The low-density limit of the Lorentz gas: periodic, aperiodic and random

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Jens Marklof

School of Mathematics, University of Bristol
http://www.maths.bristol.ac.uk

based on joint work with
Andreas Strömbergsson (Uppsala) and Bálint Tóth (Bristol & Budapest)

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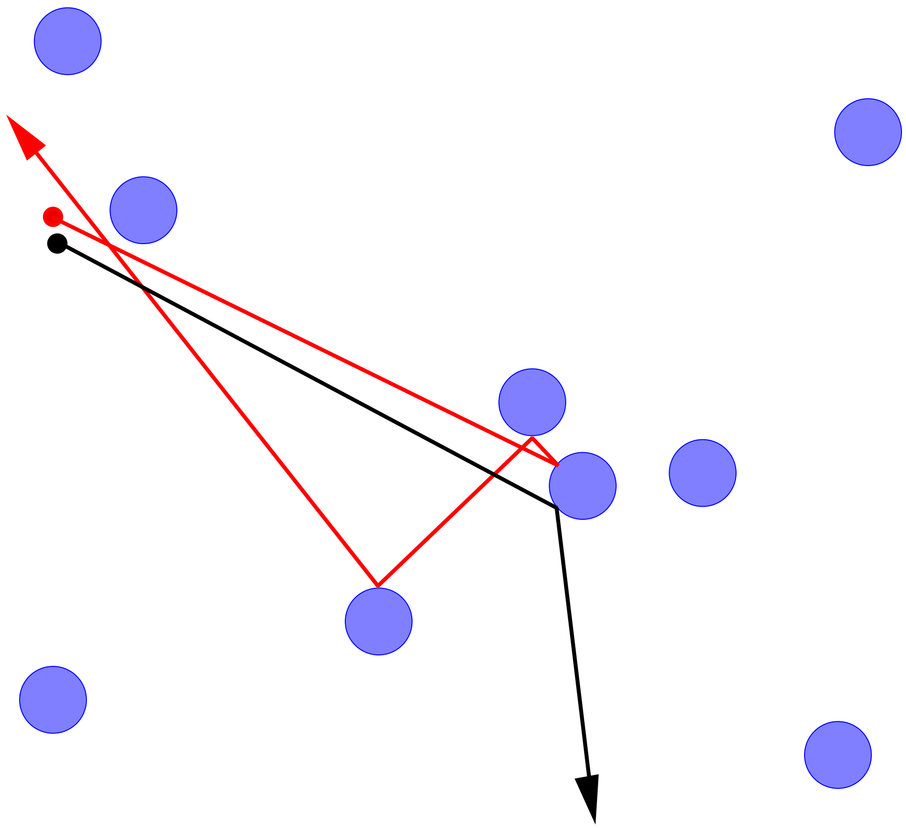
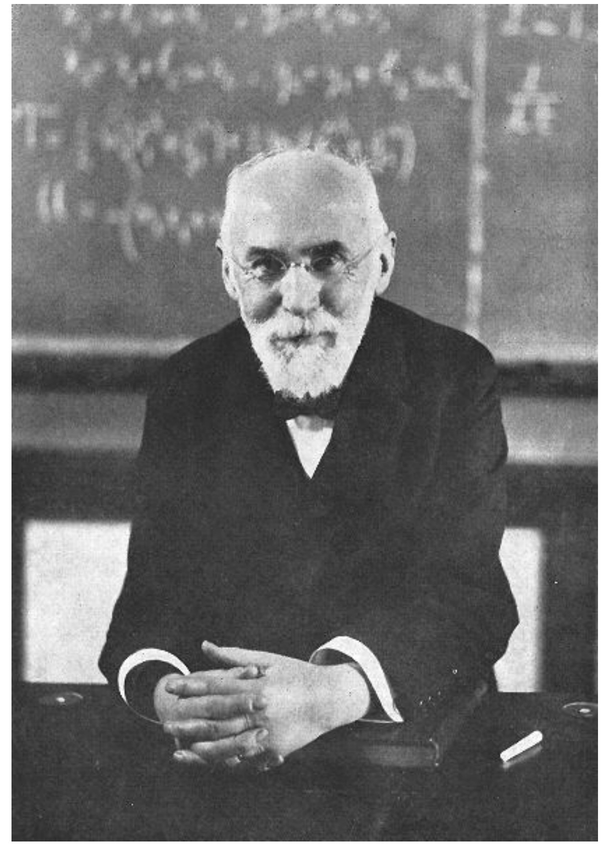
The Lorentz gas

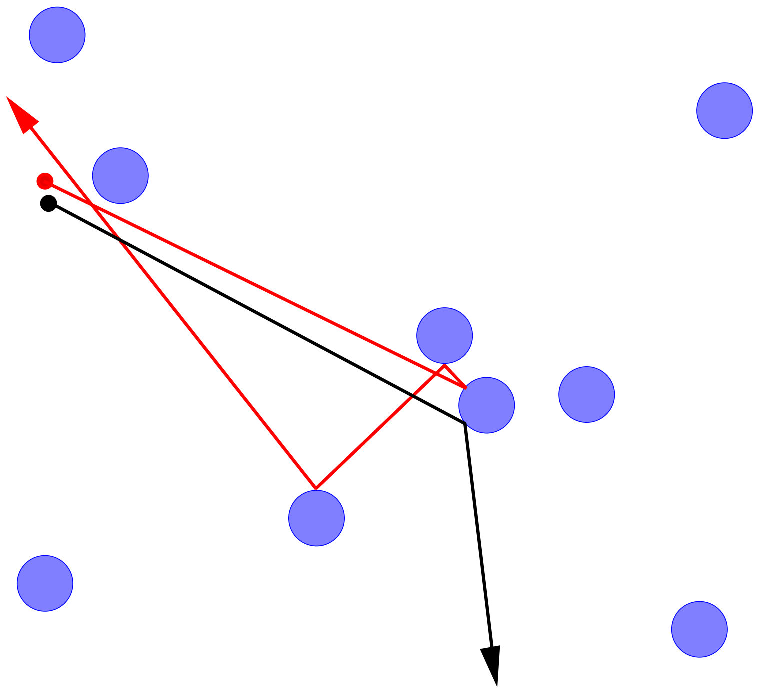


Arch. Neerl. (1905)



Hendrik Lorentz (1853-1928)

The Lorentz gas



- $\mathcal{P}$ — locally finite subset of $\mathbb{R}^d$ with unit density*
- scatterers are fixed open balls of radius r centered at the points in $\mathcal{P}$
- the particles are assumed to be non-interacting
- each test particle moves with constant velocity $\boldsymbol{v}(t)$ between collisions
- the scattering is elastic; we may assume w.l.o.g. $\|\boldsymbol{v}(t)\| = 1$

**unit density* means that $\lim_{R\to\infty} \frac{\#(\mathcal{P} \cap R\mathcal{D})}{R^d \,\mathrm{vol}(\mathcal{D})} = 1$ for all “nice” sets $\mathcal{D} \subset \mathbb{R}^d$

Examples

Example 1: $\mathcal{P} =$ a realization of the Poisson process in $\mathbb{R}^d$ with intensity 1

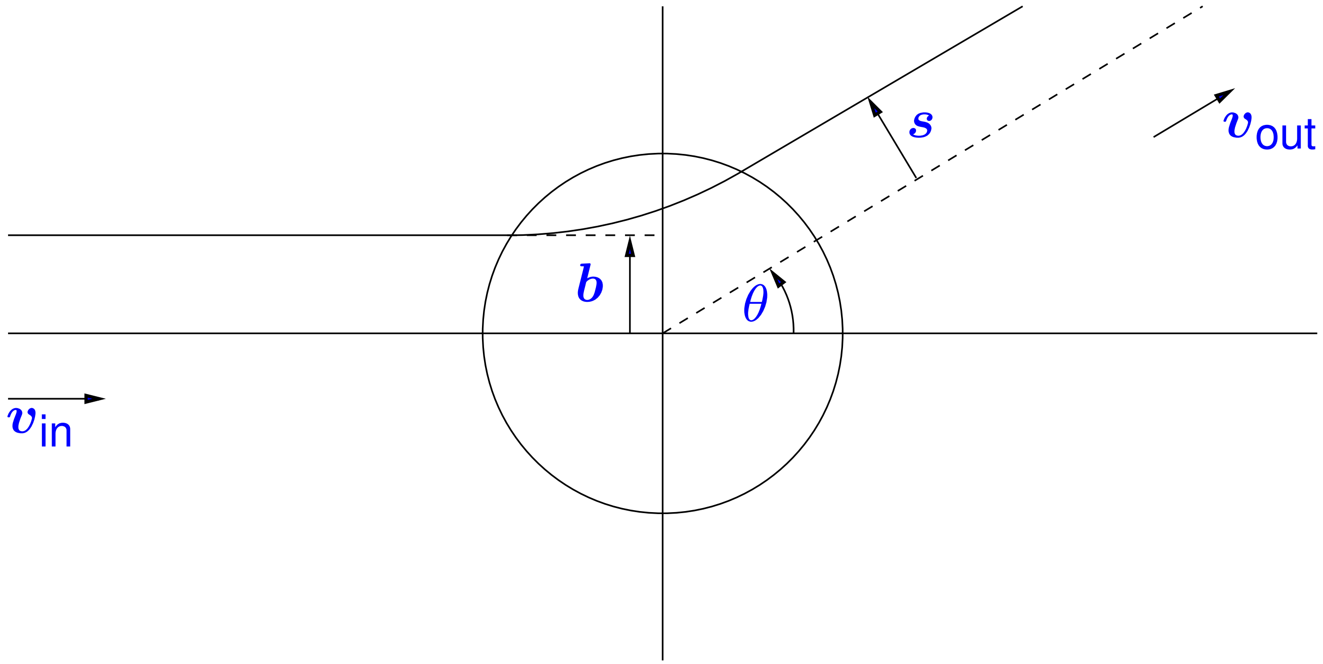
Example 2: $\mathcal{P} = \mathbb{Z}^d$ (periodic Lorentz gas)

Example 3: $\mathcal{P} =$ the vertex set of a Penrose tiling (quasicrystal)

In the case of fixed scattering radius r, allmost all results to-date on the diffusion of a test-particle in the Lorentz gas are restricted to the 2-dim periodic setting:

- Bunimovich & Sinai (Comm Math Phys 1980)
- Bleher (J Stat Phys 1992)
- Szász & Varjú (J Stat Phys 2007)
- Dolgopyat & Chernov (Russ Math Surveys, 2009)

The scattering map



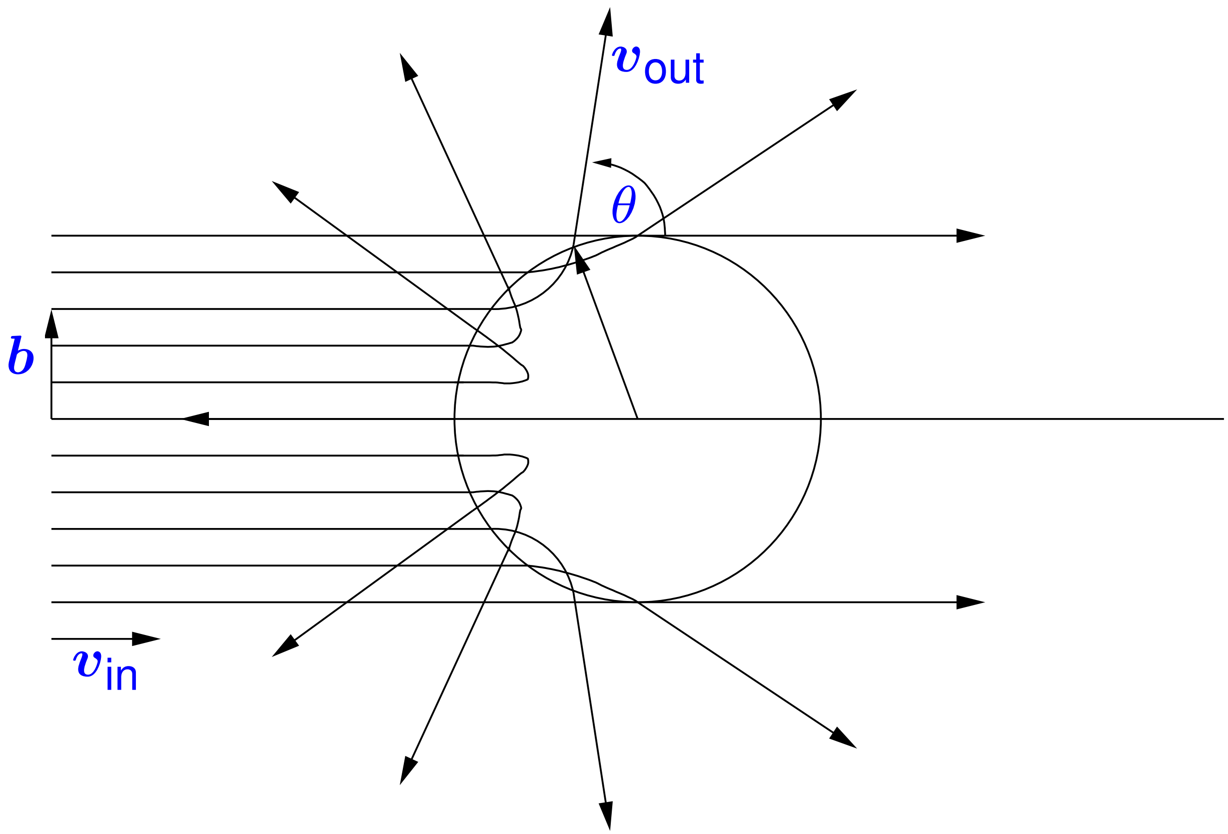
$\boldsymbol{v}_{\text{in}}$, $\boldsymbol{v}_{\text{out}}$ — incoming/outgoing velocity

$\boldsymbol{b}$, $\boldsymbol{s}$ — impact/exit parameter

(=the orthogonal projection of the point of impact onto the plane orthogonal to resp. $\boldsymbol{v}_{\text{in}}$, $\boldsymbol{v}_{\text{out}}$, measured in units of the scattering radius r)

$\theta = \theta(w)$ — the scattering angle, $w := \|\boldsymbol{b}\| \in [0, 1[$

The scattering map



Assume:

(A) $\theta \in C^1([0, 1[\,)$ is strictly decreasing with $\theta(0) = \pi$ and $\theta(w) > 0$ (as in figure) or

(B) $\theta \in C^1([0, 1[\,)$ is strictly increasing with $\theta(0) = -\pi$ and $\theta(w) < 0$

Examples

Example 1: In the classical setting of elastic hard-sphere scatterers,

$$\theta(w) = \pi - 2\,\mathrm{arcsin}(w)$$

and thus condition (A) holds

Example 2: Scattering by "muffin-tin" Coulomb potential

$$V(\boldsymbol{q}) = \begin{cases} \alpha\left(\frac{r}{\|\boldsymbol{q}\|} - 1\right) & (\|\boldsymbol{q}\| < r) \\ 0 & (\|\boldsymbol{q}\| \geq r) \end{cases}$$

with $\alpha \notin \{-2E, 0\}$ and E the total particle energy

The Boltzmann-Grad limit

- Consider the dynamics in the limit of small scatterer radius r

- $\big(\boldsymbol{q}(t), \boldsymbol{v}(t)\big)$ = "microscopic" phase space coordinate at time t

- A dimensional argument shows that, in the limit $r \to 0$, the mean free path length (i.e., the average time between consecutive collisions) scales like $r^{-(d-1)}$ (= 1/total scattering cross section)

- We thus measure position and time the "macroscopic" coordinates

$$\big(\boldsymbol{Q}(t), \boldsymbol{V}(t)\big) = \big(r^{d-1}\boldsymbol{q}(r^{-(d-1)}t), \boldsymbol{v}(r^{-(d-1)}t)\big)$$

- Time evolution of initial data $(\boldsymbol{Q}, \boldsymbol{V})$:

$$\big(\boldsymbol{Q}(t), \boldsymbol{V}(t)\big) = \Phi_r^t(\boldsymbol{Q}, \boldsymbol{V})$$

The linear Boltzmann equation

- Time evolution of a particle cloud with initial density $f \in \mathrm{L}^1$:

$$f_t^{(r)}(\boldsymbol{Q}, \boldsymbol{V}) := f\Big(\Phi_r^{-t}(\boldsymbol{Q}, \boldsymbol{V})\Big)$$

In his 1905 paper Lorentz suggested that $f_t^{(r)}$ is governed, as $r \to 0$, by the linear Boltzmann equation:

$$\left[\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{Q}}\right] f_t(\boldsymbol{Q}, \boldsymbol{V}) = \int_{\mathrm{S}_1^{d-1}} \Big[f_t(\boldsymbol{Q}, \boldsymbol{V}') - f_t(\boldsymbol{Q}, \boldsymbol{V})\Big] \sigma(\boldsymbol{V}, \boldsymbol{V}') d\boldsymbol{V}'$$

where $\sigma(\boldsymbol{V}, \boldsymbol{V}')$ is the differential cross section of the individual scatterer.
E.g.: $\sigma(\boldsymbol{V}, \boldsymbol{V}') = \frac{1}{4}\|\boldsymbol{V} - \boldsymbol{V}'\|^{3-d}$ for specular reflection at a hard sphere

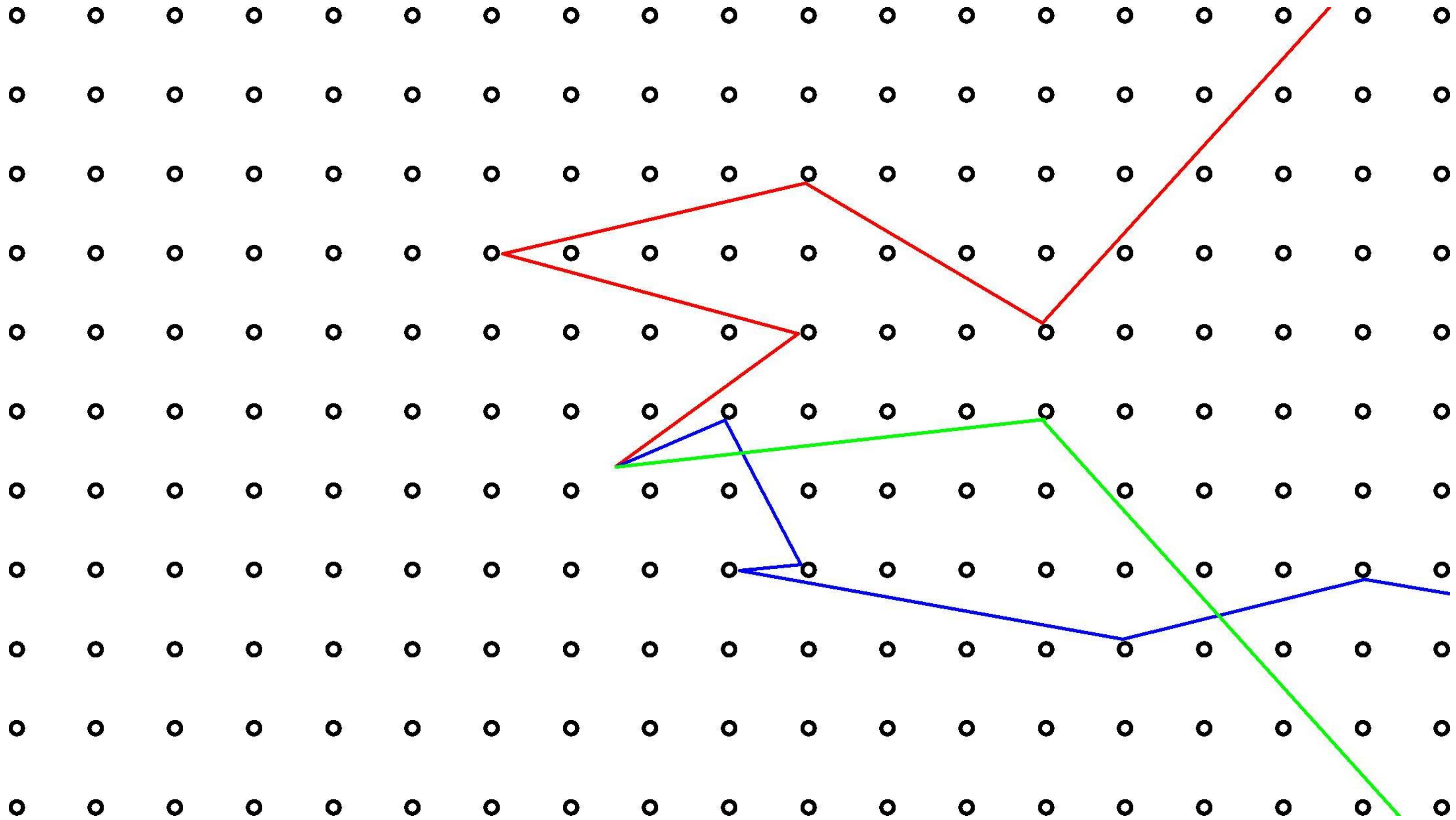
Applications: Neutron transport, radiative transfer, . . .

The linear Boltzmann equation—rigorous proofs

- Galavotti (Phys Rev 1969 & report 1972): Poisson distributed hard-sphere scatterer configuration $\mathcal{P}$
- Spohn (Comm Math Phys 1978): extension to more general random scatterer configurations $\mathcal{P}$ and potentials
- Boldrighini, Bunimovich and Sinai (J Stat Phys 1983): prove convergence for almost every scatterer configuration $\mathcal{P}$ (w.r.t. the Poisson random measure)
- Quantum: Spohn (J Stat Phys 1977): Gaussian random potentials, weak coupling limit & small times; Erdös and Yau (Contemp Math 1998, Comm Pure Appl Math 2000): General random potentials, weak coupling limit; Eng and Erdös (Rev Math Phys 2005): Low density limit

. . . but what about non-random scatterer configurations?

The periodic Lorentz gas



The distribution of free path length

For random exit parameter and exit velocity, consider the probability $F_r(t)$ of hitting the next scatterer after time t (measured in units of the mean collision time).

- Bourgain, Golse, Wennberg (CMP 1998); Golse, Wennberg (CMP 2000):

$$t^{-2} \ll F_r(t) \ll t^{-2}$$

- Golse (ICM Madrid, 2006): The above lower bound implies that **the linear Boltzmann equation fails in the periodic setting**

Note:

For random scatterer configurations the path length distribution is exponential, which is consistent with the linear Boltzmann equation

The distribution of free path length

- Boca, Zaharescu (CMP 2007): proof of convergence as $r \to 0$ and explicit formula in dimension $d = 2$

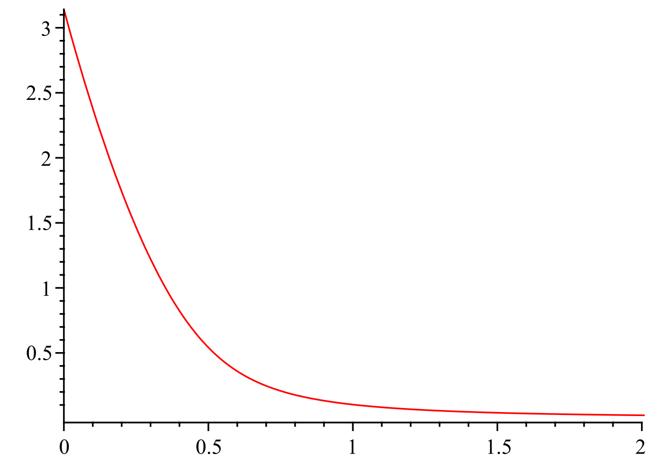
- JM & Strömbergsson (Annals of Math 2010, GAFA 2011): proof of convergence

$$F_r(t) \to D(t) = \int_t^{\infty} \Psi_0(x)dx$$

in arbitrary dimension, with continuous limit density and tail ($t \to \infty$)

$$\Psi_0(t) \sim \frac{A_d}{t^3}, \qquad A_d = \frac{2^{2-d}}{d(d+1)\zeta(d)}$$

$\Rightarrow$ No second moment!



$D(t)$ in dim. $d = 3$

A limiting random process

A cloud of particles with initial density $f(\boldsymbol{Q},\boldsymbol{V})$ evolves in time t to

$$f_t^{(r)}(\boldsymbol{Q},\boldsymbol{V}) = [L_r^t f](\boldsymbol{Q},\boldsymbol{V}) = f\Big(\Phi_r^{-t}(\boldsymbol{Q},\boldsymbol{V})\Big).$$

Theorem A [JM & Strömbergsson, Annals of Math 2011].
For every $t > 0$ there exists a linear operator

$$L^t : \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d)) \to \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$$

such that for every $f \in \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$ and any set $\mathcal{A} \subset \mathrm{T}^1(\mathbb{R}^d)$ with boundary of Liouville measure zero,

$$\lim_{r\to 0} \int_{\mathcal{A}} [L_r^t f](\boldsymbol{Q},\boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V} = \int_{\mathcal{A}} [L^t f](\boldsymbol{Q},\boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V}.$$

The operator L^t thus describes the macroscopic diffusion of the Lorentz gas in the Boltzmann-Grad limit $r \to 0$.

Note: The family $\{L^t\}_{t\geq 0}$ does *not* form a semigroup.

A generalized linear Boltzmann equation

Consider extended phase space coordinates $(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+)$:

$(\boldsymbol{Q}, \boldsymbol{V}) \in \mathrm{T}^1(\mathbb{R}^d)$ — usual position and momentum
$\xi \in \mathbb{R}_+$ — flight time until the next scatterer
$\boldsymbol{V}_+ \in \mathrm{S}_1^{d-1}$ — velocity after the next hit

$$\left[\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{Q}} - \frac{\partial}{\partial \xi}\right] f_t(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+) = \int_{\mathrm{S}_1^{d-1}} f_t(\boldsymbol{Q}, \boldsymbol{V}', 0, \boldsymbol{V})\, p_{\boldsymbol{0}}(\boldsymbol{V}', \boldsymbol{V}, \xi, \boldsymbol{V}_+) d\boldsymbol{V}'$$

with a new collision kernel $p_{\boldsymbol{0}}(\boldsymbol{V}', \boldsymbol{V}, \xi, \boldsymbol{V}_+)$, which can be expressed as a product of the scattering cross section of an individual scatterer and a ceratin transition probability for hitting a given point the next scatterer after time ξ.

A generalized linear Boltzmann equation

We obtain the original particle density via the projection

$$\overline{f}_t(\boldsymbol{Q},\boldsymbol{V}) = \int_0^\infty \int_{\mathsf{S}_1^{d-1}} f_t(\boldsymbol{Q},\boldsymbol{V},\xi,\boldsymbol{V}_+)\, d\boldsymbol{V}_+\, d\xi.$$

where $f_t(\boldsymbol{Q},\boldsymbol{V},\xi,\boldsymbol{V}_+)$ is the solution of the generalized linear Boltzmann equation subject to the initial condition

$$\lim_{t\to 0} f_t(\boldsymbol{Q},\boldsymbol{V},\xi,\boldsymbol{V}_+) = f(\boldsymbol{Q},\boldsymbol{V})\, p(\boldsymbol{V},\xi,\boldsymbol{V}_+)$$

and

$$p(\boldsymbol{V},\xi,\boldsymbol{V}_+) := \int_\xi^\infty \int_{\mathsf{S}_1^{d-1}} p_{\boldsymbol{0}}(\boldsymbol{V}',\boldsymbol{V},\xi,\boldsymbol{V}_+)\,\sigma(\boldsymbol{V},\boldsymbol{V}')\, d\boldsymbol{V}'\, d\xi'.$$

The latter is a stationary solution of the generalized linear Boltzmann equation.

Application: Superdiffusive central limit theorem

The divergent second moment of the path length distribution leads to $t \log t$ superdiffusion:

Theorem B [JM & B. Toth, preprint 2014]
Let $d \geq 2$ and fix a Euclidean lattice $\mathcal{L} \subset \mathbb{R}^d$ of covolume one. Assume $(\boldsymbol{Q}_0, \boldsymbol{V}_0)$ is distributed according to an absolutely continuous Borel probability measure Λ on $\mathrm{T}^1(\mathbb{R}^d)$. Then, for any bounded continuous $f : \mathbb{R}^d \to \mathbb{R}$,

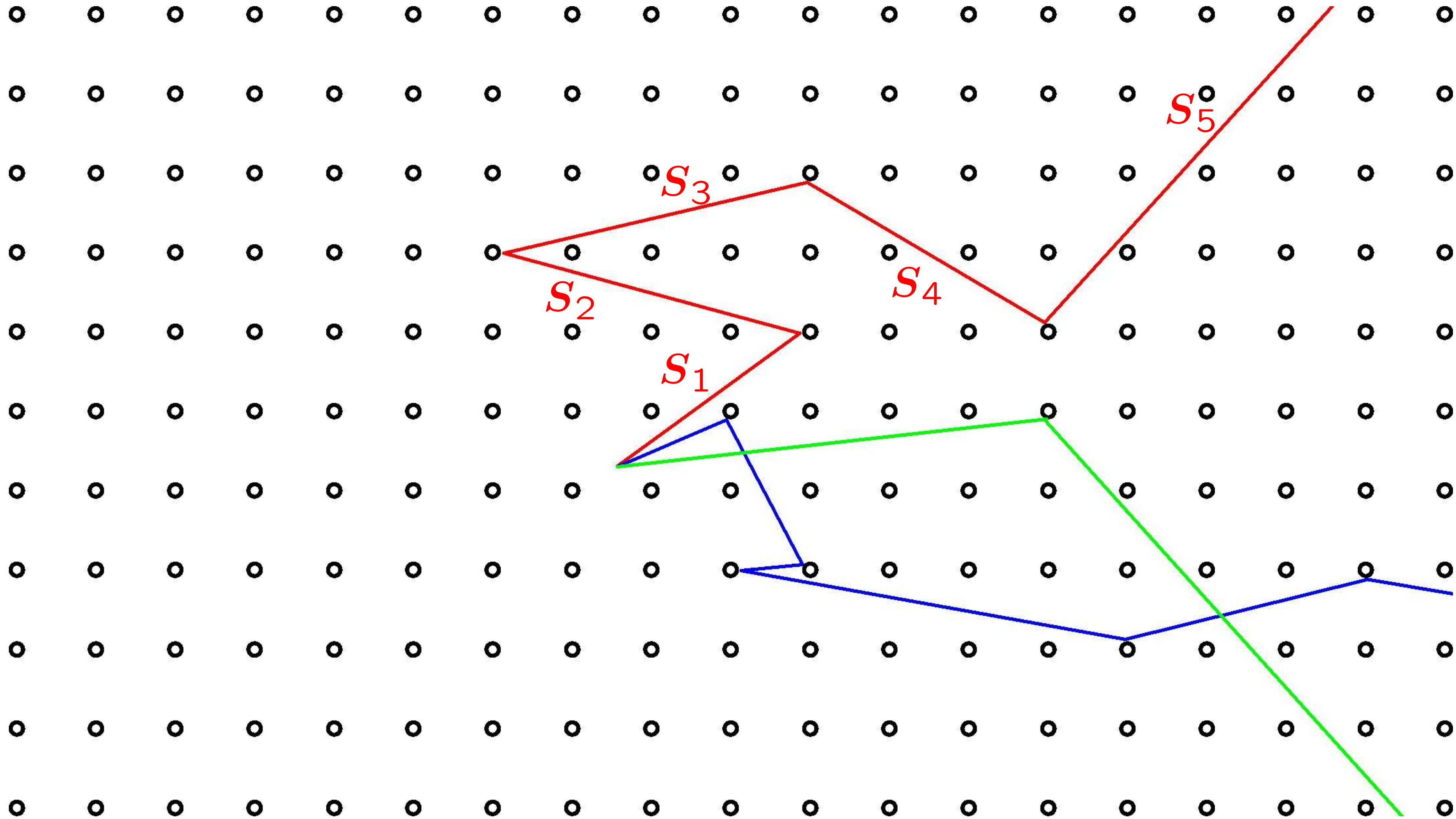
$$\lim_{t\to\infty} \lim_{r\to 0} \mathbb{E} f\left(\frac{\boldsymbol{Q}(t) - \boldsymbol{Q}_0}{\Sigma_d \sqrt{t \log t}}\right) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \mathrm{e}^{-\frac{1}{2}\|\boldsymbol{x}\|^2} d\boldsymbol{x},$$

with

$$\Sigma_d^2 := \frac{2^{1-d}\overline{\sigma}}{d^2(d+1)\zeta(d)}.$$

For fixed r the analogous result is currently known only in dimension $d = 2$, see Szász & Varjú (J Stat Phys 2007), Chernov & Dolgopyat (Russ. Math Surveys 2009).

Key ingredient for Theorem A: Joint distribution of path segments



Joint distribution for path segments

The following theorem proves the existence of a Markov process that describes the dynamics of the Lorentz gas in the Boltzmann-Grad limit.

Theorem C [JM & Strömbergsson, Annals of Math 2011]. Fix an a.c. Borel probability measure Λ on $\mathrm{T}^1(\mathbb{R}^d)$. Then, for each $n \in \mathbb{N}$ there exists a probability density $\Psi_{n,\Lambda}$ on $\mathbb{R}^{nd}$ such that, for any set $\mathcal{A} \subset \mathbb{R}^{nd}$ with boundary of Lebesgue measure zero,

$$\lim_{r\to 0} \Lambda\Big(\Big\{(\boldsymbol{Q}_0, \boldsymbol{V}_0) \in \mathrm{T}^1(\mathbb{R}^d) : (\boldsymbol{S}_1, \ldots, \boldsymbol{S}_n) \in \mathcal{A}\Big\}\Big) = \int_{\mathcal{A}} \Psi_{n,\Lambda}(\boldsymbol{S}_1', \ldots, \boldsymbol{S}_n')\, d\boldsymbol{S}_1' \cdots d\boldsymbol{S}_n',$$

and, for $n \geq 3$,

$$\Psi_{n,\Lambda}(\boldsymbol{S}_1, \ldots, \boldsymbol{S}_n) = \Psi_{2,\Lambda}(\boldsymbol{S}_1, \boldsymbol{S}_2) \prod_{j=3}^{n} \Psi(\boldsymbol{S}_{j-2}, \boldsymbol{S}_{j-1}, \boldsymbol{S}_j),$$

where Ψ is a continuous probability density independent of Λ (and the lattice).

Theorem A follows from Theorem C by standard probabilistic arguments.

First step: The distribution of free path lengths

Lattices

- $\mathcal{L} \subset \mathbb{R}^d$—euclidean lattice of covolume one

- recall $\mathcal{L} = \mathbb{Z}^d M$ for some $M \in \mathsf{SL}(d, \mathbb{R})$, therefore the homogeneous space $X_1 = \mathsf{SL}(d, \mathbb{Z}) \backslash \mathsf{SL}(d, \mathbb{R})$ parametrizes the space of lattices of covolume one

- μ_1—right-$\mathsf{SL}(d, \mathbb{R})$ invariant prob measure on X_1 (Haar)

Affine lattices

- $\mathsf{ASL}(d,\mathbb{R}) = \mathsf{SL}(d,\mathbb{R}) \ltimes \mathbb{R}^d$—the semidirect product group with multiplication law
$$(M,\boldsymbol{x})(M',\boldsymbol{x}') = (MM', \boldsymbol{x}M' + \boldsymbol{x}').$$
An action of $\mathsf{ASL}(d,\mathbb{R})$ on $\mathbb{R}^d$ can be defined as
$$\boldsymbol{y} \mapsto \boldsymbol{y}(M,\boldsymbol{x}) := \boldsymbol{y}M + \boldsymbol{x}.$$

- the space of affine lattices is then represented by $X = \mathsf{ASL}(d,\mathbb{Z})\backslash \mathsf{ASL}(d,\mathbb{R})$ where $\mathsf{ASL}(d,\mathbb{Z}) = \mathsf{SL}(d,\mathbb{Z}) \ltimes \mathbb{Z}^d$, i.e.,
$$\mathcal{L} = (\mathbb{Z}^d + \boldsymbol{\alpha})M = \mathbb{Z}^d(1,\boldsymbol{\alpha})(M,\boldsymbol{0})$$

- μ—right-$\mathsf{ASL}(d,\mathbb{R})$ invariant prob measure on X

Let us denote by $\tau_1 = \tau(\boldsymbol{q}, \boldsymbol{v})$ the free path length corresponding to the initial condition $(\boldsymbol{q}, \boldsymbol{v})$.

Theorem D [JM & Strömbergsson, Annals of Math 2010]. Fix a lattice $\mathcal{L}_0$ and the initial position $\boldsymbol{q}$. Let λ be any a.c. Borel probability measure on S_1^{d-1}. Then, for every $\xi > 0$, the limit

$$F_{\mathcal{L}_0,\boldsymbol{q}}(\xi) := \lim_{r \to 0} \lambda(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} : r^{d-1}\tau_1 \geq \xi\})$$

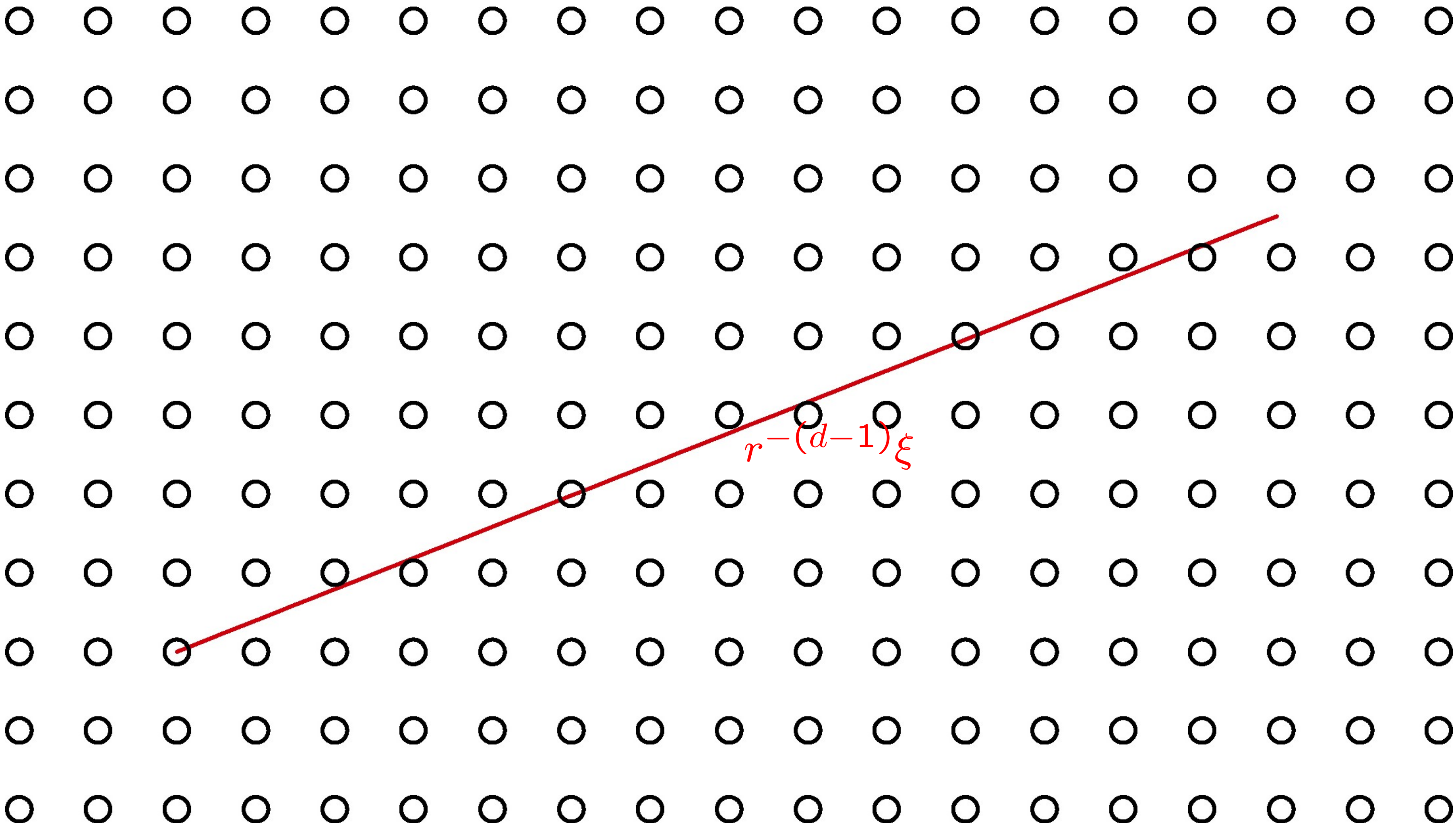
exists, is continuous in ξ and independent of λ. Furthermore

$$F_{\mathcal{L}_0,\boldsymbol{q}}(\xi) = \begin{cases} F_0(\xi) := \mu_1(\{\mathcal{L} \in X_1 : \mathcal{L} \cap \mathcal{Z}(\xi) = \emptyset\}) & \text{if } \boldsymbol{q} \in \mathcal{L}_0 \\ F(\xi) := \mu(\{(\mathcal{L} \in X : \mathcal{L} \cap \mathcal{Z}(\xi) = \emptyset\}) & \text{if } \boldsymbol{q} \notin \mathbb{Q}\mathcal{L}_0. \end{cases}$$

with the cylinder

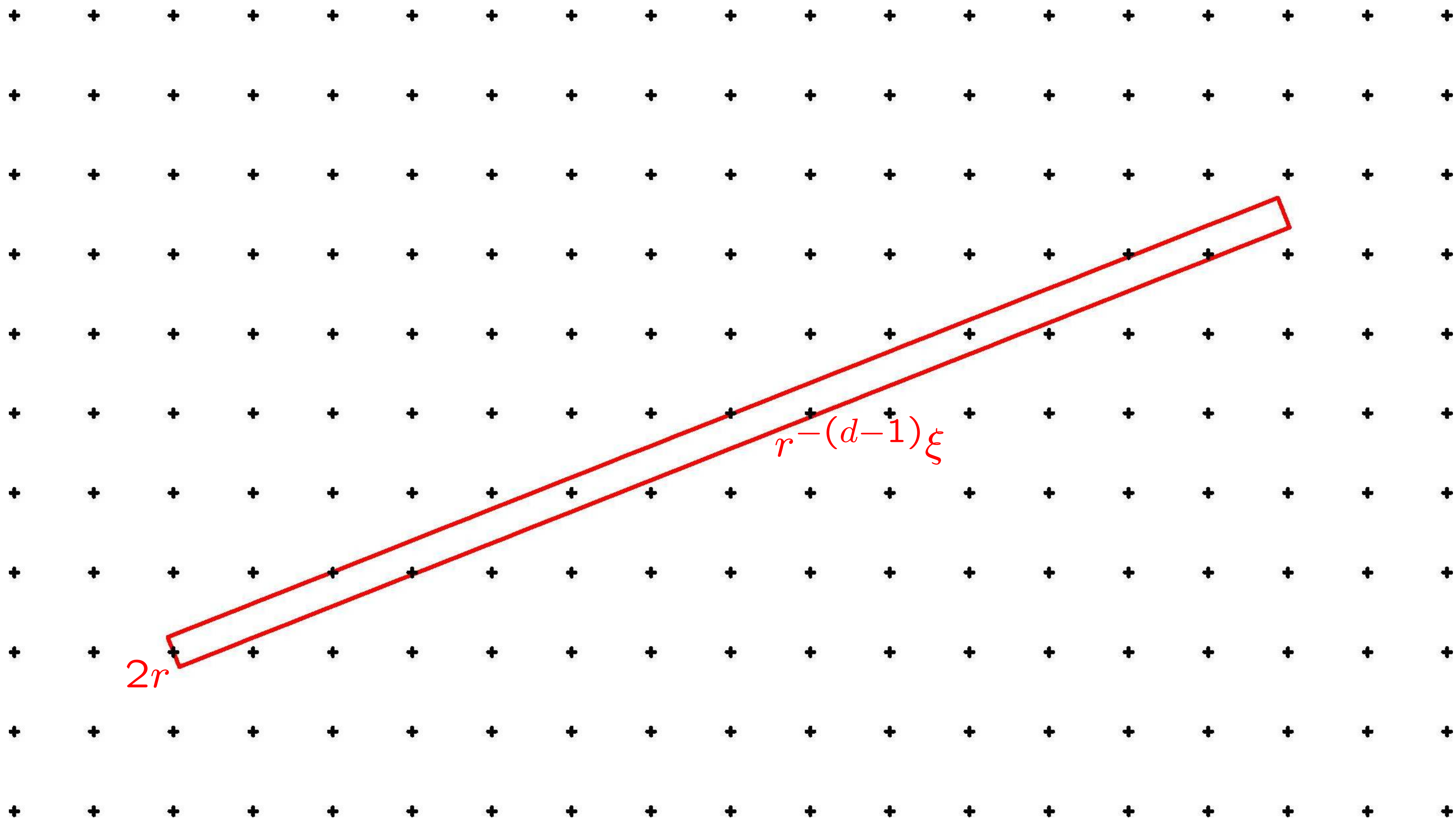
$$\mathcal{Z}(\xi) = \Big\{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 < x_1 < \xi, x_2^2 + \ldots + x_d^2 < 1\Big\}.$$

Idea of proof ($\boldsymbol{q} = \mathbf{0}$, $\mathcal{L}_0 = \mathbb{Z}^d$)



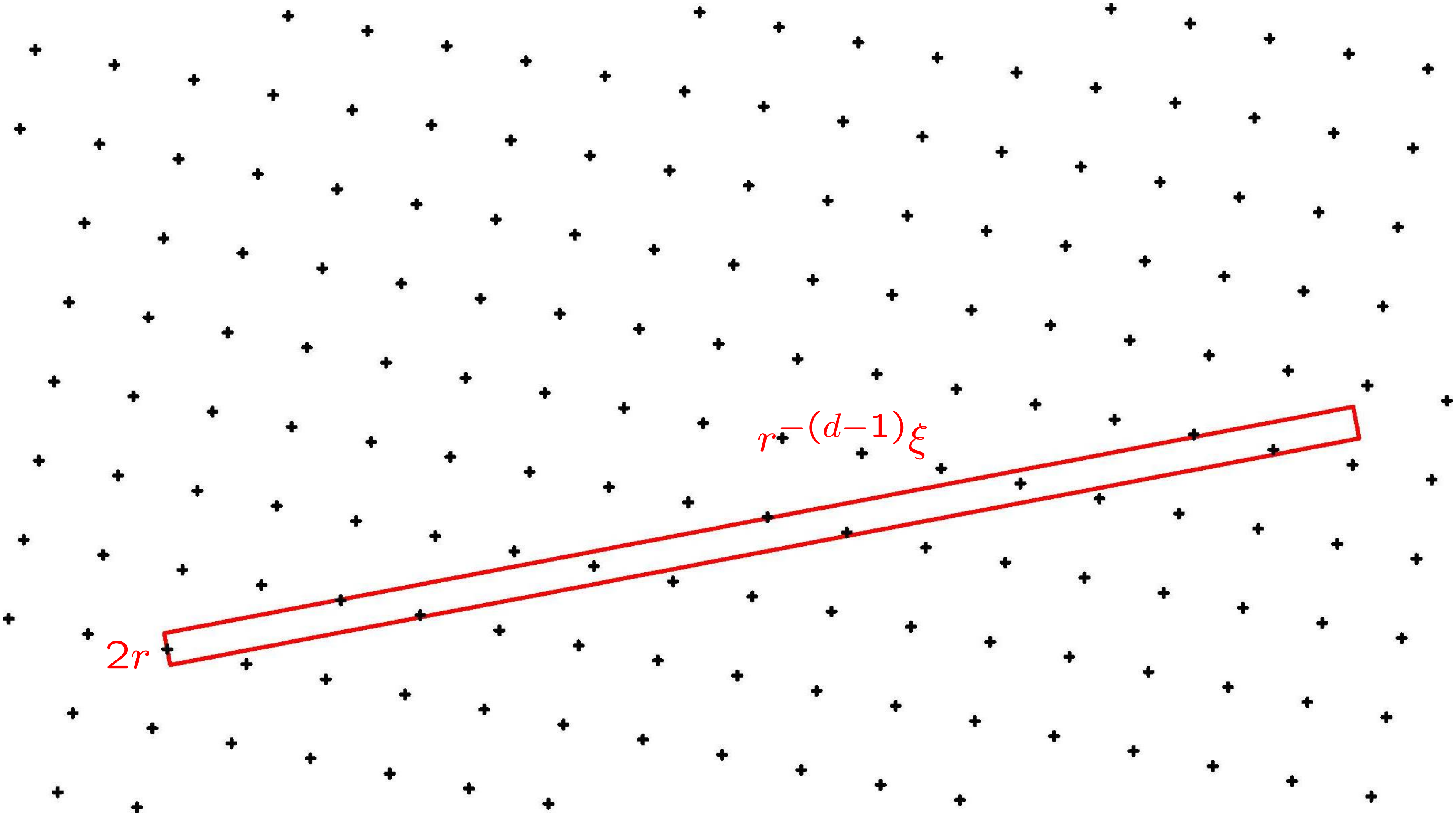
$\lambda\Big(\Big\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} : \text{ no scatterer intersects } \mathrm{ray}(\boldsymbol{v}, r^{-(d-1)}\xi)\Big\}\Big)$

Idea of proof ($\boldsymbol{q} = \mathbf{0}$, $\mathcal{L}_0 = \mathbb{Z}^d$)



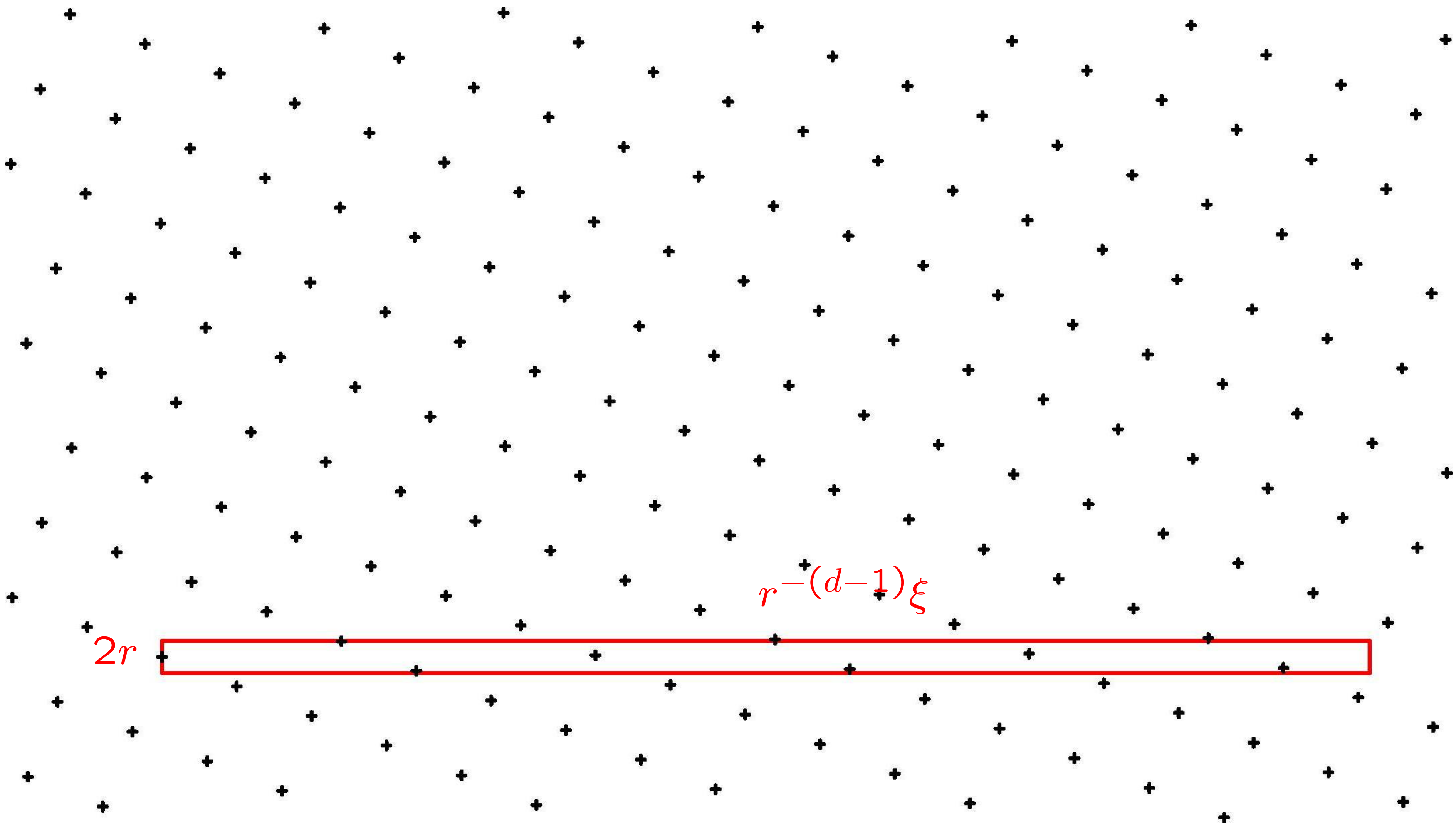
$$\approx \lambda\Big(\Big\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \mathbb{Z}^d \cap \mathcal{Z}(\boldsymbol{v}, r^{-(d-1)}\xi, r) = \emptyset\Big\}\Big)$$

Idea of proof ($\boldsymbol{q} = \mathbf{0}$, $\mathcal{L}_0 = \mathbb{Z}^d$)



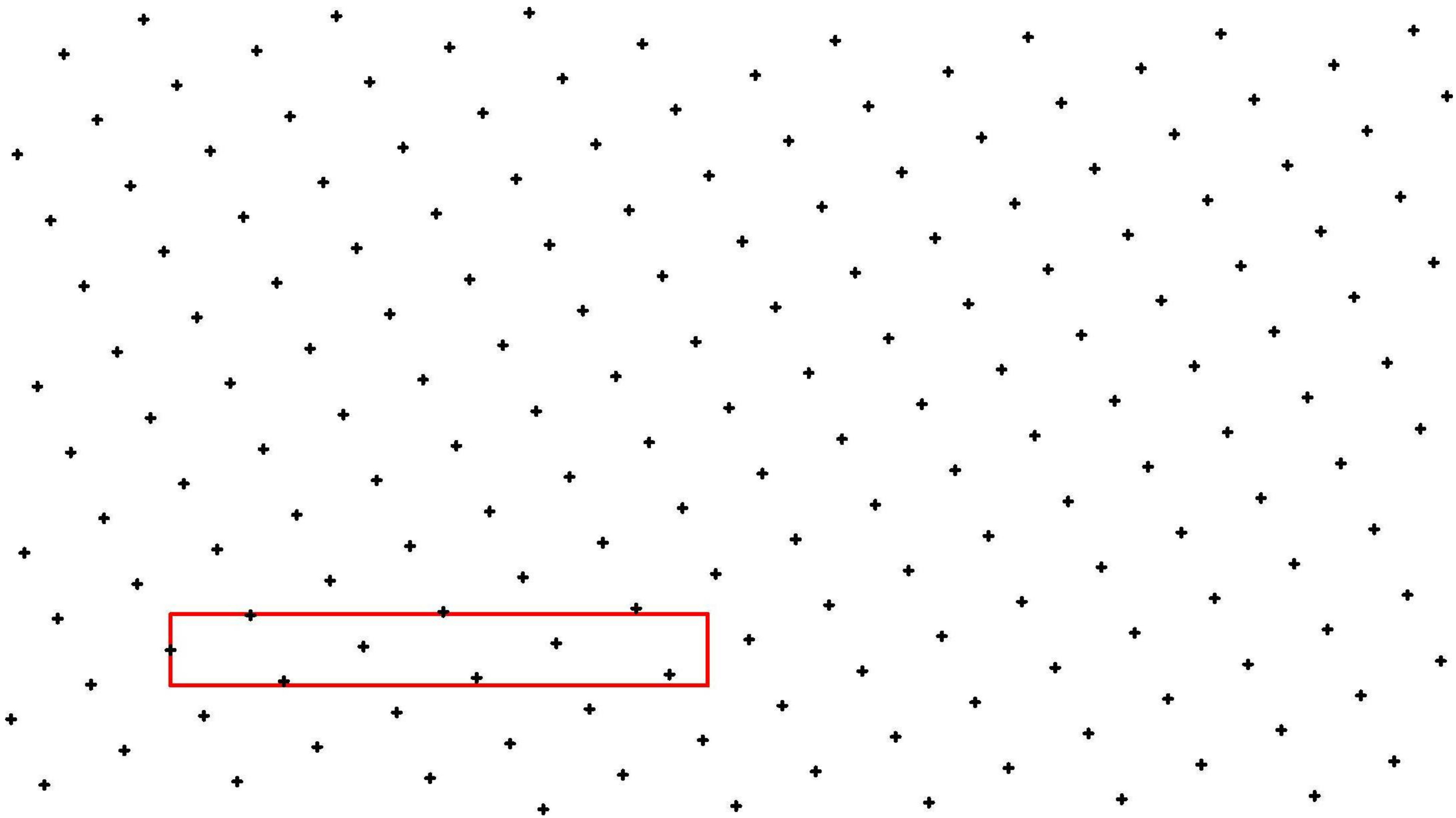
$\big($Rotate by $K(\boldsymbol{v}) \in \mathrm{SO}(d)$ such that $\boldsymbol{v} \mapsto \boldsymbol{e}_1\big)$

Idea of proof ($q = 0$, $\mathcal{L}_0 = \mathbb{Z}^d$)



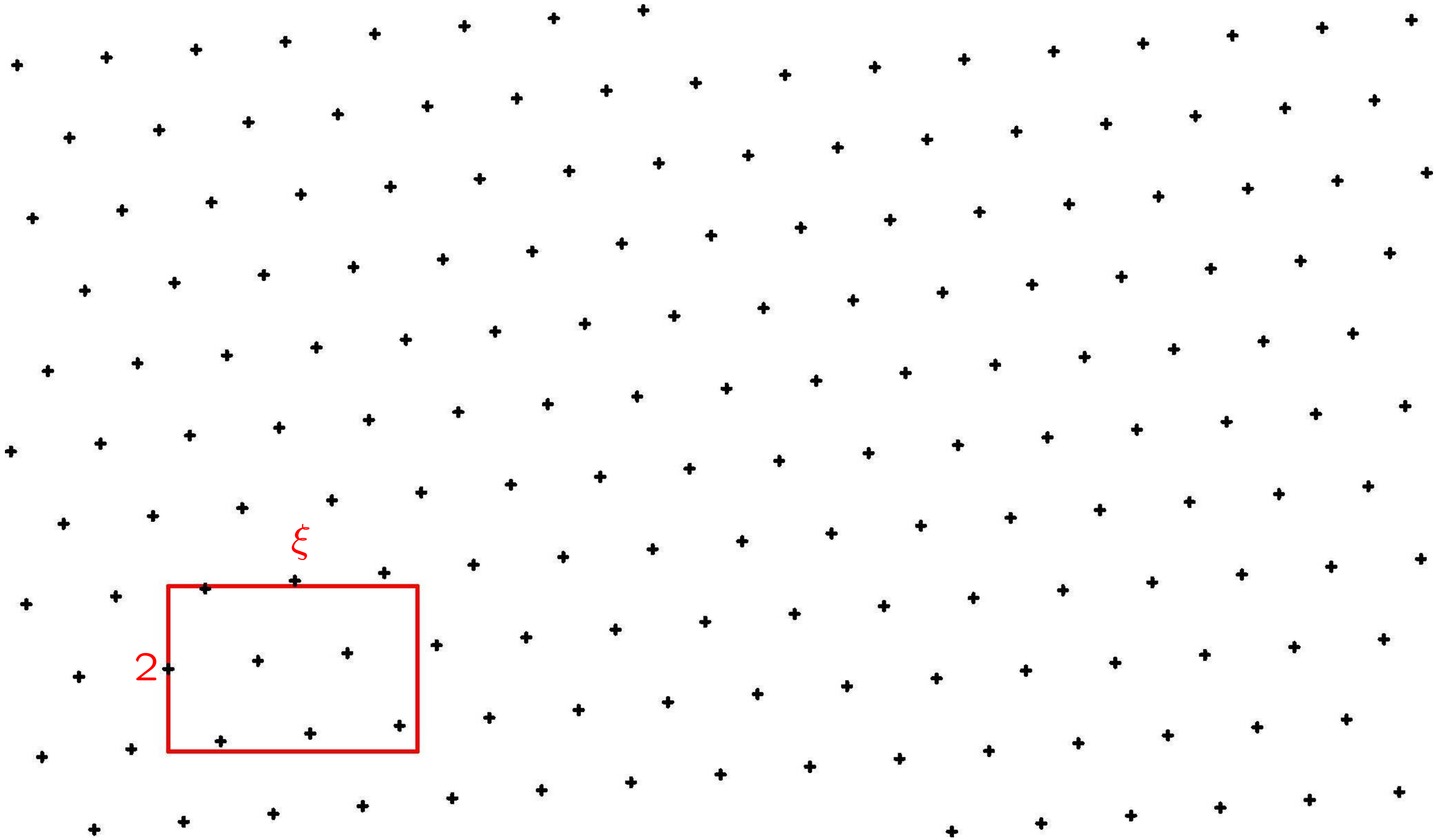
$$= \lambda\Big(\Big\{ v \in S_1^{d-1} : \mathbb{Z}^d K(v) \cap \mathcal{Z}(e_1, r^{-(d-1)}\xi, r) = \emptyset \Big\}\Big)$$

Idea of proof ($q = 0$, $\mathcal{L}_0 = \mathbb{Z}^d$)



$\Big($Apply $D_r = \mathrm{diag}(r^{d-1}, r^{-1}, \ldots, r^{-1}) \in \mathrm{SL}(d, \mathbb{R})\Big)$

Idea of proof ($q = 0$, $\mathcal{L}_0 = \mathbb{Z}^d$)



$$= \lambda\Big(\Big\{ v \in \mathsf{S}_1^{d-1} : \mathbb{Z}^d K(v) D_r \cap \mathcal{Z}(e_1, \xi, 1) = \emptyset \Big\}\Big)$$

The following Theorem shows that in the limit $r \to 0$ the lattice

$$\mathbb{Z}^d K(\boldsymbol{v}) D_r$$

behaves like a random lattice with respect to Haar measure μ_1.

Theorem E. Let λ be an a.c. Borel probability measure on S_1^{d-1}. Then, for every bounded continuous function $f : X_1 \to \mathbb{R}$,

$$\lim_{r\to 0} \int_{\mathsf{S}_1^{d-1}} f(K(\boldsymbol{v})D_r)d\lambda(\boldsymbol{v}) = \int_{X_1} f(M)d\mu_1(M).$$

Theorem E is a direct consequence of the mixing property for the flow

$$\Phi^t := D_{\exp(-t)}.$$

This concludes the proof of Theorem D when $\boldsymbol{q} \in \mathcal{L}_0$.

The generalization of Theorem E required for the full proof of Theorem D uses Ratner's classification of ergodic measures invariant under a unipotent flow. We exploit a close variant of a theorem by N. Shah (Proc. Ind. Acad. Sci. 1996) on the uniform distribution of translates of unipotent orbits.

How about aperiodic scatterer configurations?

- $\mathcal{P}$ — general locally finite subset of $\mathbb{R}^d$ with unit density

- The above approach still formally works: instead of $\mathbb{Z}^d K(\boldsymbol{v}) D_r$ we are now faced with a random point set

$$\mathcal{P} K(\boldsymbol{v}) D_r$$

  with $\boldsymbol{v}$ distributed according to λ

- Question: **Does $\mathcal{P} K(\boldsymbol{v}) D_r$ converge (in finite-dimensional distribution) to a random point process in $\mathbb{R}^d$?**

  (In the case $\mathcal{P} = \mathbb{Z}^d$ that random process would be given by the space of random lattices)

Two case studies

Case study 1: Union of lattices

- Consider scatterer locations at the point set

$$\mathcal{P} = \bigcup_{i=1}^{N} \overline{n}_i^{-1/d} \mathcal{L}_j, \qquad \mathcal{L}_i = (\mathbb{Z}^d + \boldsymbol{\omega}_i) M_i$$

with $\boldsymbol{\omega}_i \in \mathbb{R}^d$, $M_i \in \mathrm{SL}(d,\mathbb{R})$ and $\overline{n}_i > 0$ such that $\overline{n}_1 + \ldots + \overline{n}_N = 1$

- The analogue of the equidistribution Theorem E is:

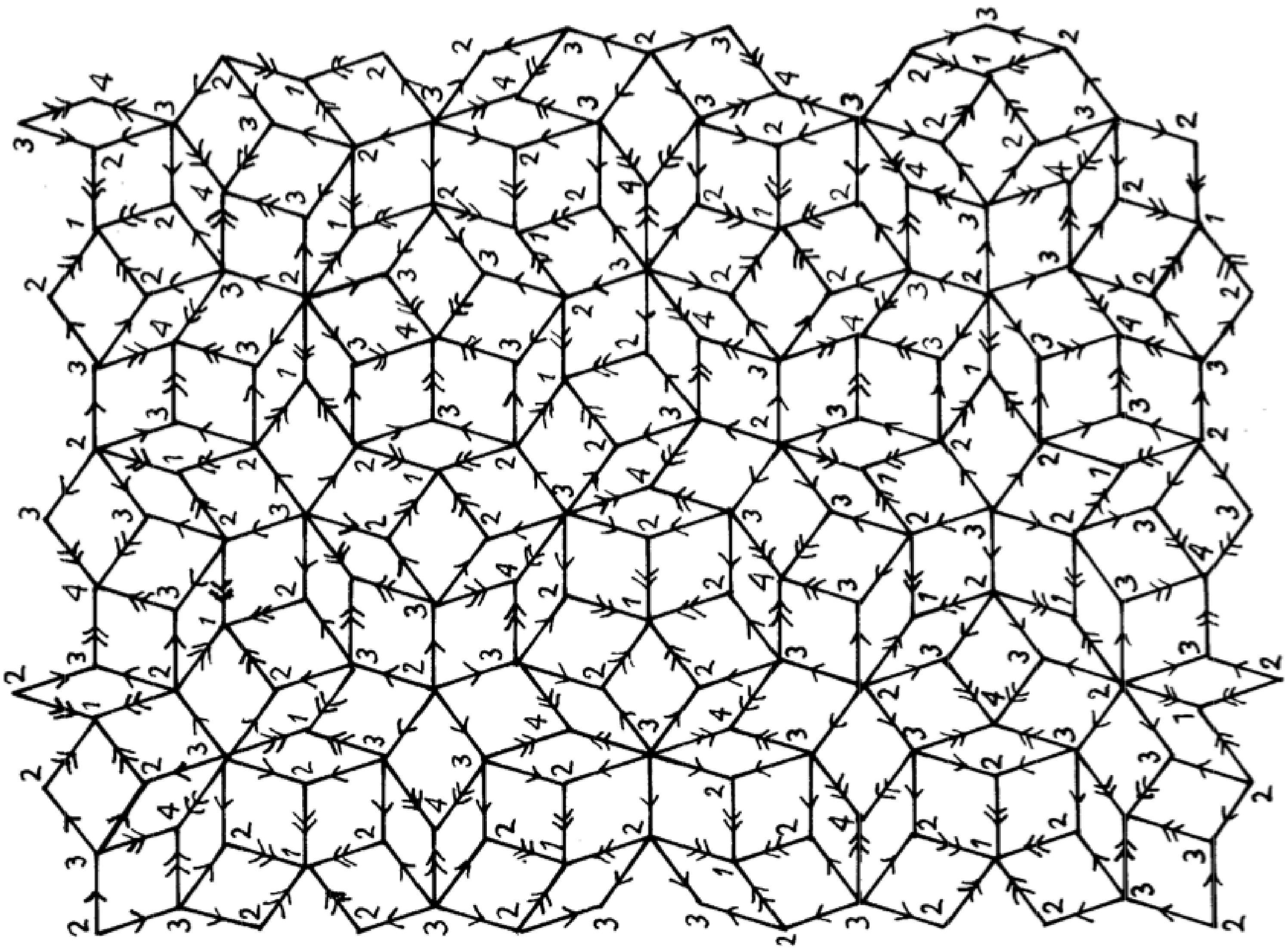
Theorem F. [JM & Strömbergsson, J Stat Phys 2014]. If $M_1, \ldots, M_N \in \mathrm{SL}(d,\mathbb{R})$ are incommensurable, then for every bdd cont $f : X_1^N \to \mathbb{R}$,

$$\lim_{r\to 0} \int_{\mathrm{S}_1^{d-1}} f(M_1 K(\boldsymbol{v}) D_r, \ldots, M_N K(\boldsymbol{v}) D_r) d\lambda(\boldsymbol{v}) = \int_{X_1^N} f(M_1', \ldots, M_N') d\mu_1(M_1') \cdots \mu_1(M_N').$$

- Interesting consequence—the path length distribution decays faster:

$$\Psi_0(t) \sim \frac{C}{t^{N+2}}$$

Case study 2: Quasicrystals



Penrose tiling
(from: de Bruijn, Kon Nederl Akad Wetensch Proc Ser A, 1981)

Cut and project

- $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^m$, π and π_{int} orthogonal projections onto $\mathbb{R}^d$, $\mathbb{R}^m$

- $\mathcal{L} \subset \mathbb{R}^n$ a lattice of full rank

- $\mathcal{A} := \overline{\pi_{\text{int}}(\mathcal{L})}$ is an abelian subgroup of $\mathbb{R}^m$, with Haar measure $\mu_{\mathcal{A}}$

- $\mathcal{W} \subset \mathcal{A}$ a "regular window set"
  (i.e. bounded with non-empty interior, $\mu_{\mathcal{A}}(\partial\mathcal{W}) = 0$)

- $\mathcal{P}(\mathcal{W},\mathcal{L}) = \{\pi(\boldsymbol{y}) : \boldsymbol{y} \in \mathcal{L},\ \pi_{\text{int}}(\boldsymbol{y}) \in \mathcal{W}\} \subset \mathbb{R}^d$
  is called a "regular cut-and-project set"

- $\mathcal{P}(\mathcal{W},\mathcal{L})$ defines the locations of scatterers in our quasicrystal

Recall $\tau_1 = \tau(\boldsymbol{q}, \boldsymbol{v})$ denotes the free path length corresponding to the initial condition $(\boldsymbol{q}, \boldsymbol{v})$.

Theorem G [JM & Strömbergsson, Comm Math Phys 2014]. Fix a regular cut-and-project set $\mathcal{P}$ and the initial position $\boldsymbol{q}$. Let λ be any a.c. Borel probability measure on S_1^{d-1}. Then, for every $\xi > 0$, the limit

$$F_{\mathcal{P},\boldsymbol{q}}(\xi) := \lim_{r\to 0} \lambda(\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : r^{d-1}\tau_1 \geq \xi\})$$

exists, is continuous in ξ and independent of λ.

In analogy with Theorem D and the space of lattices, we express $F_{\mathcal{P}_0,\boldsymbol{q}}(\xi)$ in terms of a random variable in a suitable space of quasicrystals.

Equidistribution

- Set $G = \mathsf{ASL}(n, \mathbb{R})$, $\Gamma = \mathsf{ASL}(n, \mathbb{Z})$.
- Pick $g \in G$ so that $\mathcal{L} = \mathbb{Z}^n g$ (up to a multiplicative constant)
- Define an embedding of $\mathsf{SL}(d, \mathbb{R})$ in G by the map

$$\varphi_g : \mathsf{SL}(d, \mathbb{R}) \to G, \quad A \mapsto g \left(\begin{pmatrix} A & 0 \\ 0 & 1_m \end{pmatrix}, \mathbf{0} \right) g^{-1}.$$

- It follows from Ratner's theorems that there exists a closed connected subgroup H_g of G such that
  - $\Gamma \cap H_g$ is a lattice in H_g
  - $\varphi_g(\mathsf{SL}(d, \mathbb{R})) \subset H_g$
  - the closure of $\Gamma \backslash \Gamma \varphi_g(\mathsf{SL}(d, \mathbb{R}))$ in $\Gamma \backslash G$ is given by $\Gamma \backslash \Gamma H_g$.
- Denote the unique right-H_g invariant probability measure on $\Gamma \backslash \Gamma H_g$ by μ_g.

Theorem H. Let λ be an a.c. Borel probability measure on S_1^{d-1}. Then, for every bounded continuous function $f : \Gamma \backslash G \to \mathbb{R}$,

$$\lim_{r \to 0} \int_{\mathsf{S}_1^{d-1}} f(\varphi_g(K(\boldsymbol{v}) D_r)) d\lambda(\boldsymbol{v}) = \int_{\Gamma \backslash \Gamma H_g} f(h) d\mu_g(h).$$

Examples

- If $\mathcal{P} = \mathcal{P}(\mathcal{L}, \mathcal{W})$, then for almost every $\mathcal{L}$ in the space of lattices and almost every $\boldsymbol{q}$, we have $H_g = \mathsf{ASL}(n, \mathbb{R})$, $\Gamma \cap H_g = \mathsf{ASL}(n, \mathbb{Z})$.

  $F_{\mathcal{P},\boldsymbol{q}}(\xi) \asymp \xi^{-1}$ ($\xi \to \infty$) where the implied constants depend on n, $\mathcal{W}$, δ. Again $F_{\mathcal{P},\boldsymbol{q}}(\xi)$ is independent of $\mathcal{P}$ and $\boldsymbol{q}$. ($\Rightarrow$ The tail is of the same order as in the lattice case $\mathcal{P} = \mathcal{L}$.)

- If $\mathcal{P}$ is the vertex set of a Penrose tiling and $\boldsymbol{q} \in \mathcal{P}$, we have $H_g = \mathsf{SL}(2, \mathbb{R})^2$, $\Gamma \cap H_g =$ a congruence subgroup of the Hilbert modular group $\mathsf{SL}(2, \mathcal{O}_K)$, with $\mathcal{O}_K$ the ring of integers of $K = \mathbb{Q}(\sqrt{5})$.

  $F_{\mathcal{P},\boldsymbol{q}}(\xi) \asymp$ work in progress ...

Conclusions

- The linear Boltzmann equation governs the Boltzmann-Grad limit of the Lorentz gas for “typical” scatterer configurations. . .

- . . . but may fail when long-range correlations are present. New transport equations emerge, whose transition kernel is governed by non-trivial $\mathrm{SL}(d,\mathbb{R})$-invariant point processes

- Proof of convergence reduces to equidistribution of expanding spheres in the relevant moduli spaces

Future challenges

- Classify $\mathsf{SL}(d,\mathbb{R})$-invariant point processes
- Lorentz gas in force fields
- Other scaling limits
- Quantum Lorentz gas

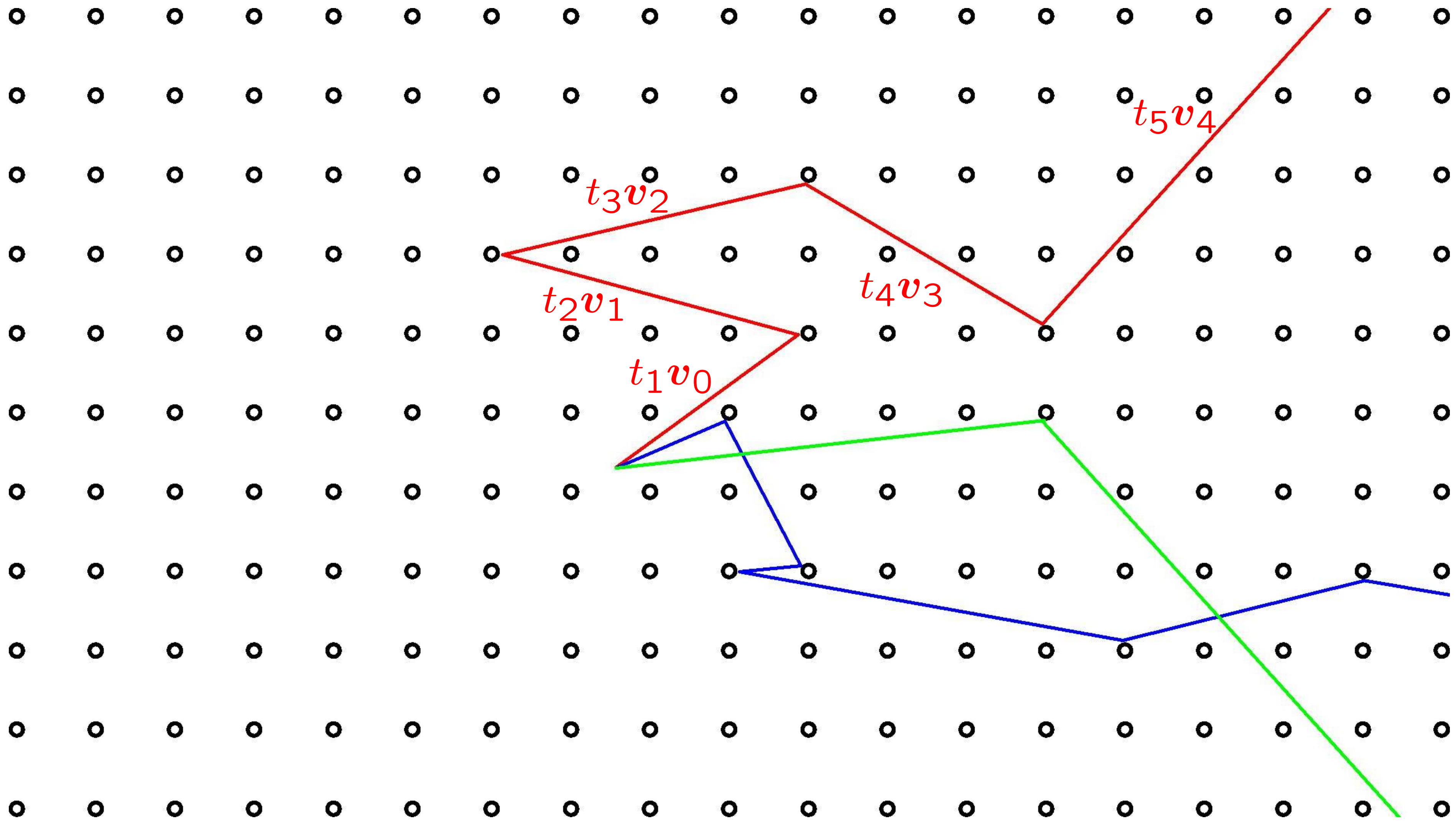
Further reading

- **The low-density limit of the Lorentz gas: periodic, aperiodic and random**
  Proceedings of the ICM 2014
  arXiv:1404.3293

- **Kinetic limits of dynamical systems**
  summer school lecture notes
  arXiv:1408.1307

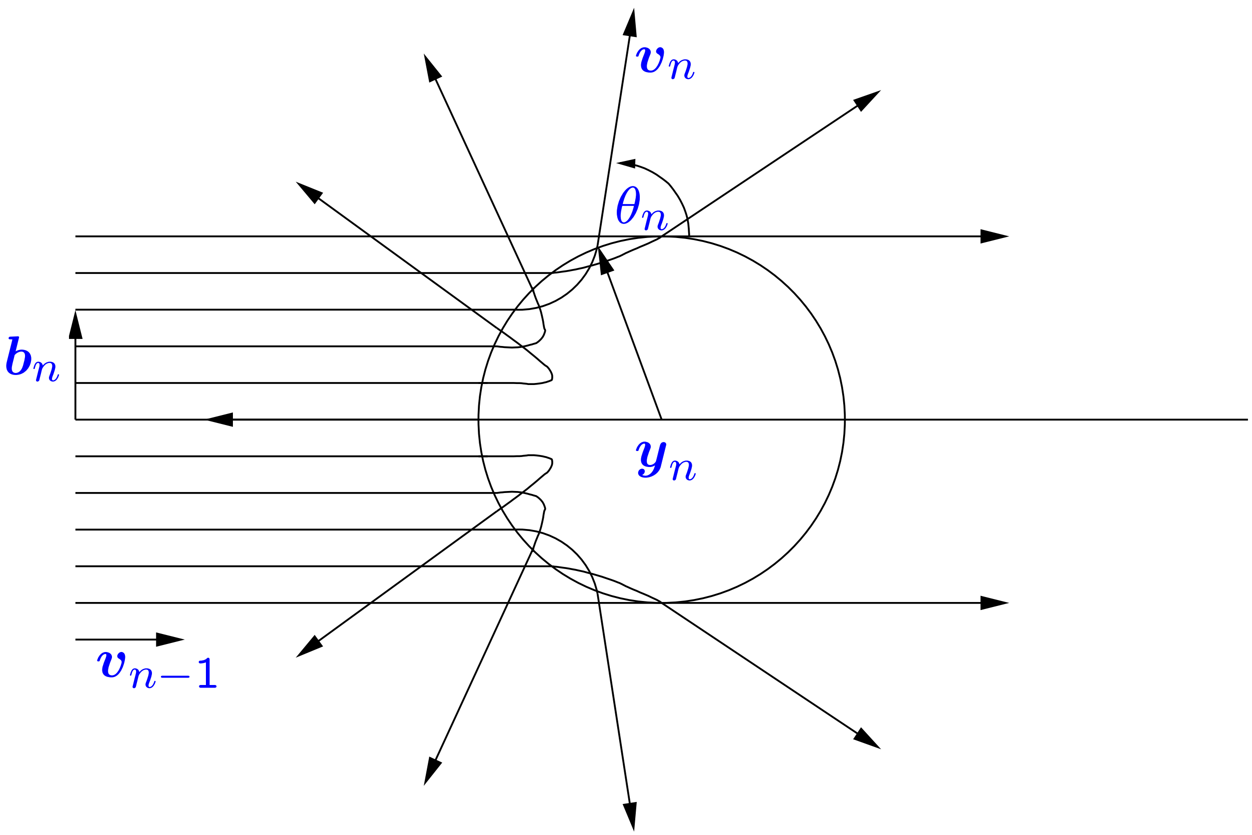
Appendix: A refined Stosszahlansatz

(for general scatterer configurations $\mathcal{P}$)



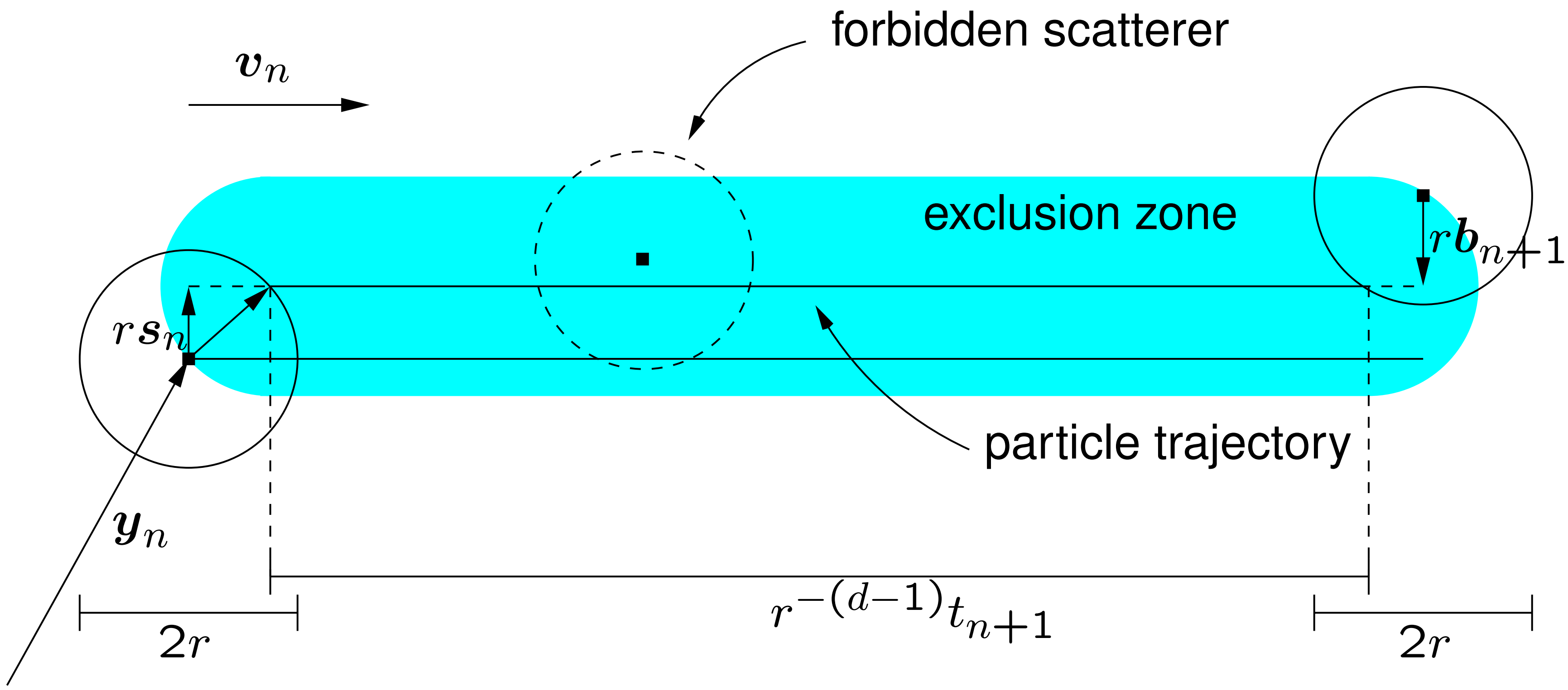
t_n — nth collision time, $\boldsymbol{v}_n$ — velocity after nth collision

The nth collision



Consider a beam of parallel particles with velocity $\boldsymbol{v}_{n-1}$ hitting a scatterer at $\boldsymbol{y}_n \in \mathcal{P}$ with random impact parameter $\boldsymbol{b}_n$.

Intercollision flights



Intercollision flight in the Lorentz gas between the nth and $(n+1)$st collision. The exclusion zone is a long and thin cylinder of radius r with spherical caps. Scatterers are centered at $\mathcal{P}$.

Collision coordinates

$$\boldsymbol{v}_n = (1, \mathbf{0}) R_n^{-1} \qquad \text{(velocity after } n\text{th collision)}$$

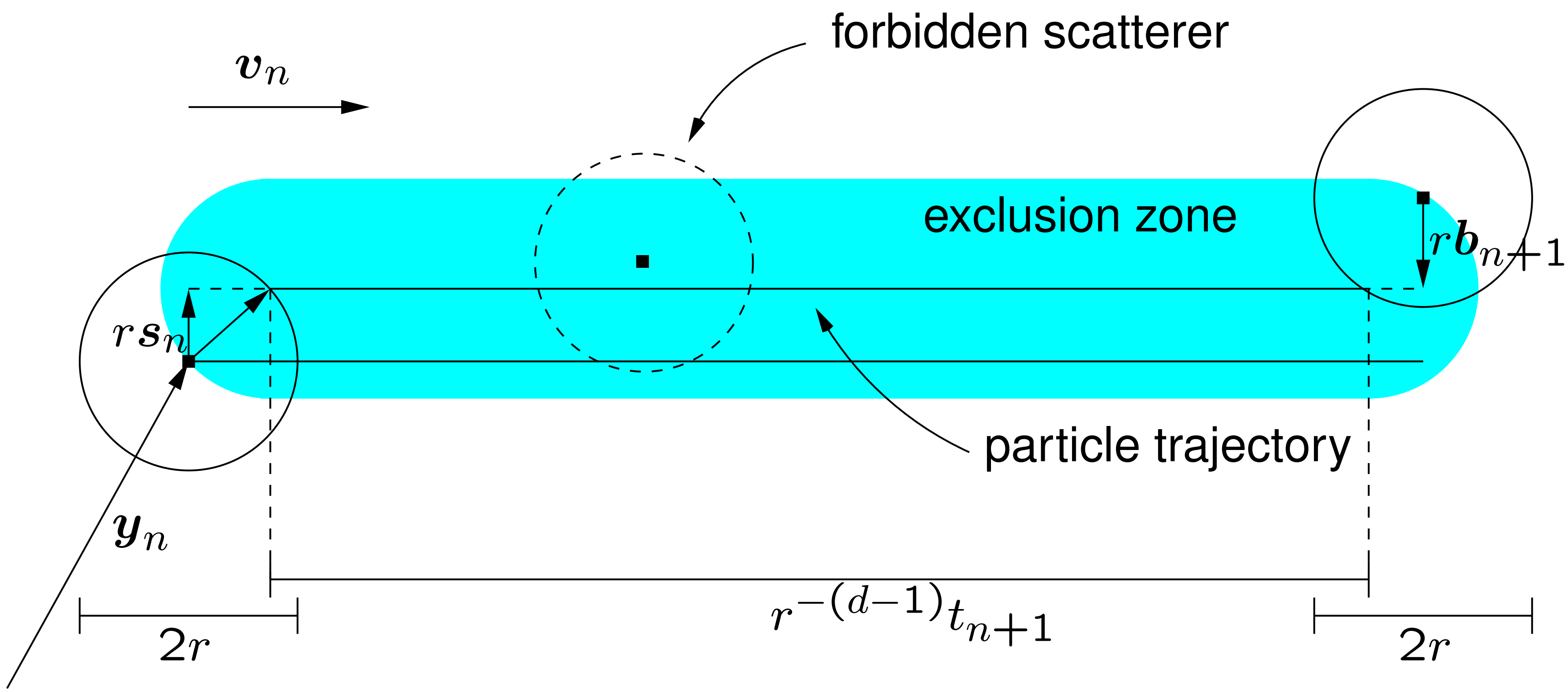
$$\boldsymbol{s}_n = (0, \boldsymbol{w}_n) R_n^{-1} \qquad \text{(exit parameter at } n\text{th collision)}$$

$$\boldsymbol{b}_{n+1} = (0, \boldsymbol{w}_{n+1}) R_n^{-1} \qquad \text{(impact parameter at } (n+1)\text{st collision)}$$

$$R_n := R_{n-1} S(\boldsymbol{w}_n) = R(\boldsymbol{v}_0) S(\boldsymbol{w}_1) \cdots S(\boldsymbol{w}_n).$$

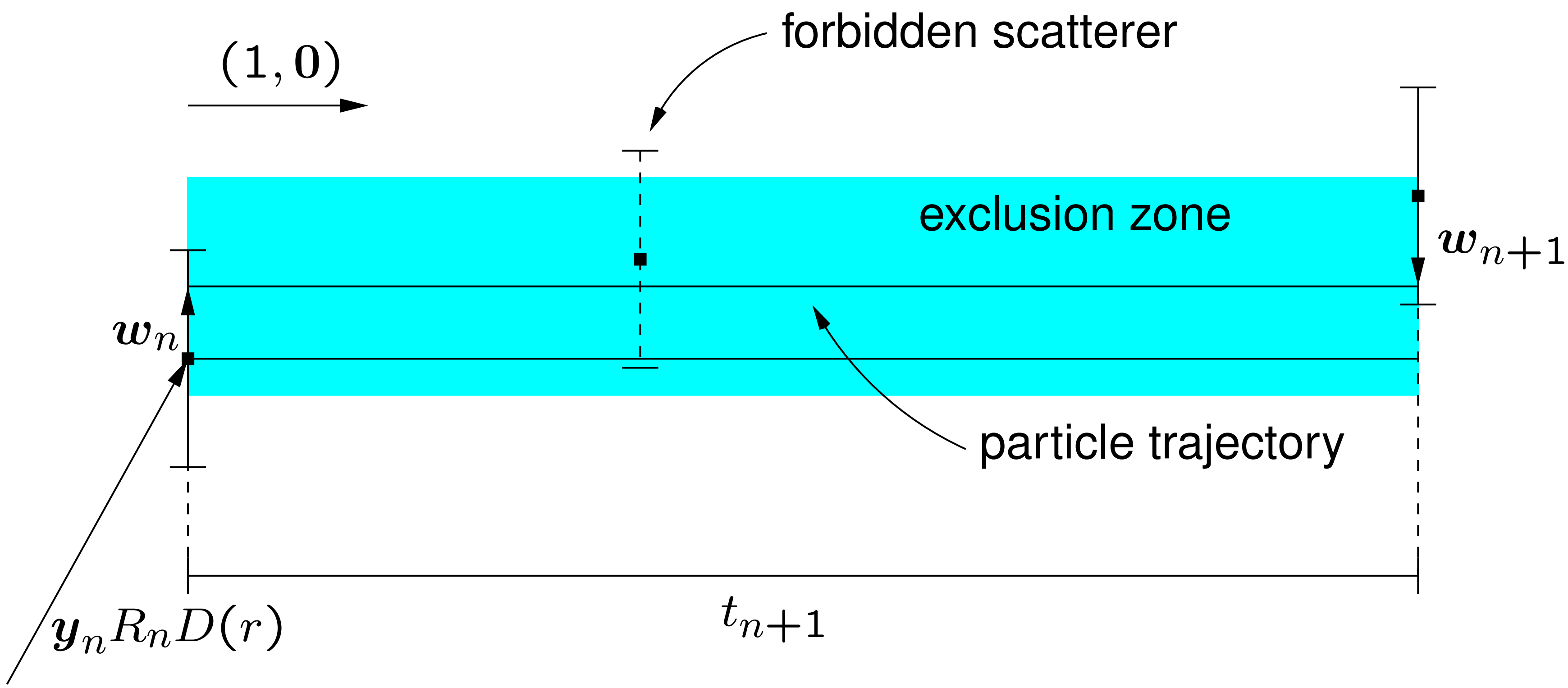
$R(\boldsymbol{v}_0) \in \mathrm{SO}(d)$ so that $\boldsymbol{v}_0 R(\boldsymbol{v}_0) = (1, \mathbf{0})$

Intercollision flights



Now apply the linear map $R_n D(r)$ to this picture, with $D(r) = \begin{pmatrix} r^{d-1} & 0 \\ 0 & r^{-1}1_{d-1} \end{pmatrix}$

Intercollision flights



The exclusion zone is now approximately a r-independent cylinder with radius 1 and flat caps. Scatterers are centered at $\mathcal{P}R_n D(r) = \mathcal{P}R_{n-1}S(\boldsymbol{w}_n)D(r)$.