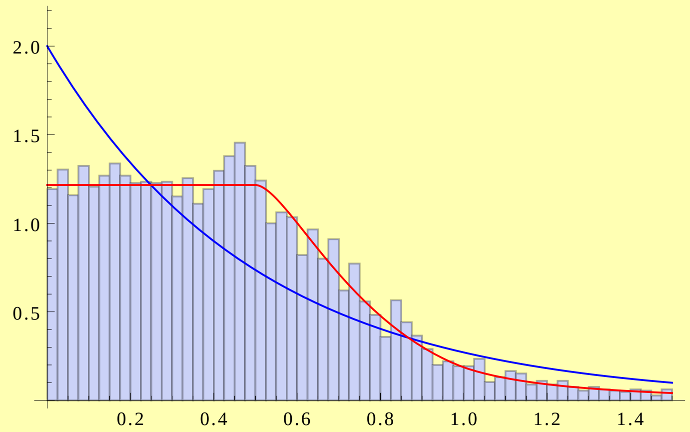
$$H(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \leq x. \end{cases}$$

- If we average the distribution over $w \in [-\frac{1}{2}, \frac{1}{2}]$ and z, we obtain the free path length distribution

$$\Psi_0(\xi) = \frac{12}{\pi^2} \times \begin{cases} 1 & (\xi \leq \frac{1}{2}) \\ \frac{1}{\xi} + 2\left(1 - \frac{1}{2\xi}\right)^2 \log\left(1 - \frac{1}{2\xi}\right) & \\ \qquad -\frac{1}{2}\left(1 - \frac{1}{\xi}\right)^2 \log\left|1 - \frac{1}{\xi}\right| & (\xi > \frac{1}{2}). \end{cases}$$

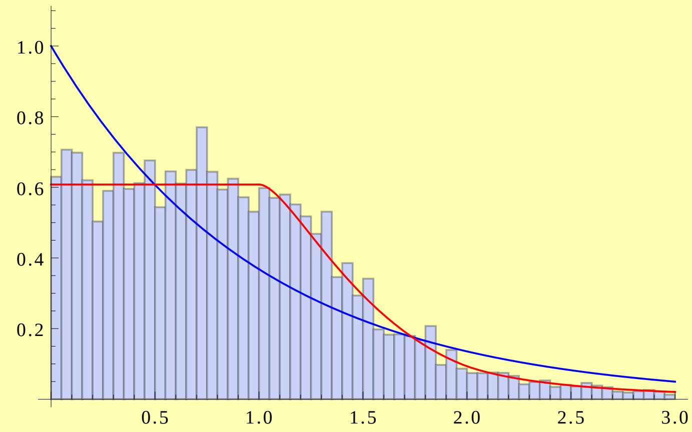
Greenman (J Phys A 1996), Dahlqvist (Nonlinearity 1997), Boca & Zaharescu (CMP 2007), JM & Strömbersson (Nonlinearity 2008)

Free paths langths vs. random three gaps*



The distribution of free path length in the periodic Lorentz gas vs. $P(s)$, e^{-s}

P. Dahlqvist 1997
F. Boca & A. Zaharescu 2007
using lattices:
JM & A. Strömbergsson 2008



The gap distribution in the energy spectrum of a two-dimensional harmonic oscillator with random frequencies = random three gaps

C. Greenman 1996
using lattices: JM 2000

*For more on this see JM, Random lattices in the wild: from Polya's orchard to quantum oscillators, LMS Newsletter, Issue 493 (2021)

Higher dimensional generalisations: Kronecker sequences

- Fix $\vec{\alpha} \in \mathbb{R}^d$, multidimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$

- Consider distances between points $\xi_n = n\vec{\alpha} \in \mathbb{T}^d$, $n = 1, \ldots, N$

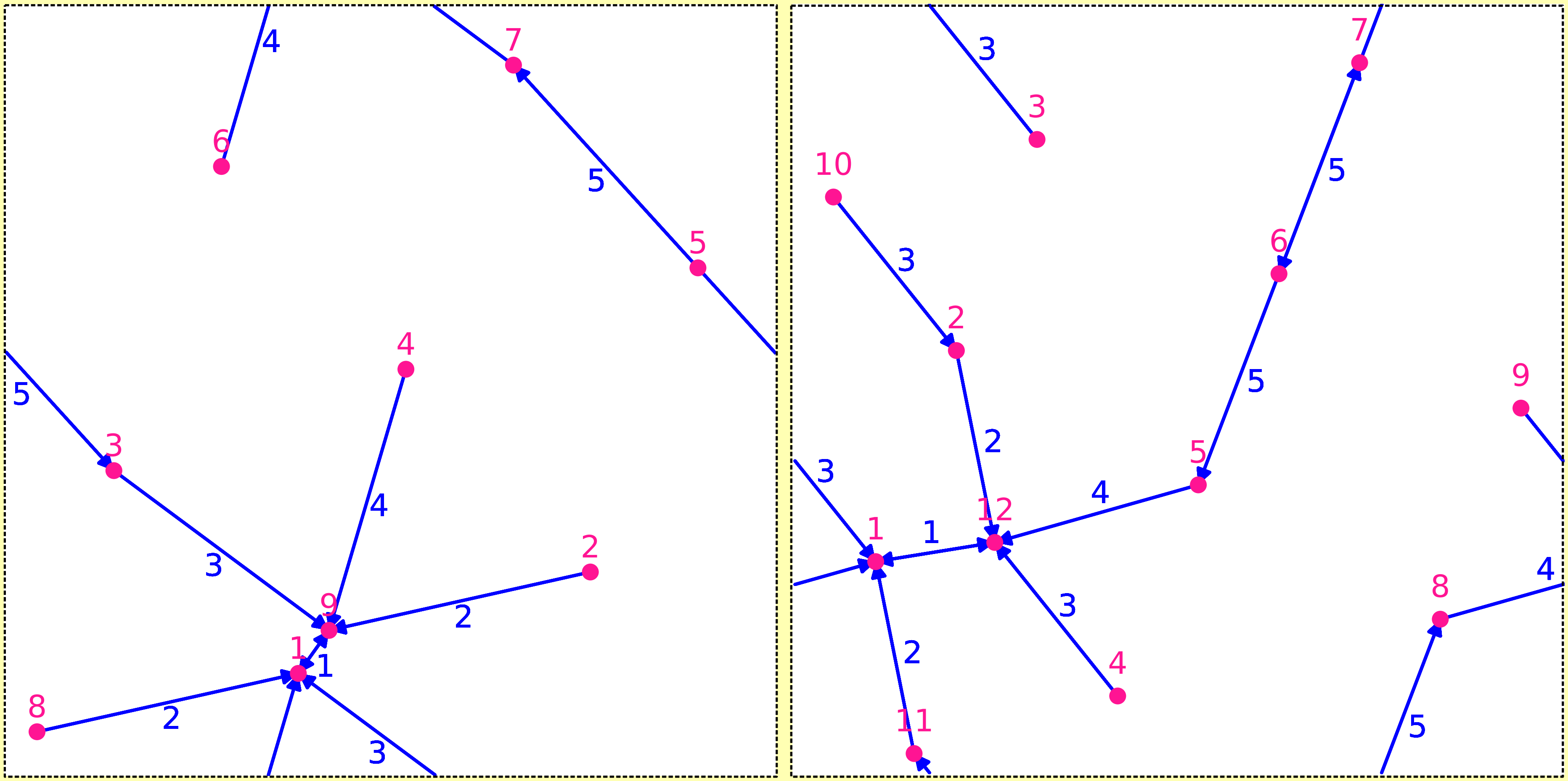
- $\delta_{n,N} = \min\{|\xi_m - \xi_n + \ell| > 0 \mid 1 \leq m \leq N,\ \ell \in \mathbb{Z}^d\}$

  (= distance of ξ_n to its nearest neighbour, $|\cdot|$ denotes Euclidean norm in $\mathbb{R}^d$)

- Number of distinct distances $g_N = |\{\delta_{n,N} \mid 1 \leq n \leq N\}|$

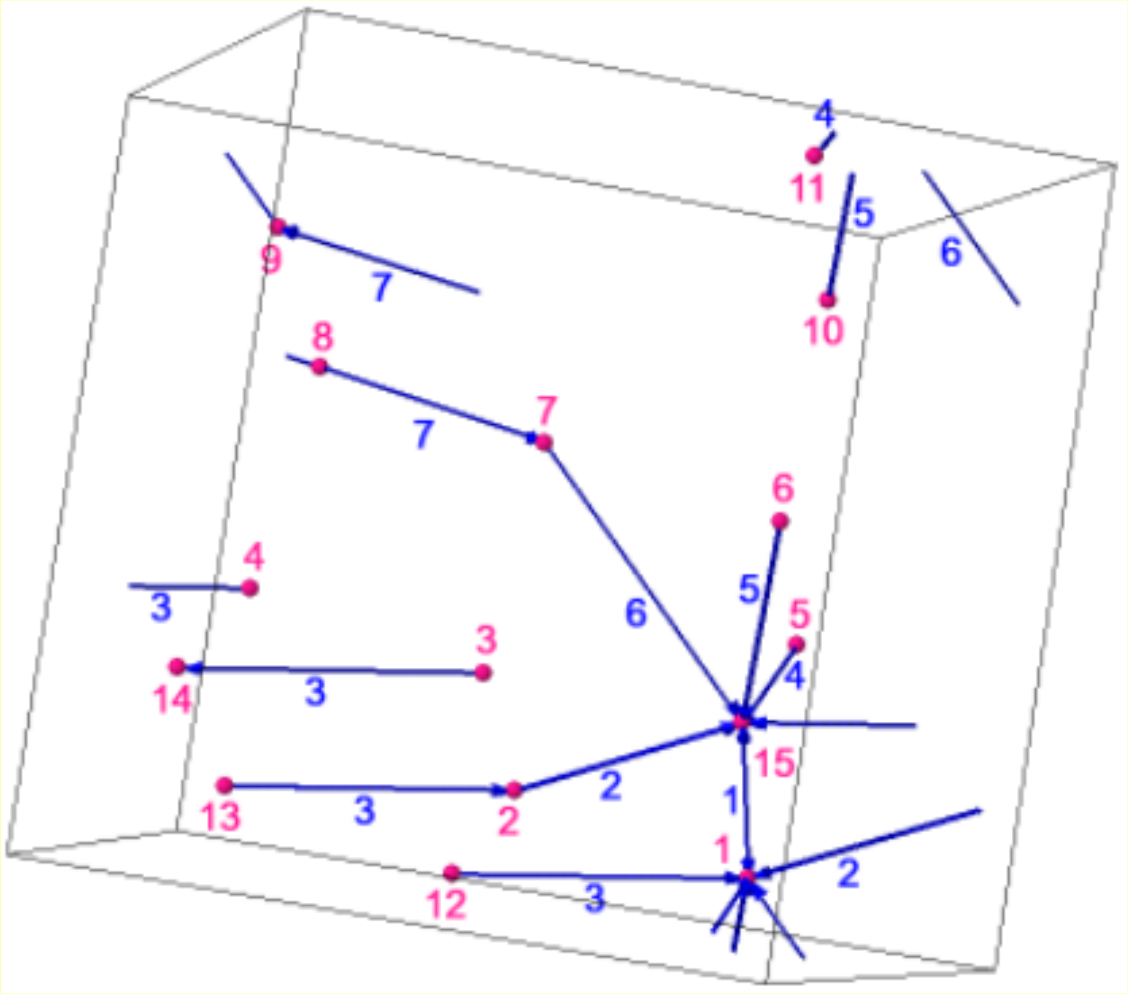
- Previous studies by Chevallier (1996, 1997, 2000, 2014) and Vijay (2008, "11 distances are enough"; see also Biringer and Schmidt for actions by isometries on general compact manifolds (2008)

Examples with 5 distances in dimension 2



$\vec{\alpha} = (0.38, 0.132)$, $N = 9$ $\vec{\alpha} = (0.105, 0.275)$, $N = 12$

Example with 7 distances in dimension 3



$$\vec{\alpha} = (\frac{46}{125}, \frac{107}{500}, \frac{43}{500}),\ N = 15$$

A five distance theorem

Theorem G. (Haynes & JM, IMRN 2022)
For every $\vec{\alpha} \in \mathbb{R}^d$ and $N \in \mathbb{N}$ we have that

$$g_N \leq \begin{cases} 3 & (d=1) \\ 5 & (d=2) \\ \sigma_d + 1 & (d \geq 3) \end{cases}$$

where σ_d is the kissing number for $\mathbb{R}^d$.

3	5	13	25	46	79	135	241	365	555

- Holds also if $\mathbb{Z}^d$ is replaced by any lattice of full rank in $\mathbb{R}^d$
- Holds also if standard Euclidean metric on $\mathbb{T}^d$ is replaced by any flat Riemannian metric
- Biringer and Schmidt (2008) showed $g_N \leq 3^d + 1$ (in fact for general isometric anctions on Riemannian manifolds with sectional curvature ≥ 0)

4	10	28	82	244	730	2188	6562	19684	59050

- If metric is given by max-norm, then $g_N \leq 2^d + 1$ (Chevallier 1996 $d = 2$, Haynes & Ramirez 2020)

3	5	9	17	33	65	129	257	513	1025

Lower bounds

We say $N_1 < N_2 < N_3 < \ldots$ of integers is sub-exponential if

$$\lim_{i\to\infty} \frac{N_{i+1}}{N_i} = 1.$$

Theorem H. (Haynes & JM, IMRN 2022)
There is a $P \subset \mathbb{R}^d$ of full Lebesgue measure, such that for every $\vec{\alpha} \in P$, $\vec{\alpha}_0 \in \mathbb{R}^d$, and for every sub-exponential sequence $(N_i)_i$, we have

$$\limsup_{i\to\infty} g_{N_i}(\vec{\alpha}) \geq \sup_{N\in\mathbb{N}} g_N(\vec{\alpha}_0).$$

- **Corollary:** For $\vec{\alpha} \in P$ we have $\limsup_{i\to\infty} g_{N_i}(\vec{\alpha}) \begin{cases} = 5 & \text{if } d = 2 \\ \geq 9 & \text{if } d = 3^* \end{cases}$

*Carl Dettmann (Exp Math 2024) found a numerical example with 9 distinct distances

Summary

- We have a detailed understanding of the Boltzmann-Grad limit of the periodic Lorentz gas, with a new type of linear Boltzmann equation

- Heavy tails, divergent moments, superdiffision, entropy asymptotics

- Explicit formulas for limit distribution only in dimension 2

- Connection with 3 gap theorem and higher dimensional variants

What’s in the final session?

- Directional statistics, dynamics on point sets
- Space of affine lattices, escape of mass estimates
- Open questions, future challenges

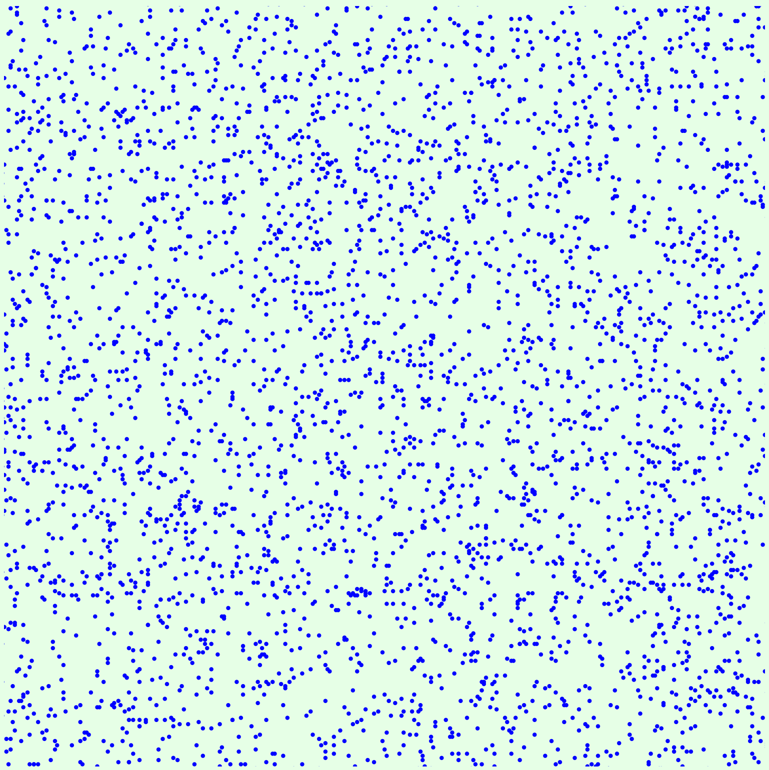
Part 3

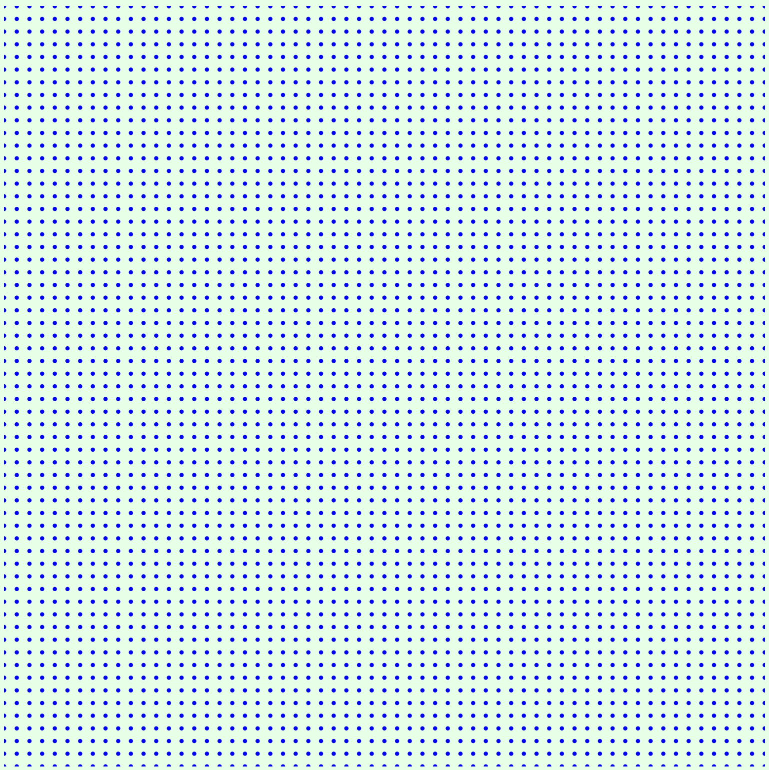
Directional statistics, dynamics on point sets, open questions and future challenges

Directional statistics

- $\mathcal{P}$ – locally finite set in $\mathbb{R}^d$
- B_T^d – ball of radius T
- Assume volume-like growth rate $\#(\mathcal{P} \cap \mathsf{B}_T^d) \leq C\,(1 + \mathrm{vol}\,\mathsf{B}_T^d)$

  (for some constant C)





Directions

- Take all points of $\mathcal{P} \cap \mathsf{B}_T^d$ and project radially onto unit sphere:

$$\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \in \mathsf{S}_1^{d-1}, \qquad \boldsymbol{y} \in \mathcal{P} \cap \mathsf{B}_T$$

  (counted with multiplicity)

- To investigate the pseudorandom properties of this sequences (as $T \to \infty$) we consider **local statistics**.

- One of the most fundamental such local statistics is the distribution of the number of points in **small discs**.

Directional statistics

- ρ – uniform probability measure on the unit sphere S^{d-1}
- $\mathfrak{D}_T(\sigma, \boldsymbol{v}) \subseteq \mathsf{S}^{d-1}$ – the open disc with center $\boldsymbol{v}$ and volume chose so that

$$\rho(\mathfrak{D}_T(\sigma, \boldsymbol{v})) = \frac{\sigma}{\operatorname{vol} \mathsf{B}_T^d}$$

- Want to study distribution of

$$\mathcal{N}_T(\sigma, \boldsymbol{v}) := \# \left\{ \boldsymbol{y} \in \mathcal{P} \cap \mathsf{B}_T^d : \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} \in \mathfrak{D}_T(\sigma, \boldsymbol{v}) \right\}$$

for random $\boldsymbol{v}$ (distributed according to ρ or a more general a.c. measure λ on S_1^{d-1})

- The scaling of the disc is chosen so that we typically expect at most a constant number of points in the disc recall $\#(\mathcal{P} \cap \mathsf{B}_T^d) \leq C\,(1 + \operatorname{vol} \mathsf{B}_T^d)$

$$\int_{\mathsf{S}_1^{d-1}} \mathcal{N}_T(\sigma, \boldsymbol{v})\, d\rho(\boldsymbol{v}) = \#(\mathcal{P} \cap \mathsf{B}_T^d) \times \rho(\mathfrak{D}_T(\sigma, \boldsymbol{v})) = \sigma\, \frac{\#(\mathcal{P} \cap \mathsf{B}_T^d)}{\operatorname{vol} \mathsf{B}_T^d}$$

Rescaling and group actions

- Define the cone

$$\mathfrak{C}(\sigma) = \left\{ \boldsymbol{x} \in \mathbb{R}^d : 0 < x_1 < 1,\ \|(x_2, \ldots, x_d)\| < \left(\tfrac{d\,\sigma}{\mathrm{vol}\,\mathsf{B}_1^{d-1}}\right)^{\frac{1}{d-1}} x_1 \right\}$$

- Note that

$$\left\{ \boldsymbol{x} \in \mathsf{B}_T^d : \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \in \mathfrak{D}_T(\sigma, \boldsymbol{v}) \right\} R(\boldsymbol{v}) D(T) \approx \mathfrak{C}(\sigma)$$

where $R(\boldsymbol{v}) \in \mathsf{SO}(d)$ so that $\boldsymbol{v}R(\boldsymbol{v}) = \boldsymbol{e}_1$, and $D(T) = \begin{pmatrix} T^{-1} & 0 \\ 0 & T^{\frac{1}{d-1}} 1_{d-1} \end{pmatrix}$

- Therefore $\mathbb{P}(\mathcal{N}_T(\sigma, \boldsymbol{v}) = k) \approx \mathbb{P}(\#[\mathcal{P}R(\boldsymbol{v})D(T) \cap \mathfrak{C}(\sigma)] = k)$

- Note: $R(\boldsymbol{v}), D(T) \in \mathsf{SL}(d, \mathbb{R})$ - we have a **group action**!

Point processes and spherical averages

- Instead of $\mathfrak{C}(\sigma)$ consider general test set $\mathcal{A} \subset \mathbb{R}^d$ (bdd & meas-0 boundary)

- Would like to show that there is a random point set (point process) Ξ such that for every $k, \mathcal{A}, \lambda$

$$\lim_{T\to\infty} \mathbb{P}(\#[\mathcal{P}R(\boldsymbol{v})D(T) \cap \mathcal{A}] = k) = \mathbb{P}(\#(\Xi \cap \mathcal{A}) = k)$$

- ... or more generally $\mathcal{P}R(\boldsymbol{v})D(T) \xrightarrow{\text{d}} \Xi$ as point processes in vague topology

- These are **spherical averages** on the space of locally finite Borel measures on $\mathbb{R}^d$

- Once we understand spherical averages, we obtain all fine-scale limit distributions for the directions in $\mathcal{P}$ as a corollary! Including nearest-neighbour distributions.

Example 1: Poisson point process

Theorem I. (see e.g. JM & Strömbergsson, Memoirs AMS 2024, §5.1)

Let $\mathcal{P}$ be a realization of a Poisson point process Ξ of unit intensity. Let $\boldsymbol{v}$ be a random vector distributed according to the Borel probability measure λ on S_1^{d-1}. Then, for every absolutely continuous λ,

$$\mathcal{P} R(\boldsymbol{v}) D(T) \xrightarrow{\mathrm{d}} \Xi \quad (T \to \infty)$$

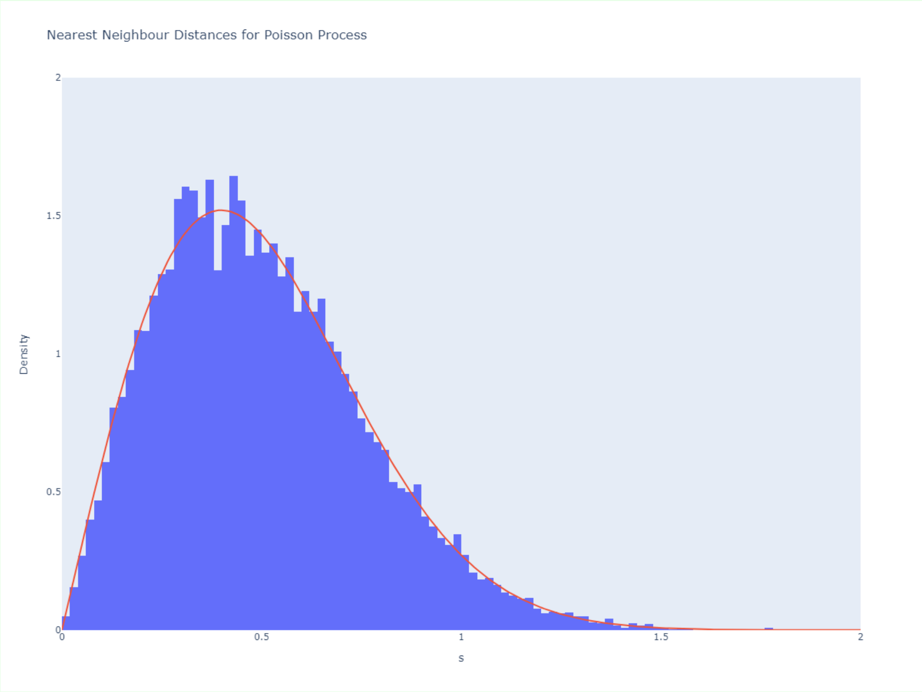
In particular (for the directions)

$$\lim_{T\to\infty} \mathbb{P}(\mathcal{N}_T(\sigma, \boldsymbol{v}) = k) = \frac{\sigma^k}{k!} \mathrm{e}^{-\sigma}.$$

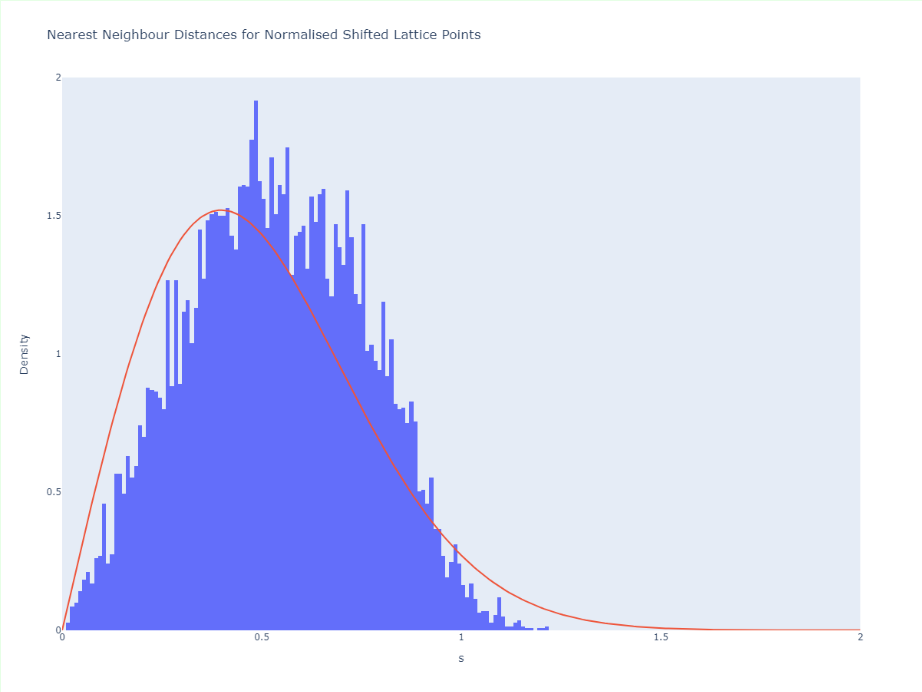
Proof inspired by Boldrighini, Bunimovich & Sinai (J Stat Phys 1983)

Example 1: Poisson point process in 3d

Nearest-neighbour distributions for directions on S_1^2 vs. $2\pi s \mathrm{e}^{-\pi s^2}$:



Fixed realisation of a Poisson point process in $\mathbb{R}^3$, $T = 15$



Affice lattice $\mathbb{Z}^3 + \boldsymbol{\alpha}$ with $\boldsymbol{\alpha} = (2^{1/4}, 3^{1/4}, 5^{1/4})$, $T = 15$

Numerics by Jory Griffin

Example 2: Affine lattices

Theorem J. (JM & Strömbergsson, Annals Math 2010)

Let $\mathcal{P} = \mathbb{Z}^d + \boldsymbol{\alpha}$, with $\boldsymbol{\alpha} \in \mathbb{R}^d$. Let $\boldsymbol{v}$ be a random vector distributed according to the Borel probability measure λ on S_1^{d-1}. Then, for every a.c. λ,

$$\mathcal{P} R(\boldsymbol{v}) D(T) \xrightarrow{\text{d}} \Xi_{\boldsymbol{\alpha}} \quad (T \to \infty)$$

where

$$\Xi_{\boldsymbol{\alpha}} = \begin{cases} \mathbb{Z}^d g \text{ with } g \text{ Haar-random in } \mathrm{ASL}(d,\mathbb{Z}) \backslash \mathrm{ASL}(d,\mathbb{R}) & (\boldsymbol{\alpha} \notin \mathbb{Q}^d) \\ (\mathbb{Z}^d + \frac{\boldsymbol{p}}{q}) M \text{ with } M \text{ Haar-random in } \Gamma_q \backslash \mathrm{SL}(d,\mathbb{R}) & (\boldsymbol{\alpha} = \frac{\boldsymbol{p}}{q} \in \mathbb{Q}^d) \end{cases}$$

$$\mathrm{ASL}(d,\mathbb{R}) = \mathrm{SL}(d,\mathbb{R}) \ltimes \mathbb{R}^d, \qquad \Gamma_q = \{\gamma \in \mathrm{SL}(d,\mathbb{Z}) : \gamma \equiv 1 \bmod q\}$$

The proof exploits equidistribution of large spheres on $\mathrm{ASL}(d,\mathbb{Z}) \backslash \mathrm{ASL}(d,\mathbb{R})$. Required Ratner's measure classification theorem when $\boldsymbol{\alpha} \notin \mathbb{Q}^d$; now have effective versions (Strömbergsson, Duke Math J 2015 $d = 2$; Wooyeon Kim, preprint 2021 $d \geq 2$)

For $\boldsymbol{\alpha} = 0$, can also consider lattice point **on** the sphere (Bourgain, Rudnick & Sarnak, Contemp. Math. 2016, Bull. Iranian Math. Soc. 2017; Kurlberg & Lester, arXiv:2112.08522)

Example 2: Affine lattices

Theorem J (cont'd). (JM & Strömbergsson, Annals Math 2010)

In particular (for the directions)

$$\lim_{T\to\infty} \mathbb{P}(\mathcal{N}_T(\sigma,\boldsymbol{v})=k)=E_{\boldsymbol{\alpha}}(k,\sigma)$$

where the limit distribution has the properties

(a) $E_{\boldsymbol{\alpha}}(k,\sigma)$ is independent of λ

(b) $\sum_{k=1}^{\infty} kE_{\boldsymbol{\alpha}}(k,\sigma)=\sigma$

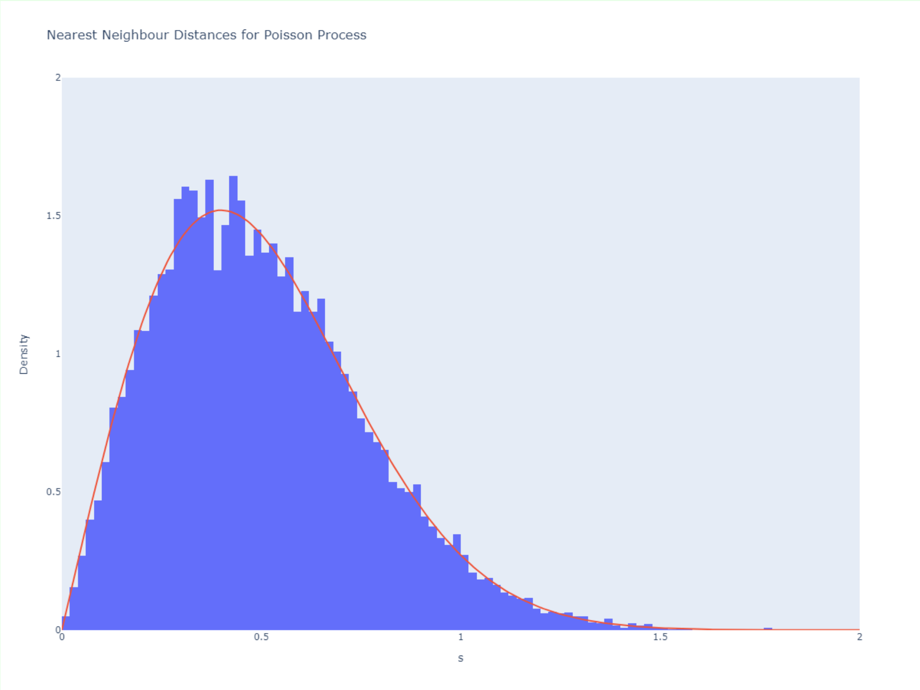
(c) For $\boldsymbol{\alpha}\in\mathbb{Q}^d$, $\sum_{k=1}^{\infty} k^\eta E_{\boldsymbol{\alpha}}(k,\sigma) \begin{cases} <\infty & (0\leq\eta<d) \\ =\infty & (\eta\geq d)\end{cases}$

(d) For $\boldsymbol{\alpha}\notin\mathbb{Q}^d$, $E_{\boldsymbol{\alpha}}(k,\sigma)=: E(k,\sigma)$ is independent of $\boldsymbol{\alpha}$

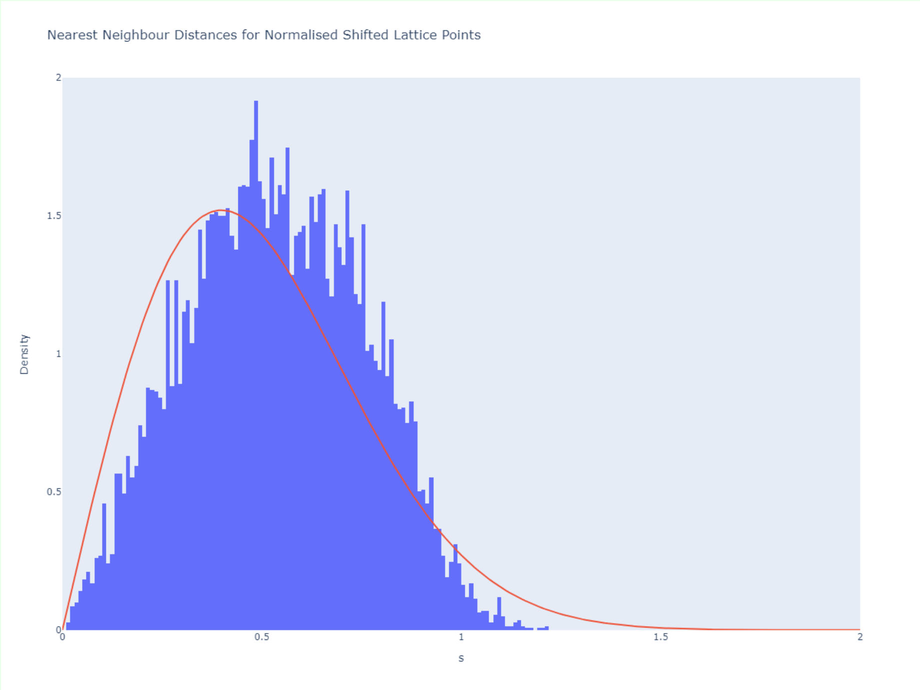
(e) For $\boldsymbol{\alpha}\notin\mathbb{Q}^d$, $\sum_{k=1}^{\infty} k^\eta E(k,\sigma) \begin{cases} <\infty & (0\leq\eta<d+1) \\ =\infty & (\eta\geq d+1)\end{cases}$

Example 2: Affine lattices in 3d

Nearest-neighbour distributions for directions on S_1^2 vs. $2\pi s e^{-\pi s^2}$:



Fixed realisation of a Poisson point process in $\mathbb{R}^3$, $T = 15$

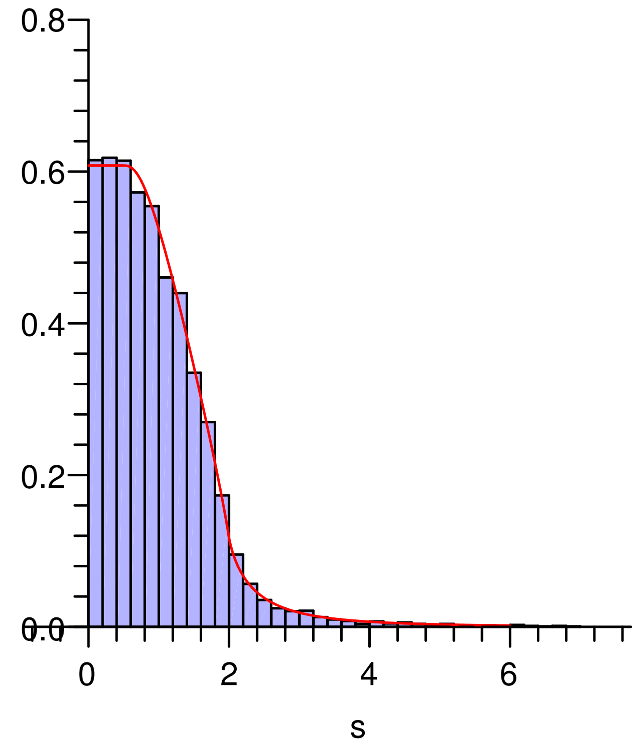


Affice lattice $\mathbb{Z}^3 + \boldsymbol{\alpha}$ with $\boldsymbol{\alpha} = (2^{1/4}, 3^{1/4}, 5^{1/4})$, $T = 15$

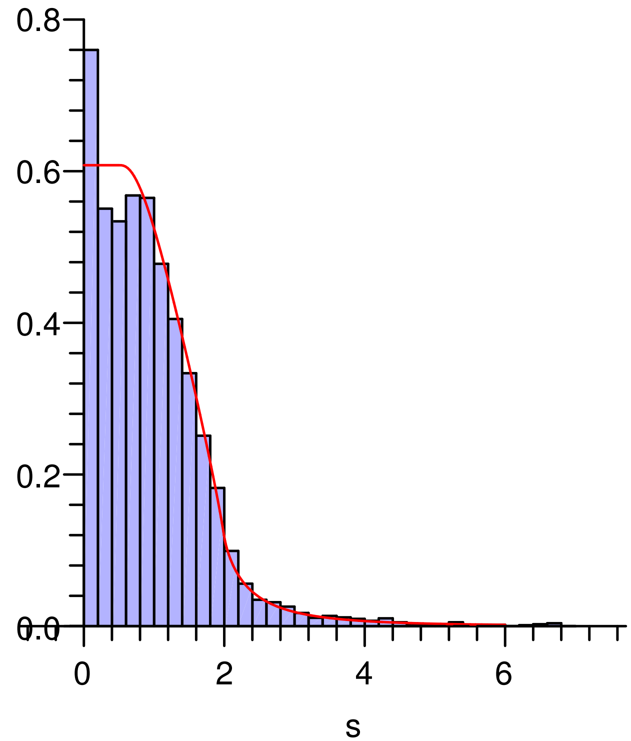
Numerics by Jory Griffin

Example 2: Affine lattices in 2d

Gap distribution of directions in 2d affine lattice vs. Elkies-McMullen distribution:



Affice lattice $\mathbb{Z}^2 + \boldsymbol{\alpha}$ with $\boldsymbol{\alpha} = (\sqrt{2}, 0)$
$T = 4900$ (JM & AS, Annals 2010)



$\sqrt{n}$ mod 1 $n = 1, \ldots, 7765$
Elkies & McMullen Duke 2005

Both proofs use Ratner's measure classification theorem on the same space for same test function – but for different unipotent flows! Tail is $\sim \frac{3}{\pi^2} s^{-3}$. Note that for $\boldsymbol{\alpha} = 0$ we would (taking only the visible lattice points) recover the classical Hall distribution (Hall, J LMS 1970) for the gaps between Farey points. Hall density has tail $\sim \frac{36}{\pi^4} s^{-3}$.

Moments

- The above convergence in distribution/vague topology does not necessarily imply the convergence of moments

$$\lim_{T\to\infty} \int_{\mathsf{S}_1^{d-1}} \mathcal{N}_T(\sigma, \boldsymbol{v})^\eta \, d\lambda(\boldsymbol{v}) = \mathbb{E}(\#[\Xi \cap \mathfrak{C}(\sigma)])^\eta$$

- For $\eta = 1$, if Ξ independent of λ and $\mathbb{E}[\#(\Xi \cap \mathcal{A})] = c \operatorname{vol} \mathcal{A}$, then the above convergence (for sufficiently large class of λ) implies the counting asymptotics

$$\lim_{T\to\infty} \frac{\#(\mathcal{P} \cap T\mathcal{D})}{T^d} = c \operatorname{vol} \mathcal{D}$$

for every bdd $\mathcal{D} \subset \mathbb{R}^d$ with meas-0 boundary.*

*This strategy was for example employed in context of counting periodic trajectories on flat surfaces (Veech, Eskin-Masur, Eskin-Mirzhakani-Mohammadi), quantitative Oppenheim (Eskin-Margulis-Mozes), for visible points in quasicrystals (JM-Strömbergsson); but cf. also counting in negative curvature (Margulis)

Pair correlation (Ripley's K-function)

- Particular popular fine-scale statistics

$$R_T^2(s) = \frac{\#\left\{(\boldsymbol{y}_1, \boldsymbol{y}_2) \in (\mathcal{P} \cap \mathsf{B}_T^d)^2_{\neq} \;:\; c_d N^{\frac{1}{d-1}} \,\mathsf{dist}_{\mathsf{S}^{d-1}}\left(\frac{\boldsymbol{y}_1}{\|\boldsymbol{y}_1\|}, \frac{\boldsymbol{y}_2}{\|\boldsymbol{y}_2\|}\right) \leq s\right\}}{N},$$

$$N = \#(\mathcal{P} \cap \mathsf{B}_T^d), \quad c_d = \mathsf{vol}_{\mathsf{S}^{d-1}}(\mathsf{S}^{d-1})^{-\frac{1}{d-1}}, \quad \mathsf{dist}_{\mathsf{S}^{d-1}} = \text{arc length}$$

- Equivalent to studying second mixed moment $\int_{\mathsf{S}_1^{d-1}} \mathcal{N}_T(\sigma, \boldsymbol{v}) \mathcal{N}_T(\epsilon, \boldsymbol{v}) \, d\lambda(\boldsymbol{v})$

- If second moment converges to second moment of Ξ then the limit pair correlation function converges and can be expressed via the intensity measure of the Palm distribution of Ξ

Example 1: Pair correlation for Poisson process

Theorem K.

Let $\mathcal{P}$ be a realization of a Poisson point process Ξ of unit intensity. Then almost surely we have that, for all $s > 0$,

$$\lim_{T\to\infty} R_T^2(s) = \frac{\pi^{\frac{d-1}{2}} s^{d-1}}{\Gamma(\frac{d+1}{2})}$$

Example 2: Pair correlation for affine lattices

Theorem L. (Wooyeon Kim & JM, ETDS 2024)

Let $\mathcal{P} = \mathbb{Z}^d + \boldsymbol{\alpha}$, with $\boldsymbol{\alpha} \notin \mathbb{Q}^d$ (plus being (0,0,2)-vaguely Diophantine* if $d = 2$). Then, for all $s > 0$,

$$\lim_{T\to\infty} R_T^2(s) = \frac{\pi^{\frac{d-1}{2}} s^{d-1}}{\Gamma(\frac{d+1}{2})}$$

- Previously known for $d = 2$ (El Baz, Vinogradov & JM, IMRN 2015)
- The key points in the proof are: (1) escape of mass estimates for embedded $\mathrm{SL}(d,\mathbb{R})$-horospheres in $\mathrm{ASL}(d,\mathbb{Z})\backslash \mathrm{ASL}(d,\mathbb{R})$, and (2) a Rogers type volume formula that shows that the limit variance is Poissonian.
- We can establish convergence of the ηth moment if $\eta < d$, and for $d \leq \eta < d+1$ if $\boldsymbol{\alpha}$ is $(0, \eta-2, 2)$-vaguely Diophantine ($d = 2$) or $(d-1, \eta-d, 1)$-vaguely Diophantine ($d \geq 3$). **All moments – except** $\eta = 2$ **– are not Poisson** and diverge for $\eta \geq d+1$.

Fact sheet on Diophantine condition

For $\kappa \geq d$, we say that $\boldsymbol{\alpha} \in \mathbb{R}^d$ is Diophantine of type κ if there exists $C_\kappa > 0$ such that

$$|\boldsymbol{\alpha} \cdot \boldsymbol{m}|_{\mathbb{Z}} > C_\kappa |\boldsymbol{m}|^{-\kappa}$$

for any $\boldsymbol{m} \in \mathbb{Z}^d \setminus \{0\}$, where $|\cdot|$ denotes the supremum norm of $\mathbb{R}^d$, and $|\cdot|_{\mathbb{Z}}$ denotes the supremum distance from $0 \in \mathbb{T}^d$. We will in fact only require a milder Diophantine condition. Define the function $\zeta : \mathbb{R}^d \times \mathbb{R}_{>0} \to \mathbb{N}$ by

$$\zeta(\boldsymbol{\alpha}, T) := \min \left\{ N \in \mathbb{N} : \min_{\substack{\boldsymbol{m} \in \mathbb{Z}^d \setminus \{0\} \\ 0 < |\boldsymbol{m}| \leq N}} |\boldsymbol{\alpha} \cdot \boldsymbol{m}|_{\mathbb{Z}} \leq \frac{1}{T} \right\}.$$

In view of Dirichlet's pigeon hole principle, we have that $\zeta(\boldsymbol{\alpha}, T) \leq T^{1/d}$ and, if $\boldsymbol{\alpha}$ is of Diophantine type $\kappa \geq d$, then $\zeta(\boldsymbol{\alpha}, T) > (C_\kappa T)^{\frac{1}{\kappa}}$.

We say $\boldsymbol{\alpha} \in \mathbb{R}^d$ is (ρ, μ, ν)-*vaguely* Diophantine, if $\displaystyle\sum_{l=1}^{\infty} l^\rho \, 2^\mu \zeta(\boldsymbol{\alpha}, 2^{l-1})^{-\nu} < \infty$.

Thus, if $\boldsymbol{\alpha}$ is Diophantine type κ, then it is also (ρ, μ, ν)-vaguely Diophantine for $\kappa\mu < \nu$.

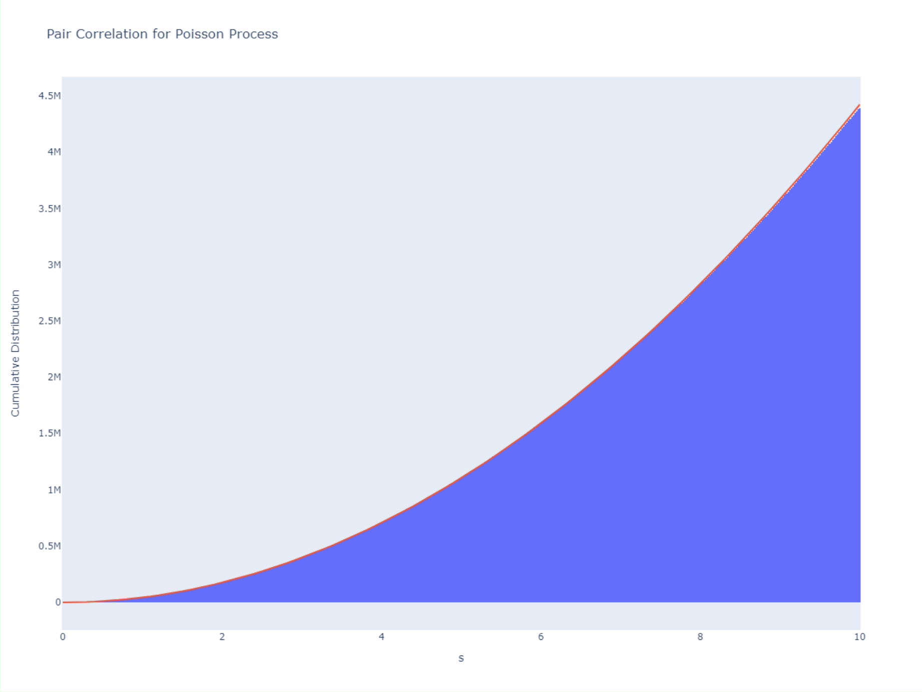
If $\boldsymbol{\alpha}$ satisfies the generalised s-Brjuno Diophantine condition[a] $\displaystyle\sum_{n=0}^{\infty} 2^{-\frac{n}{s}} \max_{\substack{\boldsymbol{m} \in \mathbb{Z}^d \setminus \{0\} \\ 0 < |\boldsymbol{m}| \leq 2^n}} \log \frac{1}{|\boldsymbol{\alpha} \cdot \boldsymbol{m}|_{\mathbb{Z}}} < \infty$

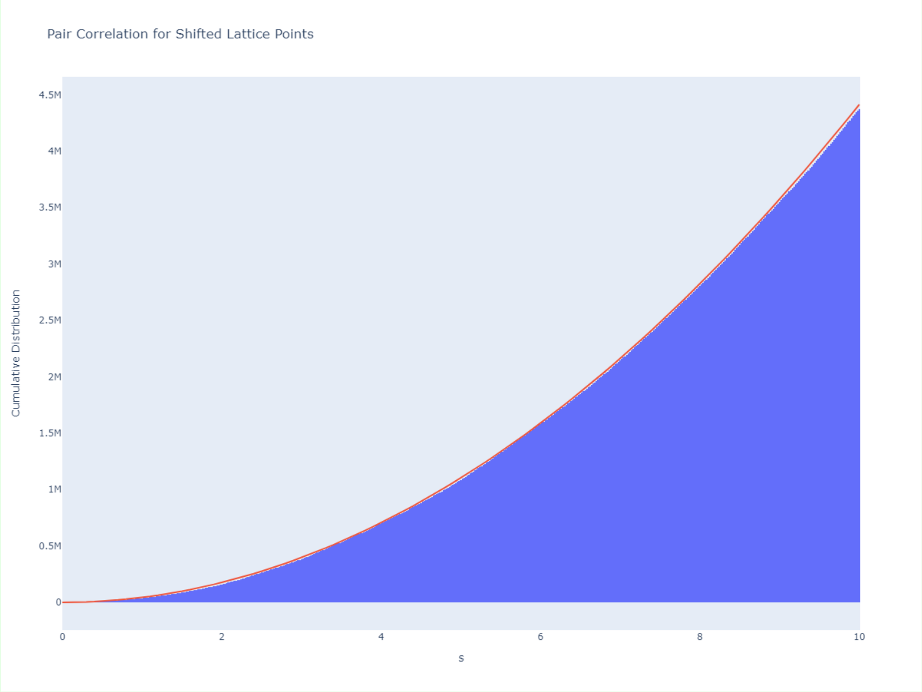
then it is $(\rho, 0, \nu)$-vaguely Diophantine for $s > \frac{\rho+1}{\nu}$.

[a]Bounemoura & Féjoz, Ann Sc Norm Super Pisa Cl Sci (2019)
Lopes Dias & Gaivão, J Diff Equ 267 (2019)

Example 1 vs. 2: Pair correlation in 3d

Pair correlation numerics vs. $\dfrac{\pi^{\frac{d-1}{2}} s^{d-1}}{\Gamma(\frac{d+1}{2})}$:





Fixed realisation of a Poisson point process in $\mathbb{R}^3$, $T = 15$

Affice lattice $\mathbb{Z}^3 + \boldsymbol{\alpha}$ with $\boldsymbol{\alpha} = (2^{1/4}, 3^{1/4}, 5^{1/4})$, $T = 15$

Numerics by Jory Griffin

The space of affine lattices

- $G = \mathsf{SL}(d,\mathbb{R})$, $\Gamma = \mathsf{SL}(d,\mathbb{Z})$, $\mathcal{X} = \Gamma\backslash G$ "the space of lattices"

- $G' = \mathsf{ASL}(d,\mathbb{R}) = G \ltimes \mathbb{R}^d$, $\Gamma' = \mathsf{ASL}(d,\mathbb{Z}) = \Gamma \ltimes \mathbb{Z}^d$ with multiplication law

$$(M,\mathbf{b})(M',\mathbf{b}') = (MM', \mathbf{b}M' + \mathbf{b}')$$

  $\mathcal{X}' = \Gamma'\backslash G'$ "the space of affine lattices" (also "space of grids")

- Embed G in G' via the homomorphism $M \mapsto (M,0)$;
  this yields embedding of $\mathcal{X}$ in $\mathcal{X}'$ as submanifold

- Right action of $g = (M,\mathbf{b}) \in G'$ on $\mathbb{R}^d$ is defined by $\mathbf{x}g := \mathbf{x}M + \mathbf{b}$

Counting and dynamics

- Recall

$$\mathbb{P}(\mathcal{N}_T(\sigma,\boldsymbol{v})=k)\approx\mathbb{P}(\#[\mathcal{P}R(\boldsymbol{v})D(T)\cap\mathfrak{C}(\sigma)]=k)$$

with $\mathcal{P}=\mathbb{Z}^d+\boldsymbol{\alpha}$ and the cone

$$\mathfrak{C}(\sigma):=\{(x_1,x')\in\mathbb{R}\times\mathbb{R}^{d-1}:0<x<1,\|x'\|<\sigma\}$$

- Theorem B follows from equidistribution of translates of spheres on $\mathcal{X}'$: For any **bounded** continuous $F:\mathcal{X}'\to\mathbb{R}$ and λ a probability measure (a.c.w.r.t. Lebesgue)*

$$\lim_{T\to\infty}\int F((1,\boldsymbol{\alpha})R(\boldsymbol{v})D(T))d\lambda(\boldsymbol{v})=\int F(g)d\mu_{\boldsymbol{\alpha}}(g)$$

- Here $\mu_{\boldsymbol{\alpha}}$ is the Haar probability measure on $\mathcal{X}'$ if $\boldsymbol{\alpha}\notin\mathbb{Q}^d$ and the Haar probability measure of an embedded $\Gamma_q\backslash G$ otherwise.

*Ratner 1991, Shah 1996, JM & Strömbergsson 2010; effective versions Strömbergsson 2015, Wooyeon Kim 2021

Convergence of moments

- We can extend the equidistribution of speherical averages to **u**nbounded test functions that grow in the cusp, under certain Diophantine conditions on $\boldsymbol{\alpha}$.
- This then gives the following convergence of moments ...

Theorem M. (Wooyeon Kim & JM, ETDS 2024)
Let λ be a Borel probability measure on the sphere with piecewise continuous density. Assume
(B1) $\eta < d$ or
(B2) $\eta < d + 1$ and $\boldsymbol{\alpha}$ is $(0, \eta - 2, 2)$-vaguely Diophantine if $d = 2$, and $(d-1, \eta - d, 1)$-vaguely Diophantine if $d \geq 3$. Then

$$\lim_{T\to\infty} \int_{\mathsf{S}_1^{d-1}} \mathcal{N}_T(\sigma, \boldsymbol{v})^\eta \, d\lambda(\boldsymbol{v}) = \sum_{k=1}^{\infty} k^\eta E(k, \sigma)$$

...as well as the convergence of more general mixed moments and that of the pair correlation density stated in Theorem B.

Sketch of proof

- **Problem:** The quantity $\#[\mathbb{Z}^d g \cap \mathfrak{C}(\sigma)]$ is unbounded as a function on $\mathcal{X}'$
- Write $\mathbb{E}\mathcal{N}_T(\sigma, \boldsymbol{v})^\eta = \int_{\mathrm{S}_1^{d-1}} \mathcal{N}_T(\sigma, \boldsymbol{v})^\eta \, d\lambda(\boldsymbol{v})$ as

$$\mathbb{E}[\mathcal{N}_T(\sigma, \boldsymbol{v})^\eta \mathbb{1}(\mathcal{N}_T(\sigma, \boldsymbol{v}) \leq K)] + \mathbb{E}[\mathcal{N}_T(\sigma, \boldsymbol{v})^\eta \mathbb{1}(\mathcal{N}_T(\sigma, \boldsymbol{v}) > K)]$$

- From equidistribution (w.r.t. bounded test functions we get that)

$$\lim_{K\to\infty} \lim_{T\to\infty} \mathbb{E}[\mathcal{N}_T(\sigma, \boldsymbol{v})^\eta \mathbb{1}(\mathcal{N}_T(\sigma, \boldsymbol{v}) \leq K)] = \sum_{k=1}^{\infty} k^\eta E_{\boldsymbol{\alpha}}(k, \sigma)$$

  provided the r.h.s. converges (this means $\eta < d + 1$ for $\boldsymbol{\alpha} \notin \mathbb{Q}^d$, and $\eta < d$ for $\boldsymbol{\alpha} \in \mathbb{Q}^d$)
- Therefore, to complete the proof of convergence of moments, it remains to be shown that

$$\lim_{K\to\infty} \limsup_{T\to\infty} \mathbb{E}[\mathcal{N}_T(\sigma, \boldsymbol{v})^\eta \mathbb{1}(\mathcal{N}_T(\sigma, \boldsymbol{v}) > K)] = 0$$

Geometry of the space of lattices

- Define the following matices in $G = \mathsf{SL}(d, \mathbb{R})$:

$$n(\mathbf{u}) := \begin{pmatrix} 1 & u_{12} & \cdots & & u_{1d} \\ & \ddots & & & \vdots \\ & & & 1 & u_{(d-1)d} \\ & & & & 1 \end{pmatrix}, \quad a(\mathbf{v}) := \begin{pmatrix} v_1 & & & \\ & v_2 & & \\ & & \ddots & \\ & & & v_d \end{pmatrix}.$$

$$\mathbf{u} = (u_{12}, \ldots, u_{1d}, u_{23}, \ldots, u_{(d-1)d}) \in \mathbb{R}^{\frac{d(d-1)}{2}}$$

$$\mathbf{v} = (v_1, v_2, \ldots, v_d) \in \mathcal{T} := \{(v_1, \ldots, v_d) \in \mathbb{R}^d_{>0}, v_1 \cdots v_d = 1\}$$

- Iwasawa decomposition of $M \in G$ is given by $M = n(\mathbf{u})a(\mathbf{v})k$, where $\mathbf{u} \in \mathbb{R}^{\frac{d(d-1)}{2}}$, $\mathbf{v} \in \mathcal{T}$ and $k \in \mathsf{SO}(d)$.

Geometry of the space of lattices

- **Siegel set** = set that contains a fundamental domain of Γ action on G and can be covered with a finite number of fundamental domains.
- Here is our choice of a Siegel set:*

$$\mathcal{S} := \left\{ n(\mathbf{u})a(\mathbf{v})k : k \in \mathsf{SO}(d), 0 < v_{j+1} \leq \frac{2}{\sqrt{3}} v_j,\ \mathbf{u} \in [-\tfrac{1}{2}, \tfrac{1}{2}]^{\frac{d(d-1)}{2}} \right\}$$

- $\mathcal{F}$ = fundamental domain of Γ action on G contained in $\mathcal{S}$
- Fundamental domain and of the Γ' action on G':

$$\mathcal{F}' = \{(1, \mathbf{b})(M, 0) : \mathbf{b} \in [-\tfrac{1}{2}, \tfrac{1}{2})^d,\ M \in \mathcal{F}\},$$

$$\mathcal{S}' = \{(1, \mathbf{b})(M, 0) : \mathbf{b} \in [-\tfrac{1}{2}, \tfrac{1}{2}]^d,\ M \in \mathcal{S}\}.$$

- For $x \in \Gamma' \backslash G'$, there exists unique $g \in \mathcal{F}'$ such that $x = \Gamma g$. Define $\iota : \mathcal{X}' \to \mathcal{F}$ so that $\iota(\Gamma g) = g$.

*Its not compact, $v_d \to 0$ produces a short lattice vector (Mahler's criterion)

Two key inequalities

- **The First:** For any fixed $\epsilon > 0$ and all T sufficiently large,

$$\mathcal{N}_T(\sigma, \boldsymbol{v}) \le \#[(\mathbb{Z}^d + \boldsymbol{\alpha})R(\boldsymbol{v})D(T) \cap \mathfrak{C}(\sigma + \epsilon)]$$

- Set

$$r = r(\mathfrak{C}) := \max\{\delta_d, \sup\{\|\mathbf{x}\| : \mathbf{x} \in \mathfrak{C}\}, \qquad \delta_d = d4^d$$

$$s_r(g) := \max\{1 \le i \le d-1 : v_i(g) > 2c_d r\}, \qquad c_d = d\left(\frac{2}{\sqrt{3}}\right)^d$$

- **The Second:**

$$\left(\#[\mathbb{Z}^d g \cap \mathfrak{C}]\right)^\eta \le (C_d r^d)^\eta \prod_{i=1}^s \left(v_i^\eta \#\left([-c_d r v_i^{-1}, c_d r v_i^{-1}] \cap (\mathbb{Z} + b_i)\right)\right)$$

where $\mathbf{v} = \mathbf{v}(g)$, $\mathbf{b} = \mathbf{b}(g)$, $r = r(\mathfrak{C})$, $s = s_r(g)$, $C_d = 4c_d$

Model test function

- χ_I – characteristic function of a subset $I \subseteq \mathbb{R}$
- For $R \geq 1$, $\eta, r > 0$, define the Γ'-invariant function

$$F_{R,\eta,r}(g) := \chi_{[R,\infty)}\left(\prod_{i=1}^{s_r(g)} v_i(g)\right) \prod_{i=1}^{s_r(g)} v_i(g)^\eta \chi_{[-c_d r, c_d r]}\big(v_i(g) b_i(g)\big).$$

- From the "second inequality": for all $g \in G$ such that $\prod_{i=1}^{s_r(g)} v_i(g) \geq R$ with R sufficiently large, we have that

$$\big(\#[\mathbb{Z}^d g \cap \mathfrak{C}]\big)^\eta \leq (C_d r^d)^\eta F_{R,\eta,r}(g)$$

- $F_{R,\eta,r}(g)$ is thus our model test function that grows in the cusp at the required rate

Controlling escape of mass

$$\tilde{n}(\mathbf{y}) := \begin{pmatrix} 1 & y_2 & \cdots & y_d \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad \Phi_t := \begin{pmatrix} e^{-\frac{d-1}{d}t} & & & \\ & e^{\frac{t}{d}} & & \\ & & \ddots & \\ & & & e^{\frac{t}{d}} \end{pmatrix}$$

Note: For compact $\mathcal{K} \subset \mathbb{R}^{d-1}$
$\{\Gamma'(1, \boldsymbol{\xi}) M_0 \tilde{n}(\mathbf{y}) \Phi_t : \mathbf{y} \in \mathcal{K}\}$ is a $\mathrm{SL}(d, \mathbb{R})$-horosphere embedded in $\mathcal{X}'$.

Theorem N. (Wooyeon Kim & JM, ETDS 2024)
Let $\boldsymbol{\xi} \in \mathbb{R}^d$, $M_0 \in G$, $\eta, r > 0$, and $\psi \in \mathrm{C}_0(\mathbb{R}^{d-1})$.
Assume
(B1) $\eta < d$ or
(B2) $\eta < d+1$ and $\boldsymbol{\xi}$ is $(0, \eta - 2, 2)$-vaguely Diophantine if $d = 2$, and $(d-1, \eta - d, 1)$-vaguely Diophantine if $d \geq 3$. Then

$$\lim_{R \to \infty} \limsup_{t \to \infty} \left| \int_{\mathbf{y} \in \mathbb{R}^{d-1}} F_{R,\eta,r}\left(\Gamma'(1, \boldsymbol{\xi}) M_0 \tilde{n}(\mathbf{y}) \Phi_t\right) \psi(\mathbf{y}) d\mathbf{y} \right| = 0.$$

Open questions, future challenges

- Derive more properties of the limit distributions in the above: analyticity, tail estimates, behaviour near zero, explicit formulas for moments etc.

- *We have seen all of the above limit processes are* $\mathrm{SL}(d,\mathbb{R})$ *or* $\mathrm{ASL}(d,\mathbb{R})$ *invariant.* Can we classify all such point processes in $\mathbb{R}^d$, or at least produce some new interesting examples that are not based on lattices & Poisson?

- Prove the convergence of spherical averages (or parabolic shears) on such spaces (cf. Eskin, Mirzakhani & Mohammadi 2015)

- Are there examples of spherical averages of the above form that converge but the limit process is not $\mathrm{SL}(d,\mathbb{R})$ invariant?

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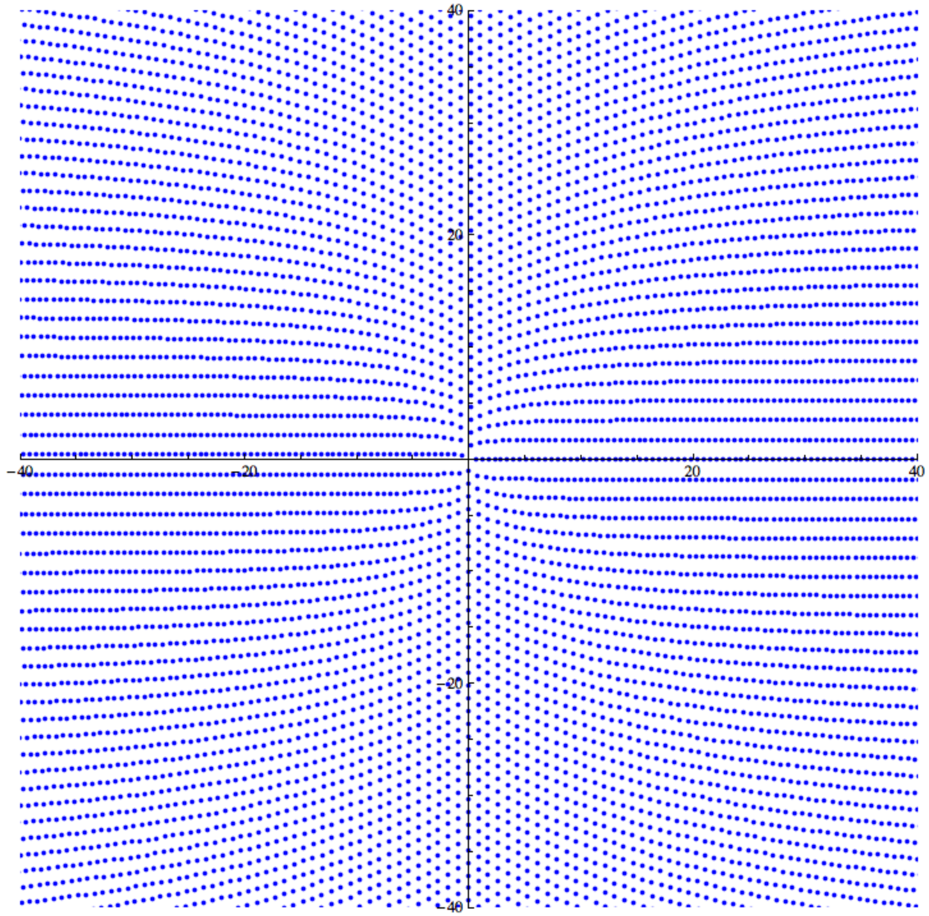
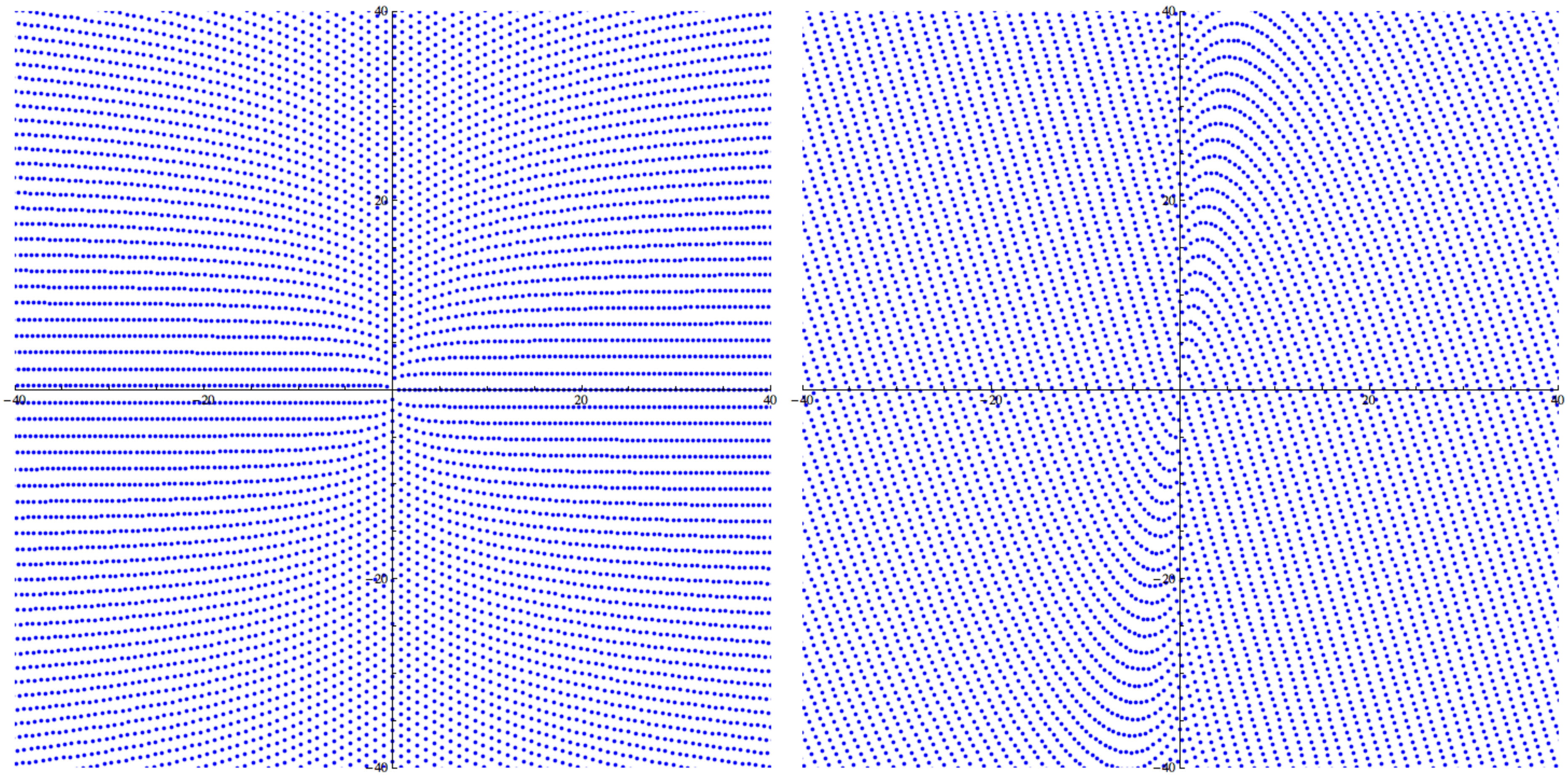
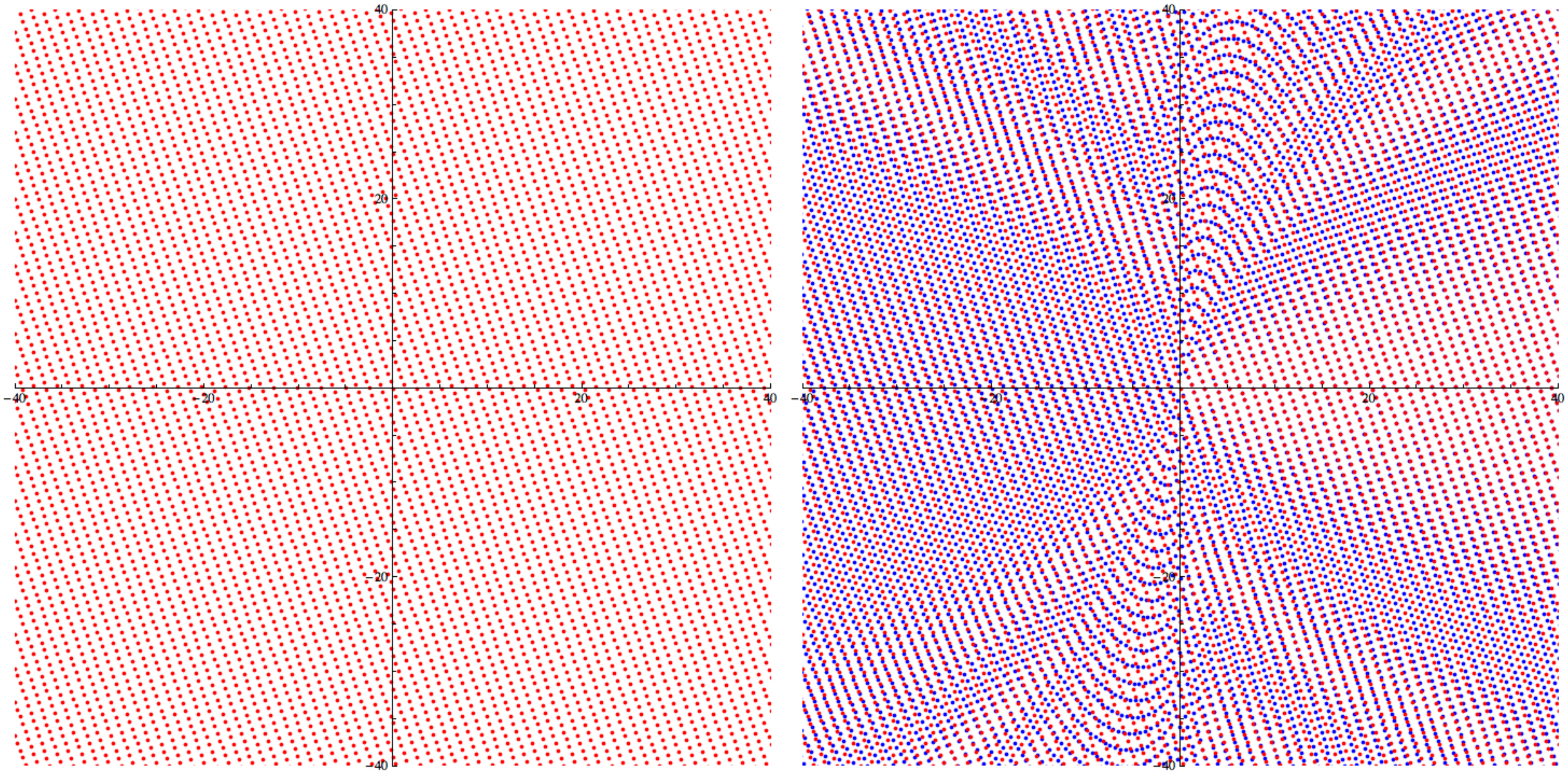


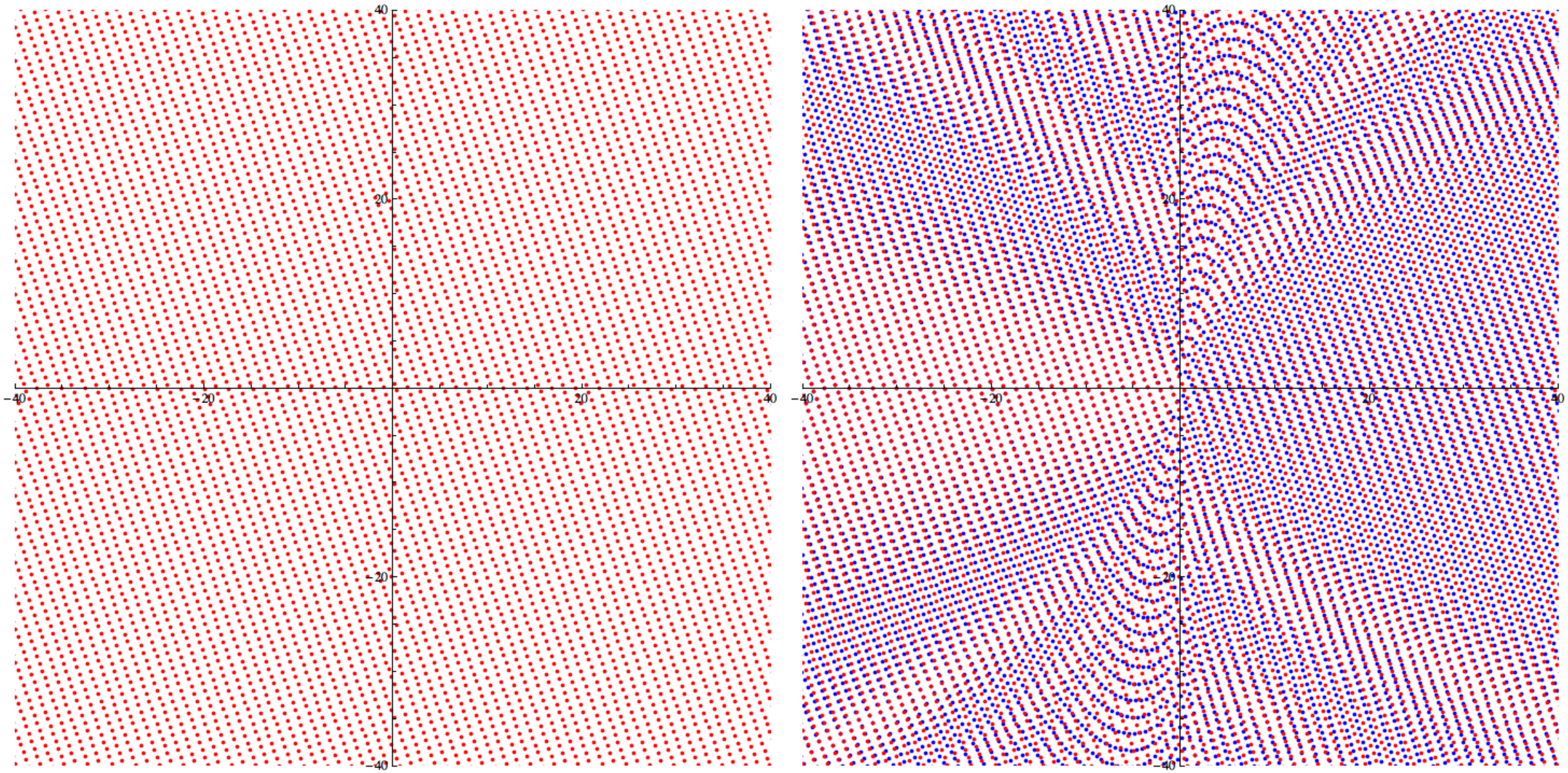
$$\mathcal{P} = \left\{ \left(\sqrt{\frac{n}{\pi}} \cos\left(2\pi\sqrt{n}\right), \sqrt{\frac{n}{\pi}} \sin\left(2\pi\sqrt{n}\right) \right) \,\middle|\, n \in \mathbb{N} \right\}$$



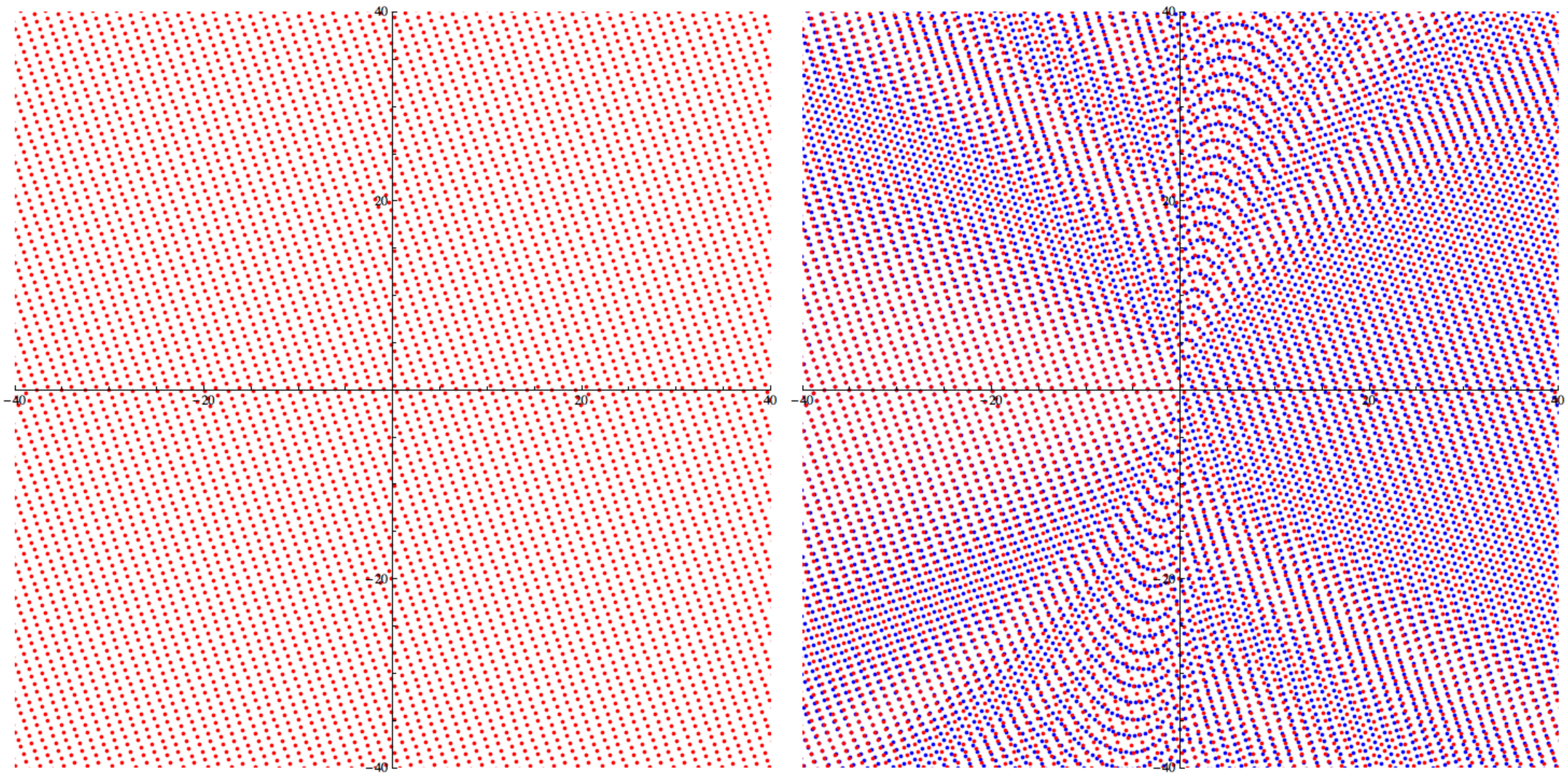
The point sets $\mathcal{P}$ and $\mathcal{P}R(\theta)D(T)$ with $T = 4$ and $\theta = 0.7$.



The approximation of $\mathcal{P}R(\theta)D(T)$ by an affine lattice in right halfplane.



The approximation of same $\mathcal{P}R(\theta)D(T)$ by different affine lattice in left halfplane.



Proof of convergence to limiting point process (two coupled but different random affine lattices in left and right half plane) follows closely Elkies & McMullen (Duke 2005), see “Square roots and lattices” (preprint arXiv:2406.09107)

Open questions, future challenges (cont'd)

- (Super-) diffusive limits in various scaling limits (Boltzmann-Grad vs. long time)

- Boltzmann-Grad limit of Lorentz gas in force fields; trajectories will be curved $\Rightarrow$ subtle lattice point counting problems, expect little difference between random and periodic scatterer configurations

- Extend to long-range potentials, e.g. Coulomb

- Boltzmann-Grad limit of quantum Lorentz gas (cf. work by Erdös, Yau and others on random Lorentz gas; extension to periodic and quasiperiodic seems out of reach); "quantum chaos"

Further reading

- J. Marklof and A. Strömbergsson
  Kinetic theory for the low-density Lorentz gas
  Memoirs of the American Mathematical Society Volume 294, Number 1464 (2024), https://doi.org/10.1090/memo/1464

- J. Marklof
  Random lattices in the wild: from Polya's orchard to quantum oscillators
  Feature, London Mathematical Society Newsletter, Issue 493 (2021) 42-49

- J. Marklof
  The low-density limit of the Lorentz gas: periodic, aperiodic and random
  Proceedings of the ICM 2014, Seoul