Spectral theta series

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Set-up

- X — compact C^∞ manifold with volume dx
- $P = \sqrt{-\Delta + V}$, Δ the Laplacian and $V \in C^\infty(X)$ positive potential (P is a pseudo-differential operator of order 1)

P has discrete spectrum

$$\rho_1 \le \rho_2 \le \rho_3 \le \ldots \to \infty,$$

denote by $(\varphi_n)_{n\in\mathbb{N}}$ and orthonormal basis of eigenfunctions, s.t.

$$P\varphi_n = \rho_n \varphi_n.$$

Wave trace. The asymptotic expansion of the wave trace

$$\operatorname{Tr} e^{-iPt} = \sum_n e^{-i\rho_n t}$$

in terms of singular distributions suppported at the lengths of the periodic bicharacteristics of P (i.e., the closed geodesics on X) is well understood

(Selberg, Gutzwiller, Balian-Bloch, Chazarain, Colin de Verdière, Duistermaat-Guillemin,. . .)

Spectral theta series. Objective: Replace P by P^2 and study

$$\vartheta_P(z) = \mathrm{Tr}\, e(P^2 z) = \sum_n e(\rho_n^2 z), \qquad e(z) = \mathrm{e}^{2\pi \mathrm{i} z},$$

for z in the complex upper half plane $\mathbb{H}$, when $\mathrm{Im}\, z \to 0$. Duistermaat and Guillemin have pointed out that this behaviour is expected to be much more singular than for the wave trace.

The function $Z(t) = \vartheta_P(\mathrm{i}t)$ ($t > 0$) corresponds of course to the trace of the heat kernel on X.

Note the formal relation

$$\vartheta_P(z) = \frac{1}{\sqrt{-2\mathrm{i}z}} \int_{-\infty}^{\infty} \mathrm{Tr}\, e(-\mathrm{i}Ps)\, e(-s^2/4z)\, ds. \tag{1}$$

The classical theta series. Let $P = \sqrt{-\Delta}$, Δ the Laplacian on the unit circle. Then (w.r.t. even test functions)

$$\mathrm{Tr}\, e(-\mathrm{i}Ps) = \sum_{n\in\mathbb{Z}} e(ns) \overset{\text{Poisson summation}}{=} \sum_{k\in\mathbb{Z}} \delta(s-k) \tag{2}$$

In this case $\vartheta_P(z)$ is of course the classical theta series

$$\vartheta(z) = \sum_{n\in\mathbb{Z}} e(n^2 z),$$

and the trace formula (2) is encoded, via (1), as

$$\vartheta(z) = \frac{1}{\sqrt{-2\mathrm{i}z}}\, \vartheta(-1/4z)$$

The second fundamental funct relation is

$$\vartheta(z+1) = \vartheta(z)$$

Motivation.

- *quantum dynamics:* the autocorrelation function of solutions $u(x,t)$ to the Schrödinger equation

$$-\frac{1}{2\pi \mathrm{i}}\,\partial_t u = P^2 u$$

is def by

$$C(t) = \int_X u(x,t)\,\overline{u(x,0)}\,dx$$

For the initial data $u(x,0) = \sum \mathrm{e}^{-\pi\rho_n^2 y}\varphi_n(x)$ we have

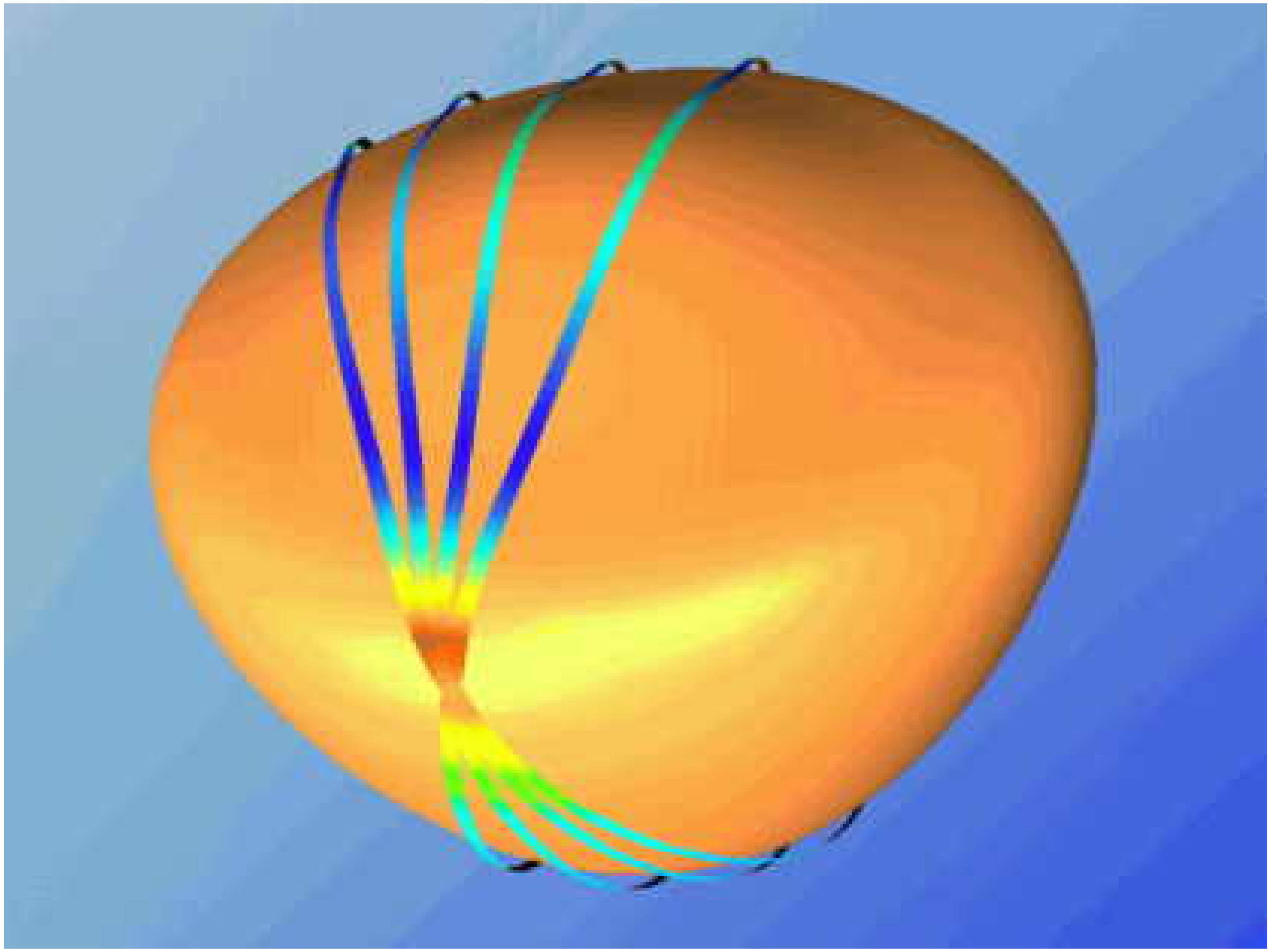
$$C(t) = \vartheta_P(-t + \mathrm{i}y)$$

- *spectral statistics:* for X a surface the correctly scaled pair correlation density

$$\sum_{i,j} \delta(s - (\rho_i^2 - \rho_j^2))\mathrm{e}^{-2\pi(\rho_i^2+\rho_j^2)y}$$

is the Fourier transform of $|\vartheta_P(x + \mathrm{i}y)|^2$ (w.r.t. x)

A Zoll surface



from http://www-sfb288.math.tu-berlin.de/Research/GEODESICS/Geodesic.html

Periodic flows and spectral clusters.

- assume X is a Zoll manifold—i.e., the geodesic flow is periodic, and all geodesics have length 2π

$\Rightarrow$ (Weinstein, Duistermaat-Guillemin, Colin de Verdiere)
There is a constant $M > 0$ such that

$$\mathrm{spec}(P^2) \subset \bigcup_k \left[\left(k + \frac{\alpha}{4}\right)^2 - M, \left(k + \frac{\alpha}{4}\right)^2 + M \right], \qquad k = 0, 1, 2, \ldots$$

where $\alpha \in \mathbb{Z}$ is the common Maslov index of the geodesics.

Relabel the eigenvalues

$$\rho_n^2 = \left(k + \frac{\alpha}{4}\right)^2 + \mu_{kl}$$

where $1 \le l \le \delta_k$ and

$$-M \le \mu_{k1} \le \ldots \le \mu_{k\delta_k} \le M$$

Define spectral density

$$\mu_k(\lambda) = \sum_{l=1}^{\delta_k} \delta(\lambda - \mu_{kl}).$$

Theorem (Colin de Verdiere 1979) *There is a distribution* $\mathcal{R}(t)$ *with support in* $[-M, M]$ *such that for every* $f \in C^\infty(\mathbb{R})$

$$\int f\, d\mu_k = \int f\, d\mathcal{R}\left(k + \frac{\alpha}{4}\right)$$

and

$$\mathcal{R}(x) \sim \nu_1 x^{d-1} + \nu_3 x^{d-3} + \nu_4 x^{d-4} + \ldots$$

Approximate functional relations

Set

$$W(t) = \int_{-M}^{M} e(t\lambda) d\nu_1(\lambda).$$

and denote by $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \simeq \mathbb{H} \times \mathbb{R}$ the universal cover of $\mathrm{SL}(2,\mathbb{R})$. A lattice $\Gamma \subset \widetilde{\mathrm{SL}}(2,\mathbb{R})$ is a discrete subgroup such that the homogeneous space $\Gamma\backslash\widetilde{\mathrm{SL}}(2,\mathbb{R})$ has finite measure (with respect to Haar).

Theorem *There is a continuous function* $\Theta : \widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \mathbb{C}$ *with the properties that*

(i) there is a lattice $\Gamma \subset \widetilde{\mathrm{SL}}(2,\mathbb{R})$ *such that* $\Theta(\gamma M) = \Theta(M)$ *for all* $\gamma \in \Gamma$,

(ii) there is a constant $C \geq 0$ *such that for all* $z = x + \mathrm{i}y \in \mathbb{H}$

$$|y^{(2d-1)/4}\vartheta_P(z) - W(x)\Theta(z,0)| \leq Cy^{1/4}.$$

The following results are consequences of the almost modularity of $\varphi_P(z)$ and ergodic properties of the geodesic flow on $\Gamma \subset \widetilde{\mathrm{SL}}(2,\mathbb{R})$.

Lagarithm laws

Theorem *Let* $\psi : (0,1] \to \mathbb{R}_+$ *be a non-increasing function such that the integral*

$$\int_0^1 \frac{dy}{y\psi(y)^4}$$

diverges (resp. converges). Then for almost every (resp. almost no) $x \in \mathbb{R}$ *there is an infinite sequence of* $y_1 > y_2 > \ldots \to 0$ *such that*

$$|\vartheta_P(x + \mathrm{i}y_j)| \geq y_j^{-(2d-1)/4}\psi(y_j).$$

The proof of this theorem exploits Sullivan's logarithm law for geodesic flows.

That is for $y < 1$ and almost all x,

$$\vartheta_P(x + \mathrm{i}y) \neq O_x\Big(y^{-(2d-1)/4}\psi(y)\Big),$$

and

$$\vartheta_P(x + \mathrm{i}y) = O_x\Big(y^{-(2d-1)/4}\psi(y)\Big)$$

respectively, if the above integral diverges or converges. Compare this behaviour with the heat kernel asymptotics ($x = 0$) where

$$\vartheta_P(\mathrm{i}y) \sim cy^{-d/2}$$

for some constant $c > 0$.

By choosing $\psi(y) = (\log y)^{(1\pm\epsilon)/4}$ in the theorem with $\epsilon > 0$ arbitrarily small we obtain the following.

Corollary *For almost all* x

$$\limsup_{y\to 0} \frac{\log(y^{(2d-1)/4}|\vartheta_P(x+\mathrm{i}y)|)}{\log\log y^{-1}} = \frac{1}{4}.$$

Limit theorems

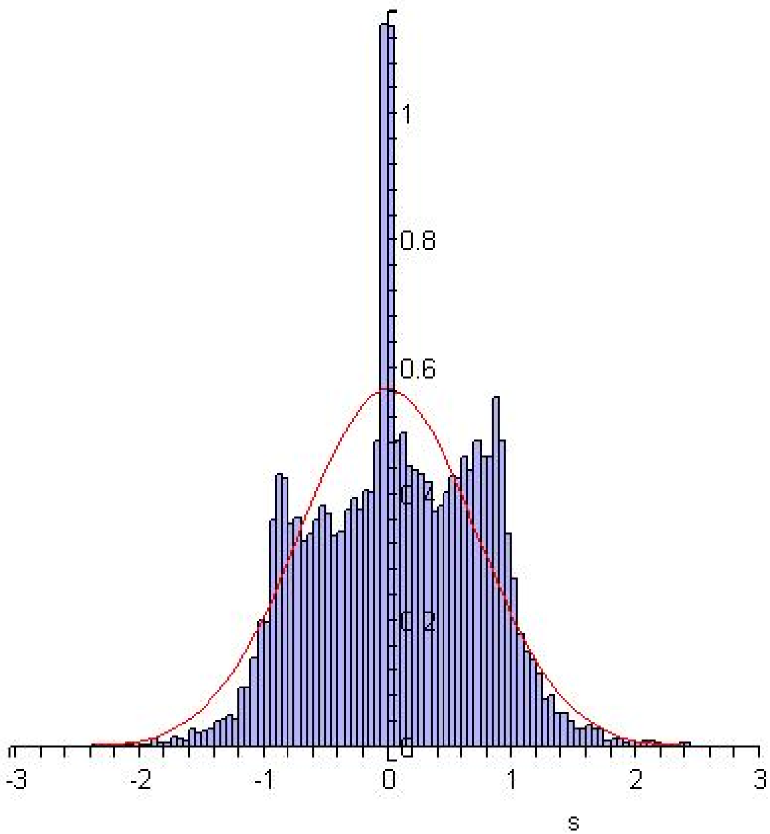
View x as a random variable uniformly distributed in $[a,b]$.

It is not hard to show that the variance has the asymptotics

$$\int_a^b |\vartheta_P(x+\mathrm{i}y)|^2 dx \sim y^{-(d-1/2)} \int_0^\infty s^{2(d-1)} \mathrm{e}^{-4\pi s^2} ds \int_a^b |W(t)|^2 dt.$$

We therefore normalize $\vartheta_P(z)$ by a factor $y^{(2d-1)/4}$.

Distribution of the real part of the spectral theta series of the circle and the sphere vs. normal distribution



Theorem *Let* $[a,b] \subset \mathbb{R}$ *and* g *a bounded continuous function* $\mathbb{C} \to \mathbb{R}$*. Then*

$$\lim_{y\to 0} \int_a^b g\Big(y^{(2d-1)/4}\vartheta_P(x+\mathrm{i}y)\Big)dx = \int_{\mathbb{C}} \int_a^b g(Z\,W(t))\,d\rho_{d,\alpha}(Z)\,dt$$

where $\rho_{d,\alpha}$ *is a probability measure on* $\mathbb{C}$ *with the tail distribution*

$$\int_{|Z|>R} d\rho_{d,\alpha}(Z) \sim c_{d,\alpha} R^{-4} \qquad (R \to \infty).$$

This theorem is a consequence of mixing of the geodesic flow, or more specifically the uniform distribution of horocycles.

Correlations

Theorem *Suppose* $\omega_1, \ldots, \omega_n \in \mathbb{R}$ *are linearly independent over* $\mathbb{Q}$*, and let* $[a,b] \subset \mathbb{R}$ *and* g *a bounded continuous function* $\mathbb{C}^n \to \mathbb{R}$*. Then*

$$\lim_{y\to 0} \int_a^b g\Big(y^{(2d-1)/4}\vartheta_P(\omega_1 x + \mathrm{i}y), \ldots, y^{(2d-1)/4}\vartheta_P(\omega_n x + \mathrm{i}y)\Big)\, dx$$
$$= \int_{\mathbb{C}^n} \int_a^b g(Z_1\, W(t), \ldots, Z_n\, W(t)) \prod_{j=1}^{n} d\rho_{d,\alpha}(Z_j)\, dt.$$

This theorem follows from Shah's theorem on the uniform distribution of translates of unipotent orbits, which is based on Ratner's theory.

Let me now sketch how the approximate functional equations for $\vartheta_P(z)$ can be established.

For $g \in \mathsf{SL}(2,\mathbb{R})$ we have the Iwasawa decomposition

$$g = n_x \, a_y \, k_\phi = (z, \phi),$$

where $z = x + \mathrm{i}y \in \mathfrak{H}$, $\phi \in [0, 2\pi)$, and

$$n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a_y = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}, \quad k_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

This can be extended to $\widetilde{\mathsf{SL}}(2,\mathbb{R})$,

$$M = N_x \, A_y \, K_\phi$$

where N_x, A_y, K_ϕ are the corresponding lifts and $x + \mathrm{i}y \in \mathbb{H}$, $\phi \in \mathbb{R}$.

Shale-Weil representation. For any $f \in \mathsf{L}^2(\mathbb{R})$ we set

$$[R(N_x)f](t) = e(t^2x)f(t),$$

$$[R(A_y)f](t) = y^{1/4}f(y^{1/2}t),$$

and

$$[R(K_\phi)f](t) = \begin{cases} e(-\sigma_\phi/8)\, f(t) & (\phi = 0 \bmod 2\pi) \\ e(-\sigma_\phi/8)\, f(-t) & (\phi = \pi \bmod 2\pi) \\ \dfrac{e(-\sigma_\phi/8)\, 2^{1/2}}{|\sin\phi|^{1/2}} \displaystyle\int_{\mathbb{R}} e\left[\frac{(t^2+t'^2)\cos\phi - 2tt'}{\sin\phi}\right] f(t')\,dt' & \\ & (\phi \neq 0 \bmod \pi) \end{cases}$$

where

$$\sigma_\phi = \begin{cases} 2\nu & \text{if } \phi = \nu\pi, \\ 2\nu+1 & \text{if } \nu\pi < \phi < (\nu+1)\pi. \end{cases}$$

For sufficently nice* f we define the theta series

$$\Theta_f(z,\phi) := \Theta_f(M) := \sum_{n\in\mathbb{Z}} [R(M)f](n),$$

with $M = N_x A_y K_\phi$. More explicitly,

$$\Theta_f(z,\phi) = y^{1/4} \sum_{n\in\mathbb{Z}} f_\phi(ny^{1/2})\, e(n^2 x),$$

where $f_\phi = R(K_\phi)f$.

Using Poisson summation and periodicity, one can show that Θ_f is continuous and invariant under a lattice Γ in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

*That is

$$\sup_{t,\phi}(1+|t|)^\eta |[R(K_\phi)f](t)| < \infty.$$

Choice of f. We require for our application (in view of Colin de Verdiere's Theorem p.9)

$$f(t) = \begin{cases} 0 & (t \leq 0) \\ t^{d-1}\mathrm{e}^{-2\pi t^2} & (t > 0). \end{cases}$$

This leads to

$$f_\phi(t) = \mathrm{e}^{-\mathrm{i}\pi/4} 2^{1/2} (\sin\phi)^{-1/2} e\left(t^2 \cot\phi\right) \int_0^\infty t'^{d-1} e\left[\frac{t'^2 \mathrm{e}^{\mathrm{i}\phi} - 2tt'}{\sin\phi}\right] dt'$$

$$= \frac{\mathrm{e}^{-\mathrm{i}\phi/2} 2^{1/2} \Gamma(d) \mathrm{e}^{-\pi t^2 (1 - \mathrm{i}\cot\phi)}}{(4\pi)^{d/2} (1 - \mathrm{i}\cot\phi)^{(d-1)/2}} D_{-d}\left(\mathrm{i}t[4\pi(1 + \mathrm{i}\cot\phi)]^{1/2}\right)$$

where $D_p(z)$ is a parabolic cylinder function.

For d odd one can in fact use symmetry to show that $f_\phi(t)$ can be expressed in terms of Hermite functions rather that parabolic cyclinder functions.