Random lattices and their applications in number theory, geometry and statistical mechanics

Jens Marklof
University of Bristol

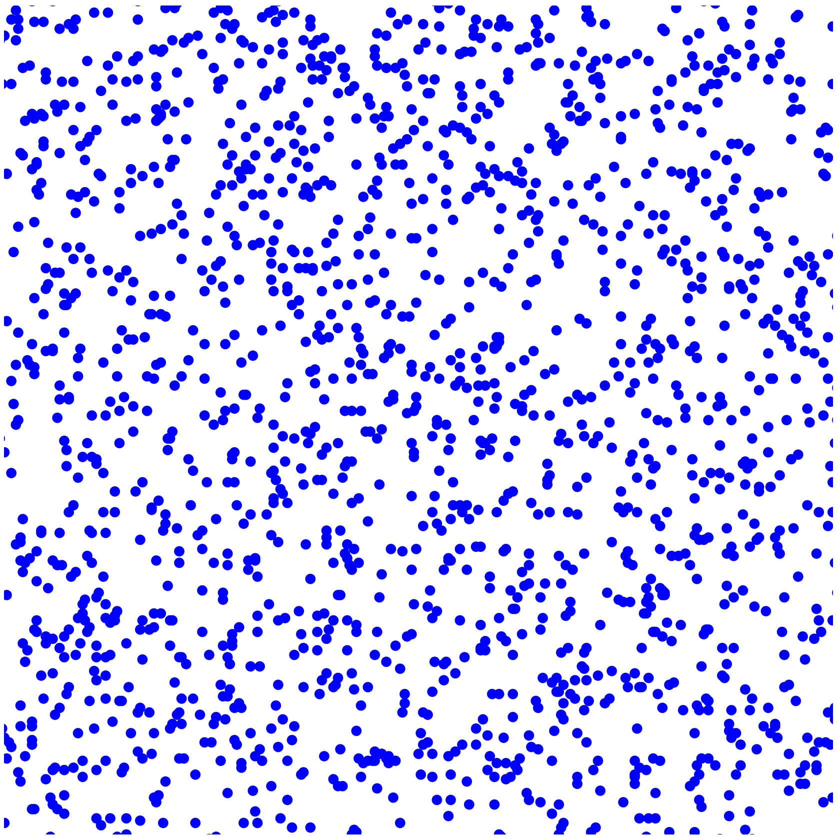
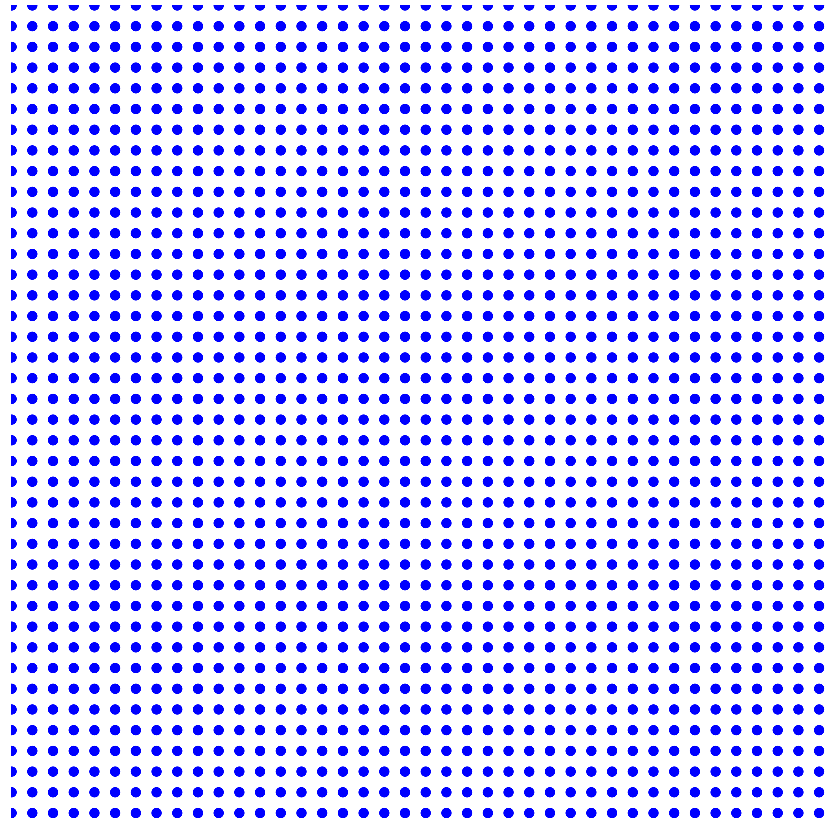
Tata Institute for Fundamental Research, Mumbai, February 2023

Outline

Lecture 1 What are random lattices? Basic features and examples

Lecture 2 Random lattices in number theory and geometry

Lecture 3 Random lattices in statistical mechanics – the Lorentz gas

Warm-up in dimension one

- **Poisson point process** in $\mathbb{R}$ (homogeneous, with intensity one): sequence of random variables $\cdots \xi_{-2} \leq \xi_{-1} \leq \xi_0 \leq \xi_1 \leq \xi_2 \cdots$ such that for any interval B

$$\mathbb{P}\Big(\#\Big\{n : \xi_n \in B\Big\} = k\Big) = \frac{|B|^k}{k!}\,\mathrm{e}^{-|B|}$$

  and the point counts in disjoint intervals are independent.

- **Random (affine) lattice** in $\mathbb{R}$: Start with the integer lattice $\mathbb{Z}$ and define sequence of random variable $\xi_n = n + \alpha$, $n \in \mathbb{Z}$, with α a random variable uniformly distributed in the unit interval $[0, 1]$. Then

$$\mathbb{P}\Big(\#\Big\{n : \xi_n \in B\Big\} = k\Big) = \max\Big(1 - \Big|k - |B|\Big|,\ 0\Big).$$

Both random sequences define stationary point processes of intensity one. But the gaps of the former are independent random variables distributed according to the exponential distribution e^{-s}, whereas the gaps of the latter are all 1 and thus completely deterministic. . . **more interesting in higher dimensions!**

Two-dimensional random lattices

- To construct two-dimensional random (affine) lattice, could start with $\mathbb{Z}^2$ and randomly shift by a vector $\boldsymbol{\alpha}$. But unlike in 1d there are now also **linear transformations**

$$\mathbb{R}^2 \to \mathbb{R}^2, \qquad \boldsymbol{x} \mapsto \boldsymbol{x} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax_1 + cx_2, bx_1 + dx_2).$$

  We'll only consider orientation- and area-preserving maps, i.e., $ad - bc = 1$.

- **Basic examples of random lattices:**

$$\mathcal{P}_1(u) = \mathbb{Z}^2 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{R}_1(\phi) = \mathbb{Z}^2 \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

  with u a random variable uniformly distributed in the unit interval $[0, 1]$ and ϕ is uniformly distributed in $[-\frac{\pi}{2}, \frac{\pi}{2}]$*

- **Is this all?**

*note the periodicity $\mathcal{P}_1(u+1) = \mathcal{P}_1(u)$, $\mathcal{R}_1(\phi+\pi) = \mathcal{R}_1(\phi)$

Two-dimensional random lattices

- We can write a general linear transformation $M \in \mathrm{SL}(2,\mathbb{R})$ as

$$M = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

- Can define random lattice

$$\mathbb{Z}^2 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

  with u random in $[0,1]$ and ϕ in $[-\frac{\pi}{2},\frac{\pi}{2}]$ as above.

  But what about v? Any natural probability measure on this variable? What happens when $v \to 0$?

- How about sequences of random lattices

$$\mathbb{Z}^2 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \qquad \mathbb{Z}^2 \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$$

  as $\epsilon \to 0$?

The space of lattices

- $G_0 = \mathsf{SL}(d, \mathbb{R})$, $\Gamma_0 = \mathsf{SL}(d, \mathbb{Z})$.

- The map $\Gamma_0 M \mapsto \mathbb{Z}^d M$ gives a one-to-one correspondence between the homogeneous space $\Gamma_0 \backslash G_0$ and the space of Euclidean lattices in $\mathbb{R}^d$ of covolume one.

- The Haar measure μ_0 on G_0 can be normalized so that it gives a probability measure on $\Gamma_0 \backslash G_0$ (H. Minkowski); also denote by μ_0

- For dimension $d = 2$ we have (in the above Iwasawa coordinates)

$$\mu_0 = \frac{3}{\pi^2} \frac{du\, dv\, d\phi}{v^2}$$

Siegel's mean value formula

- C.L. Siegel (1945): For any measurable function $f:\mathbb{R}^d \to \mathbb{R}_{\geq 0}$

$$\int_{\Gamma_0\backslash G_0} \Bigg(\sum_{\boldsymbol{x}\in\mathbb{Z}^d M} f(\boldsymbol{x}) \Bigg)\, d\mu_0(M) = f(\boldsymbol{0}) + \int_{\mathbb{R}^d} f(\boldsymbol{y})\, dy.$$

- C.A. Rogers in the 1950s calculated mean values of higher order sums

$$\int_{\Gamma_0\backslash G_0} \Bigg(\sum_{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k\in\mathbb{Z}^d M} f(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k) \Bigg)\, d\mu_0(M) = \ldots,$$

with significantly more complicated answers.

The space of affine lattices

- $G = \mathsf{SL}(d,\mathbb{R}) \ltimes \mathbb{R}^d$ the semidirect product with multiplication law

$$(M,z)(M',z') = (MM', zM' + z')$$

- Define action of $g = (M,z) \in G$ on $\mathbb{R}^d$ by $yg = yM + z$.

- $\Gamma = \mathsf{SL}(d,\mathbb{Z}) \ltimes \mathbb{Z}^d$ is a lattice in G.

- The Haar measure on G is $\mu = \mu_0 \times \mathsf{Leb}$ (the Lebesgue measure normalised so that $\mathsf{Leb}[0,1]^d = 1$); corresponding probability measure on $\Gamma\backslash G$ also denoted by μ.

Mean value formulas for affine lattices

- The analogue of Siegel's formula is much easier to prove for affine lattice (use translation invariance of Lebesgue measure):

$$\int_{\Gamma\backslash G} \left(\sum_{\boldsymbol{x}\in\mathbb{Z}^d g} f(\boldsymbol{x}) \right) d\mu(g) = \int_{\mathbb{R}^d} f(\boldsymbol{y})\, dy.$$

- Siegel's formula on the other hand gives us a "Rogers-style" second-order identity

$$\int_{\Gamma\backslash G} \left(\sum_{\boldsymbol{x}_1\neq\boldsymbol{x}_2\in\mathbb{Z}^d g} f(\boldsymbol{x}_1,\boldsymbol{x}_2) \right) d\mu(g) = \int_{\mathbb{R}^d} f(\boldsymbol{y}_1,\boldsymbol{y}_2)\, dy_1 dy_2$$

Probabilistic formulation

- Assume now $\mathcal{P} = \mathbb{Z}^d g$ is a random affine lattice with the random element $g \in \Gamma\backslash G$ distributed according to μ. Then the formulas on the previous slide become

$$\mathbb{E}\bigg(\sum_{\boldsymbol{x} \in \mathcal{P}} f(\boldsymbol{x}) \bigg) = \int_{\mathbb{R}^d} f(\boldsymbol{y})\, dy.$$

$$\mathbb{E}\bigg(\sum_{\boldsymbol{x}_1 \neq \boldsymbol{x}_2 \in \mathcal{P}} f(\boldsymbol{x}_1, \boldsymbol{x}_2) \bigg) = \int_{\mathbb{R}^d} f(\boldsymbol{y}_1, \boldsymbol{y}_2)\, dy_1 dy_2$$

- The first formula tells us that $\mathcal{P}$ is a point process in $\mathbb{R}^d$ with intensity one. It is attributed to N.R. Campbell for general stationary point processes.

- **The second formula is the same as for Poisson point process!** But higher-order moments do not agree.

Distribution functions

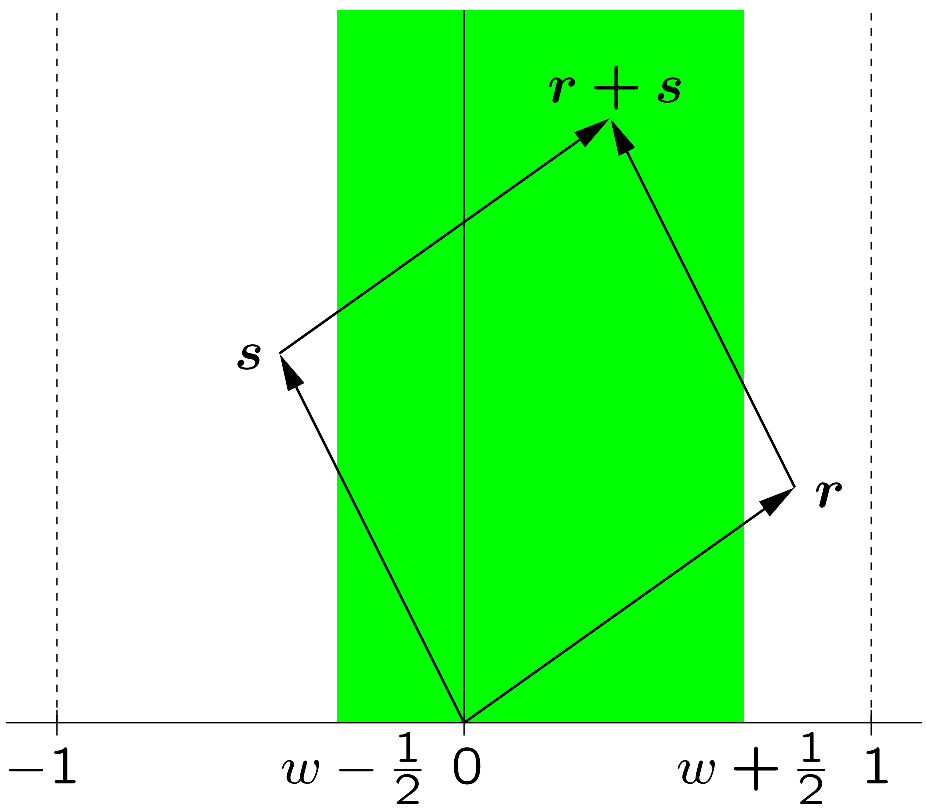
- Very difficult to work out explicit formulas for

$$\mathbb{P}\Big(|\mathbb{Z}^d M \cap B| = k\Big) \quad \text{or} \quad \mathbb{P}\Big(|\mathbb{Z}^d g \cap B| = k\Big)$$

  for general $k = 0, 1, 2, \ldots$ and $B \subset \mathbb{R}^d$, especially when $d \geq 3$

- As an illustration, let us consider the distribution of the minimal height vector of lattice points in a strip

Lattice points in a strip



The two linearly independent lattice vectors with lowest and second-lowest heights in the vertical strip between -1 and 1 form a basis. One can show that at any vertical strip of width one (in green) contains at least one of the three points, and hence the minimal height vector $\boldsymbol{q}$ is either $\boldsymbol{r}$, $\boldsymbol{s}$ or $\boldsymbol{r}+\boldsymbol{s}$

JM & A. Strömbergsson, The three gap theorem and the space of lattices, American Math. Monthly 2017

Lattice points in a strip

- If $\mathbb{Z}^2 M$ is a Haar random lattice, then the minimal height vector $\boldsymbol{q} = (q_1, q_2)$ in the **green strip** $(w-\frac{1}{2}, w+\frac{1}{2}) \times \mathbb{R}_{>0}$ is distributed according the probability density

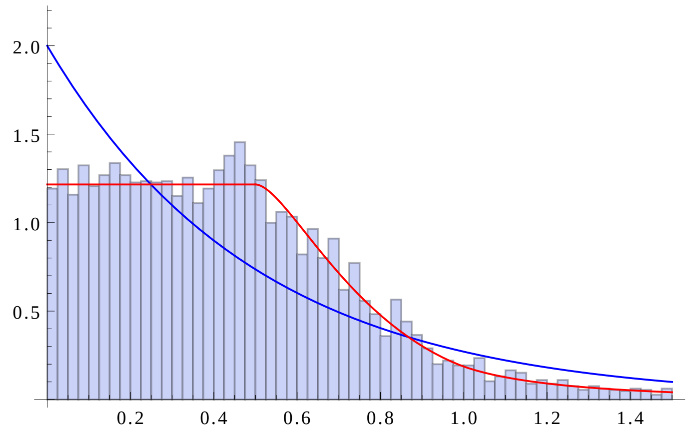
$$K_w(q_1, q_2) = \frac{6}{\pi^2}\, H\left(1 + \frac{q_2^{-1} - \max\left(|w|, |q_1 - w|\right) - \frac{1}{2}}{|q_1|}\right)$$

$$H(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \leq x. \end{cases}$$

- If we average the distribution over $q_1 \in (w - \frac{1}{2}, w + \frac{1}{2})$ and $w \in [-\frac{1}{2}, \frac{1}{2}]$, we obtain the distribution of the height of the minimal height vector,

$$P(s) = \frac{6}{\pi^2} \times \begin{cases} 1 & (s \leq 1) \\ \frac{1}{s} + 2\left(1 - \frac{1}{s}\right)^2 \log\left(1 - \frac{1}{s}\right) & \\ \qquad -\frac{1}{2}\left(1 - \frac{2}{s}\right)^2 \log\left|1 - \frac{2}{s}\right| & (s > 1). \end{cases}$$

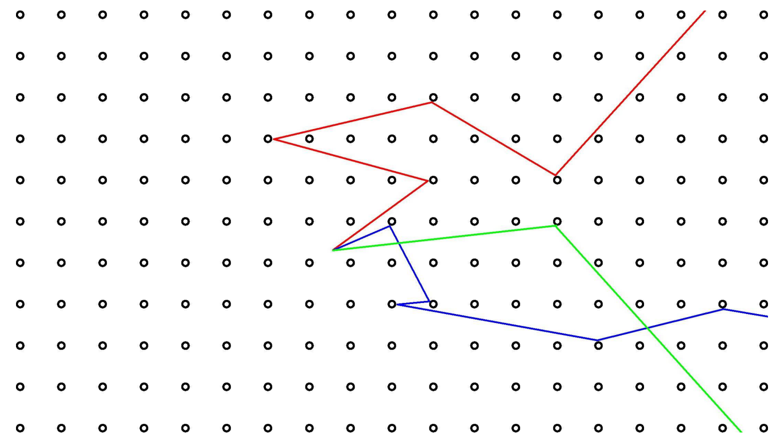
Do these distributions appear in "nature"?*



The distribution of free path length in the periodic Lorentz gas vs. $2P(2s)$, $2\mathrm{e}^{-2s}$

P. Dahlqvist 1997
F. Boca & A. Zaharescu 2007

using lattices: JM & A. Strömbergsson 2008



Particle trajectories in the periodic Lorentz gas

*For more on this see JM, Random lattices in the wild: from Polya's orchard to quantum oscillators, LMS Newsletter, Issue 493 (2021)

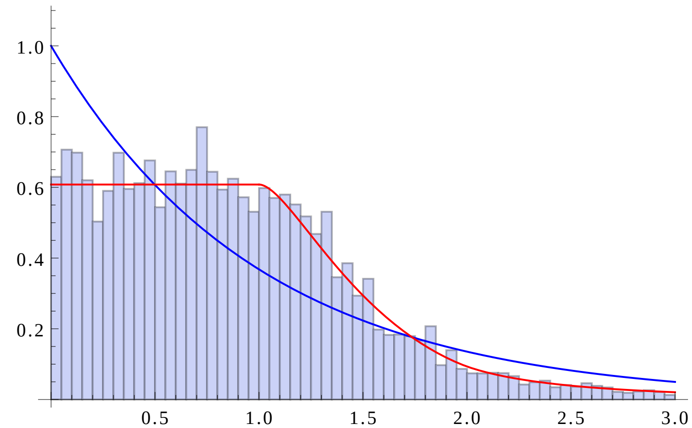
Do these distributions appear in “nature”?*

Quantum energy levels of a two-dimensional harmonic oscillator:

$$\omega_1(m+\tfrac{1}{2})+\omega_2(n+\tfrac{1}{2})$$

$$m,n=0,1,2,\ldots$$

ω_1,ω_2 frequencies

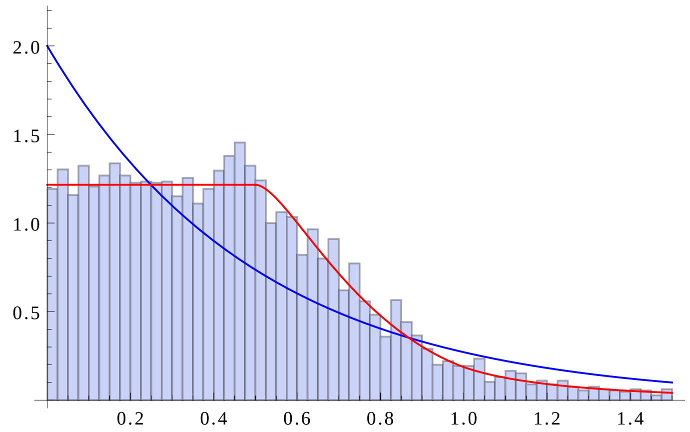


The gap distribution in the energy spectrum of a two-dimensional harmonic oscillator with random frequencies

C. Greenman 1996
using lattices: JM 2000

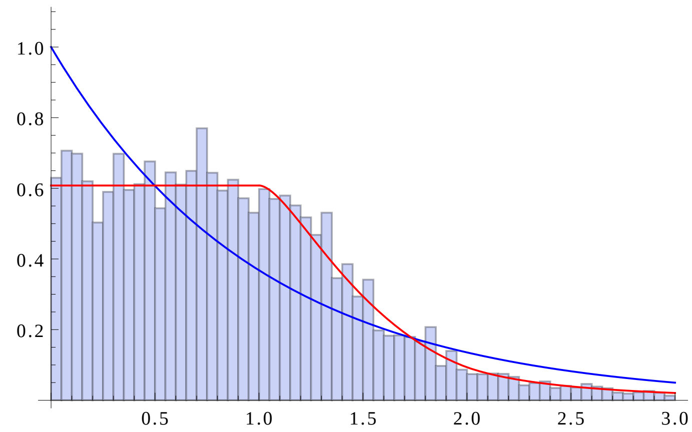
*For more on this see JM, Random lattices in the wild: from Polya’s orchard to quantum oscillators, LMS Newsletter, Issue 493 (2021)

Do these distributions appear in "nature"?*



The distribution of free path length in the periodic Lorentz gas vs. $2P(2s)$, e^{-2s}

P. Dahlqvist 1997
F. Boca & A. Zaharescu 2007
using lattices:
JM & A. Strömbergsson 2008



The gap distribution in the energy spectrum of a two-dimensional harmonic oscillator with random frequencies

C. Greenman 1996
using lattices: JM 2000

*For more on this see JM, Random lattices in the wild: from Polya's orchard to quantum oscillators, LMS Newsletter, Issue 493 (2021)

Convergence in distribution

- Previous proofs use continued fractions analytic number theory techniques (Kloosterman sums)

- Key ingredient in our proof of the path length distribution is the convergence

$$\mathbb{Z}^2 \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \xrightarrow{\mathrm{d}} \mathbb{Z}^2 M$$

  as $\epsilon \to 0$ (ϕ represents the direction of particle velocity)

- ...and in our proof of the gap distribution for harmonic oscillator

$$\mathbb{Z}^2 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \xrightarrow{\mathrm{d}} \mathbb{Z}^2 M$$

  as $\epsilon \to 0$ ($u = \omega_1/\omega_2$ represents the ratio of the oscillation frequencies)

- In both cases the limit is the *same* μ_0-distributed random lattice (a non-trivial fact)

Equidistribution $\Rightarrow$ convergence in distribution

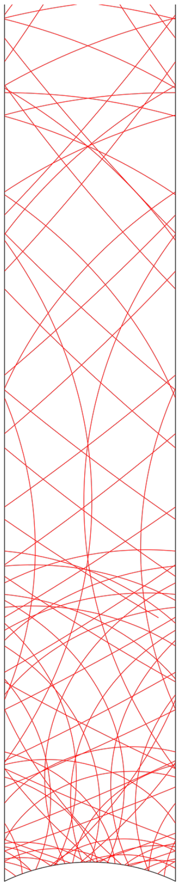
- space of lattices $\simeq \mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})$
  $\simeq T^1(\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}) =$ unit tangent bundle of modular surface
- The convergence

$$\mathbb{Z}^2 \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \xrightarrow{\mathrm{d}} \mathbb{Z}^2 M$$

  follows from the equidistribution of large circles on the modular surface (G. Margulis, A. Eskin & C. McMullen)
- The convergence

$$\mathbb{Z}^2 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \xrightarrow{\mathrm{d}} \mathbb{Z}^2 M$$

  follows from equidistribution of large closed horocycles on the modular surface (D. Zagier, P. Sarnak, D. Hejhal, L. Flaminio & G. Forni, A. Strömbergsson)



by C. Kogler

Right multiplication by the diagonal subgroup corresponds to the action of the geodesic flow, which in turn is closely related to the Gauss map and continued fractions; cf. C. Series, The modular surface and continued fractions (J. LMS 1985)

Equidistribution $\Rightarrow$ convergence in distribution

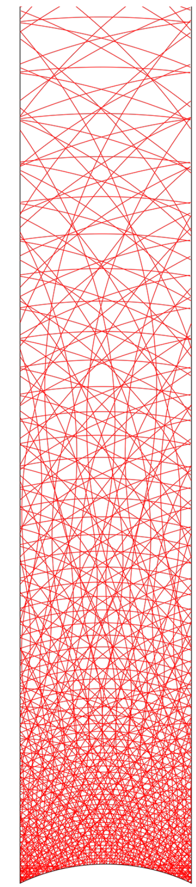
- space of lattices $\simeq \mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})$
  $\simeq T^1(\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H})$ = unit tangent bundle of modular surface
- The convergence

$$\mathbb{Z}^2 \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \xrightarrow{\mathrm{d}} \mathbb{Z}^2 M$$

  follows from the equidistribution of large circles on the modular surface (G. Margulis, A. Eskin & C. McMullen)
- The convergence

$$\mathbb{Z}^2 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \xrightarrow{\mathrm{d}} \mathbb{Z}^2 M$$

  follows from equidistribution of large closed horocycles on the modular surface (S.G. Dani & J. Smillie, D. Zagier, P. Sarnak, D. Hejhal, L. Flaminio & G. Forni, A. Strömbergsson)

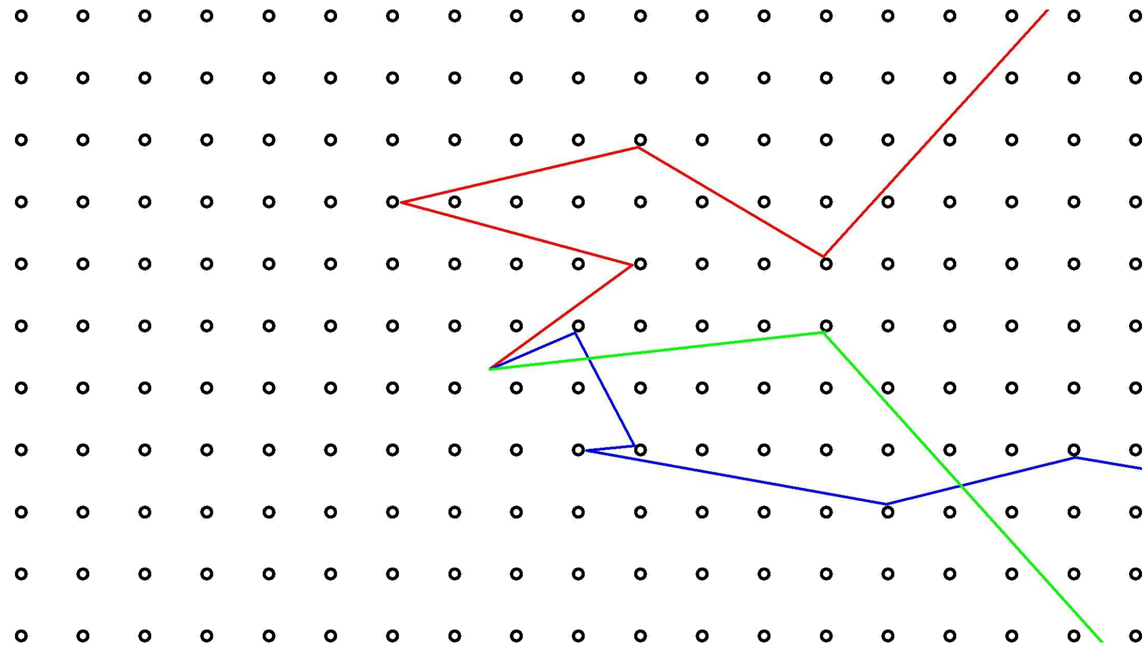


by C. Kogler

Right multiplication by the diagonal subgroup corresponds to the action of the geodesic flow, which in turn is closely related to the Gauss map and continued fractions; cf. C. Series, The modular surface and continued fractions (J. LMS 1985)

This works in higher dimensions (unlike continued fractions)

- Free path length in higher-dimensional Lorentz gases; no explicit formulas, but one prove $P(s) \sim C_d\, s^{-3}$ for any dimension $d \geq 2$ [*] as well as proof of convergence to a limit Markov process in the Boltzmann-Grad limit[†] which exhibits superdiffusion [‡]



[*] JM & A. Strömbergsson, Annals Math 2010, GAFA 2011, improving upper/lower bounds of J.-P. Bourgain, F. Golse & B. Wennberg CMP 1998/M2AN 2000

[†] JM & A. Strömbergsson, Annals Math 2011

[‡] JM & B. Toth, CMP 2017

This works in higher dimensions, and not just for lattices!

- Free path length in higher-dimensional Lorentz gases; no explicit formulas, but one prove $P(s) \sim C_d\, s^{-3}$ for any dimension $d \geq 2$ [*] as well as proof of convergence to a limit Markov process in the Boltzmann-Grad limit[†] which exhibits superdiffusion [‡]

- ... and Lorentz gases in quasicrystals[§] (deploys M. Ratner's measure classification theorem) and in lattices with defects[¶]

- Kinetic transport in Lorentz gas for general scatterer configurations[‖]

[*] JM & A. Strömbergsson, Annals Math 2010, GAFA 2011, improving upper/lower bounds of J.-P. Bourgain, F. Golse & B. Wennberg CMP 1998/M2AN 2000

[†] JM & A. Strömbergsson, Annals Math 2011

[‡] JM & B. Toth, CMP 2017

[§] JM & A. Strömbergsson, CMP 2014

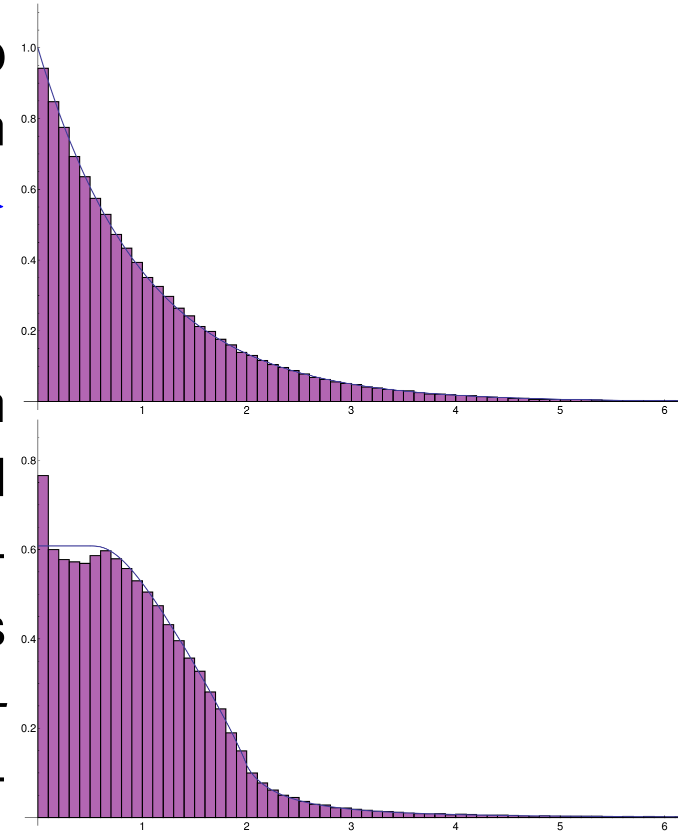
[¶] JM & I. Vinogradov, Geom. Dedicata 2017

[‖] JM & A. Strömbergsson, Memoirs AMS (in press)

Fractional parts of small powers

- For fixed $0 < \beta < 1$, $\beta \neq \frac{1}{2}$, the gap distribution of $n^\beta \bmod 1$ looks Poisson numerically—-NO PROOFS!* $\beta = \frac{1}{3} \rightarrow$

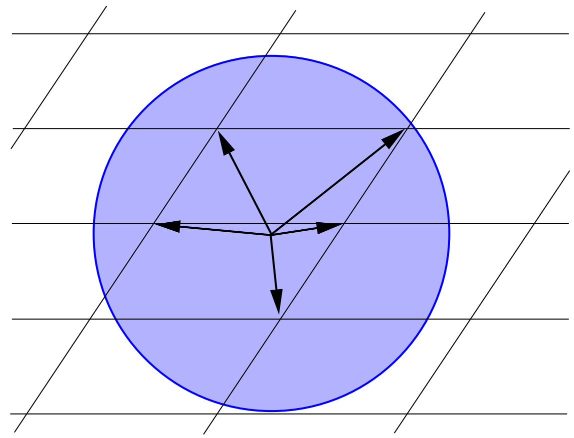
- For $\beta = \frac{1}{2}$, Elkies & McMullen have shown that the gap distribution exists, and derived an explicit formula which is clearly different from the exponential. Their proof uses equidistribution of closed *nonlinear horocycles* on the space of affine lattices and Ratner's measure classification theorem



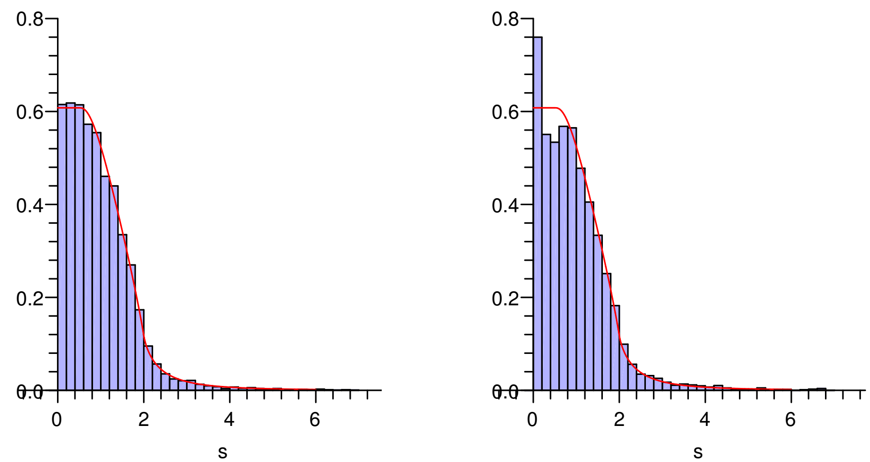
N. Elkies & C. McMullen, Duke Math J 2004

*See recent proof for two-point statistics: C. Lutsko, N. Technau & A. Sourmelidis, Pair correlation of the fractional parts of αn^θ, arXiv:2106.09800 (2021)

Statistics of directions



- Consider the directions of shifted lattice points of a Euclidean lattice $\mathcal{L}$ in a large ball



- The gap distribution exists and coincides with the Elkies-McMullen distribution if $(\alpha, \beta) \notin \mathbb{Q}\mathcal{L}$. Proof uses again Ratner's measure classification theorem—but for a different unipotent flow!

JM & Strömbergsson, Annals Math 2010

Directions in quasicrystals

Acta Crystallographica Section A
Foundations and Advances
ISSN 2053-2733

Radial spacing distributions from planar point sets

M. Baake,* F. Götze, C. Huck and T. Jakobi

Department of Mathematics, Bielefeld University, Bielefeld, Germany. Correspondence e-mail: mbaake@math.uni-bielefeld.de

Received 12 February 2014
Accepted 14 May 2014

This paper explores the radial projection method for locally finite planar point sets and provides numerical examples for different types of order. The main question is whether the method is suitable to analyse order in a quantitative way. The findings indicate that the answer is affirmative. In this context, local visibility conditions are also studied for certain types of aperiodic point sets.

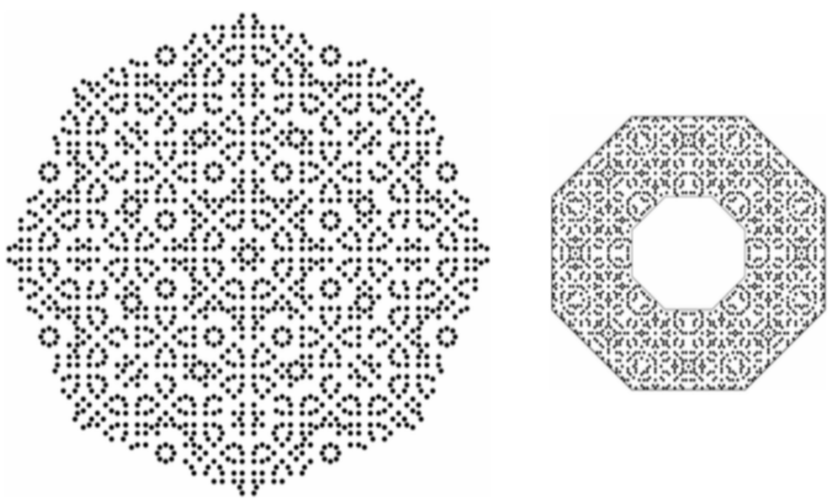


Figure 8
Visible vertices of the eightfold symmetric *Ammann–Beenker* tiling (left: direct space, right: internal space).

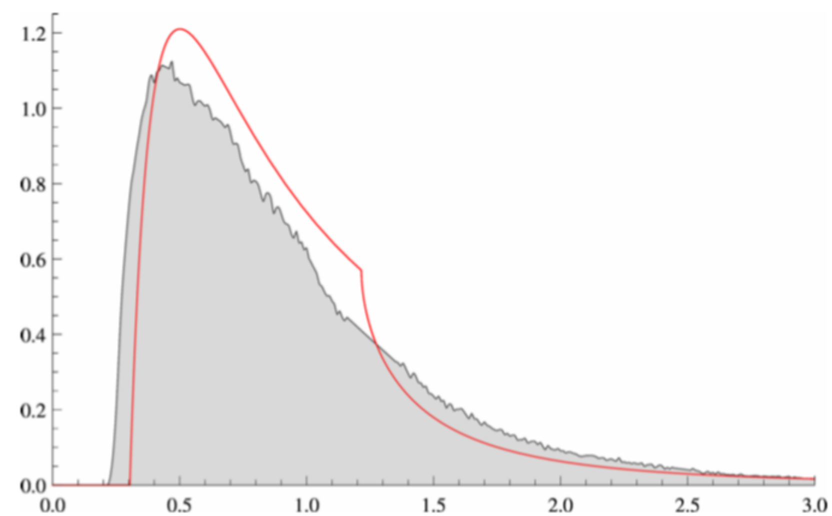


Figure 9
Distribution of the radial spacings of a large D_8-symmetric AB patch.

JM and A. Strömbergsson, Visibility and directions in quasicrystals, IMRN 2015

Small-world networks

Collective dynamics of 'small-world' networks

Duncan J. Watts* & Steven H. Strogatz

Department of Theoretical and Applied Mechanics, Kimball Hall, Cornell University, Ithaca, New York 14853, USA

Networks of coupled dynamical systems have been used to model biological oscillators[1–4], Josephson junction arrays[5,6], excitable media[7], neural networks[8–10], spatial games[11], genetic control networks[12] and many other self-organizing systems. Ordinarily, the connection topology is assumed to be either completely regular or completely random. But many biological, technological and social networks lie somewhere between these two extremes. Here we explore simple models of networks that can be tuned through this middle ground: regular networks 'rewired' to introduce increasing amounts of disorder. We find that these systems can be highly clustered, like regular lattices, yet have small characteristic path lengths, like random graphs. We call them 'small-world' networks, by analogy with the small-world phenomenon[13,14] (popularly known as six degrees of separation[15]). The neural network of the worm *Caenorhabditis elegans*, the power grid of the western United States, and the collaboration graph of film actors are shown to be small-world networks. Models of dynamical systems with small-world coupling display enhanced signal-propagation speed, computational power, and synchronizability. In particular, infectious diseases spread more easily in small-world networks than in regular lattices.

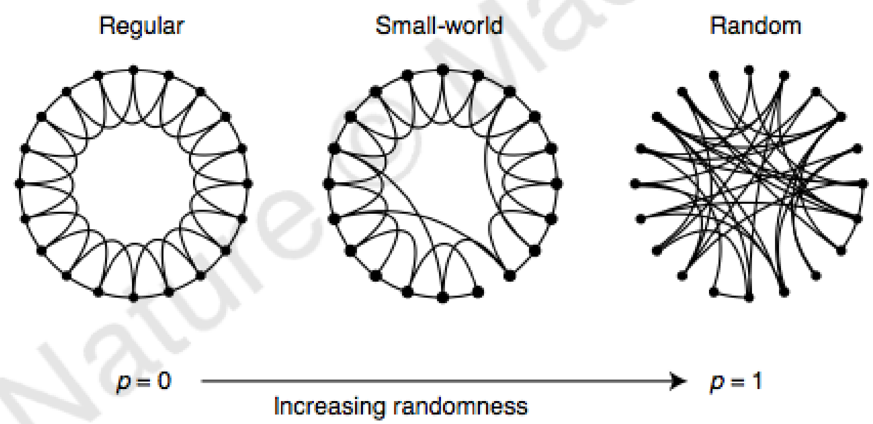


Figure 1 Random rewiring procedure for interpolating between a regular ring lattice and a random network, without altering the number of vertices or edges in the graph. We start with a ring of n vertices, each connected to its k nearest neighbours by undirected edges. (For clarity, $n = 20$ and $k = 4$ in the schematic examples shown here, but much larger n and k are used in the rest of this Letter.) We choose a vertex and the edge that connects it to its nearest neighbour in a clockwise sense. With probability p, we reconnect this edge to a vertex chosen uniformly at random over the entire ring, with duplicate edges forbidden; otherwise we leave the edge in place. We repeat this process by moving clockwise

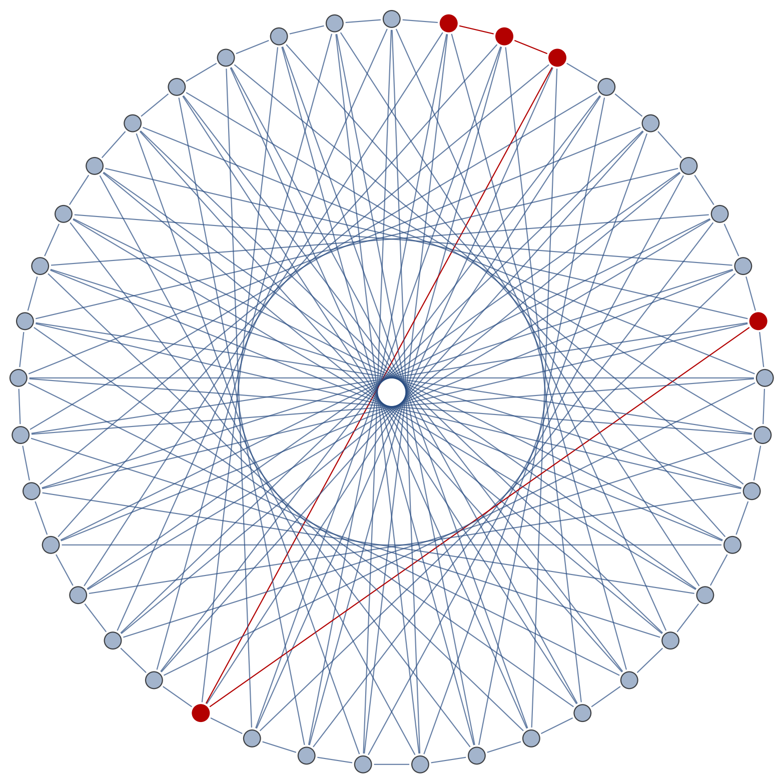
from: Watts & Strogatz, Nature 1998

Circulant graphs

1. Fix integers $0 < a_1 < \ldots < a_k \leq n/2$ with $\gcd(a_1, \ldots, a_k, n) = 1$;
2. Connect vertex i and j, if $|i - j| \equiv a_h \mod n$ for some a_h; assign length ℓ_h to this edge.

The resulting graph $C_n(\boldsymbol{\ell}, \boldsymbol{a})$ is called a "circulant graph" (its adjacency matrix is circulant), sometimes also "multiloop network". It is of course the undirected Cayley graph of the cyclic group of order n w.r.t. the generating set $\{\pm a_1, \ldots, \pm a_k\}$.

```
HighlightGraph[#, FindDiameterPath[#]] &[CirculantGraph[41, {1, 15, 20}]]
```



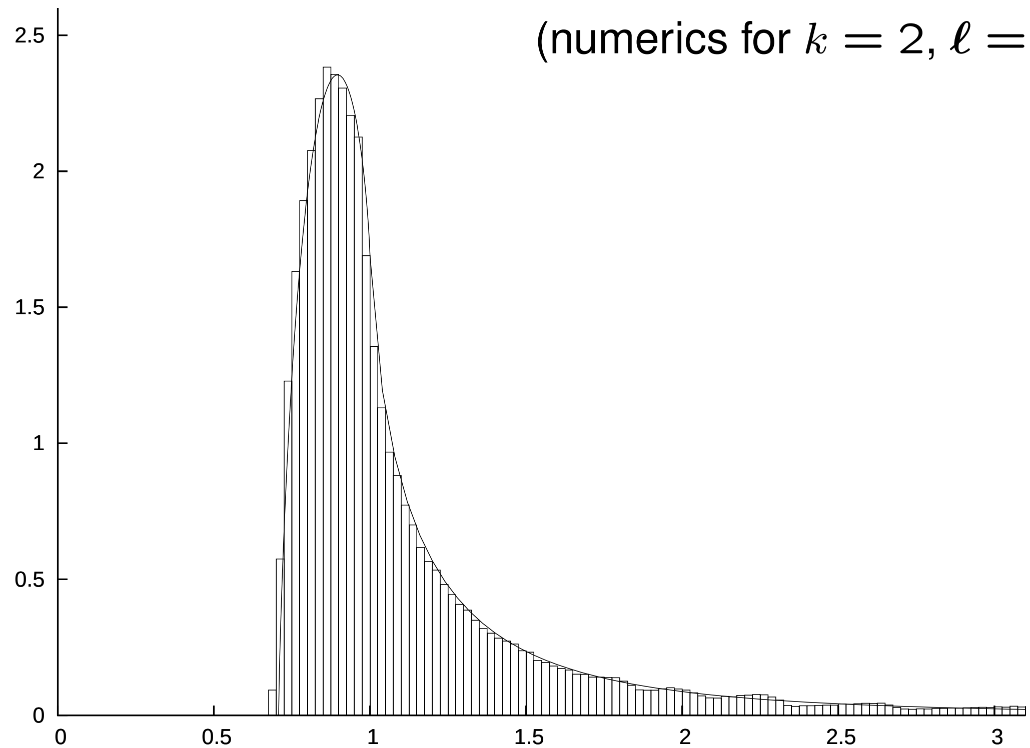
Random lattices and circulant graphs

Theorem A (Marklof & Strömbergsson, Combinatorica 2013). Let $k \geq 2$, $\mathcal{D} \subset \mathbb{R}^{k+1}$ bounded, non-empty and boundary of Lebesgue measure zero. Pick $(\boldsymbol{a}, n)$ at random in $T\mathcal{D}$. Then

$$\frac{\operatorname{diam} C_n(\boldsymbol{\ell}, \boldsymbol{a})}{(n\ell_1 \cdots \ell_k)^{1/k}} \xrightarrow{\mathrm{d}} \rho(\mathfrak{P}, L) \qquad \text{as } T \to \infty,$$

where $\rho(\mathfrak{P}, L)$ is ...

. . . a random variable distributed according to the probability density



(numerics for $k = 2$, $\boldsymbol{\ell} = (1, 1)$, $n = 1000$)

For $k = 2$:

$$\tilde{p}_2(R) = \begin{cases} 0 & (0 \le R \le \frac{1}{\sqrt{2}}) \\ \frac{24}{\pi^2}\Big(\frac{2R^2-1}{R}\log\Big(\frac{2R^2}{2R^2-1}\Big) + \frac{1-R^2}{R}\log\Big(\frac{R^2}{|1-R^2|}\Big)\Big) & (R > \frac{1}{\sqrt{2}}). \end{cases}$$

For general $k \ge 2$:

$$\tilde{p}_k(R) = 0 \ (R < \tfrac{1}{2}(k!)^{1/k}), \qquad \tilde{p}_k(R) \sim \frac{k}{2\zeta(k)} R^{-(k+1)} \ (R \to \infty)$$

Random lattices and circulant graphs

Theorem A (Marklof & Strömbergsson, Combinatorica 2013). Let $k \geq 2$, $\mathcal{D} \subset \mathbb{R}^{k+1}$ bounded, non-empty and boundary of Lebesgue measure zero. Pick $(\boldsymbol{a}, n)$ at random in $T\mathcal{D}$. Then

$$\frac{\operatorname{diam} C_n(\boldsymbol{\ell}, \boldsymbol{a})}{(n\ell_1 \cdots \ell_k)^{1/k}} \xrightarrow{\text{d}} \rho(\mathfrak{P}, L) \qquad \text{as } T \to \infty,$$

where $\rho(\mathfrak{P}, L)$ is ...

Random lattices and circulant graphs

Theorem A (Marklof & Strömbergsson, Combinatorica 2013). Let $k \geq 2$, $\mathcal{D} \subset \mathbb{R}^{k+1}$ bounded, non-empty and boundary of Lebesgue measure zero. Pick $(\boldsymbol{a}, n)$ at random in $T\mathcal{D}$. Then

$$\frac{\operatorname{diam} C_n(\boldsymbol{\ell}, \boldsymbol{a})}{(n\ell_1 \cdots \ell_k)^{1/k}} \xrightarrow{\mathrm{d}} \rho(\mathfrak{P}, L) \qquad \text{as } T \to \infty,$$

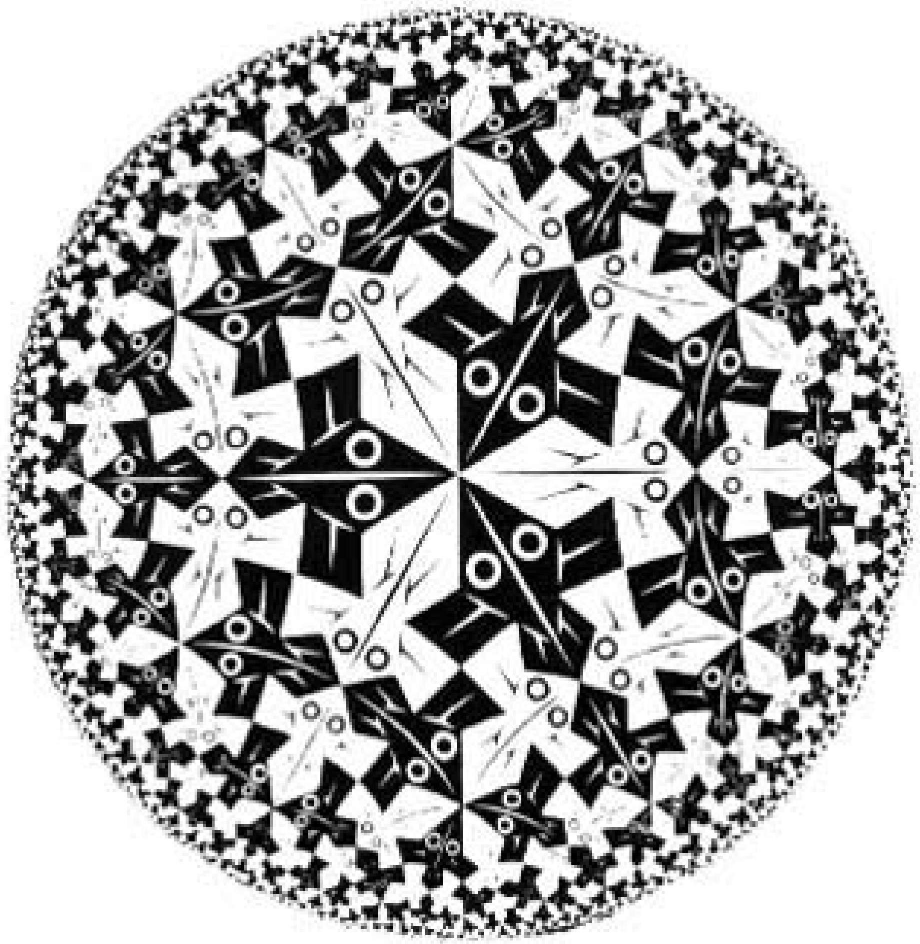
where $\rho(\mathfrak{P}, L)$ is ...the covering radius of a random lattice L in $\mathbb{R}^k$ with respect to the polytope

$$\mathfrak{P} = \left\{ \boldsymbol{x} \in \mathbb{R}^k : |x_1| + \ldots + |x_k| \leq 1 \right\}.$$

($\mathfrak{P}$ is a square for $k = 2$ and an octahedron for $k = 3$.)

Improved by Shapira & Zuck (Combinatorica 2019) who proved the above for fixed $n \to \infty$ and only $\boldsymbol{a}$ random.

How about random lattices in hyperbolic geometry?



Escher’s Circle Limit I

Outline

Lecture 1 What are random lattices? Basic features and examples

Lecture 2 Random lattices in number theory and geometry (Tomorrow)

Lecture 3 Random lattices in statistical mechanics – the Lorentz gas (Friday)