CHAOS, CHANCE AND RANDOMNESS

Jens Marklof
School of Mathematics

Inaugural Lecture, 11 October 2007

This lecture will discuss:

- Classic determinism
- How to generate randomness
- Chaos and randomness in simple models
- Visibility in a forest
- Chaotic transport in crystals

This lecture will *not* discuss (sadly):

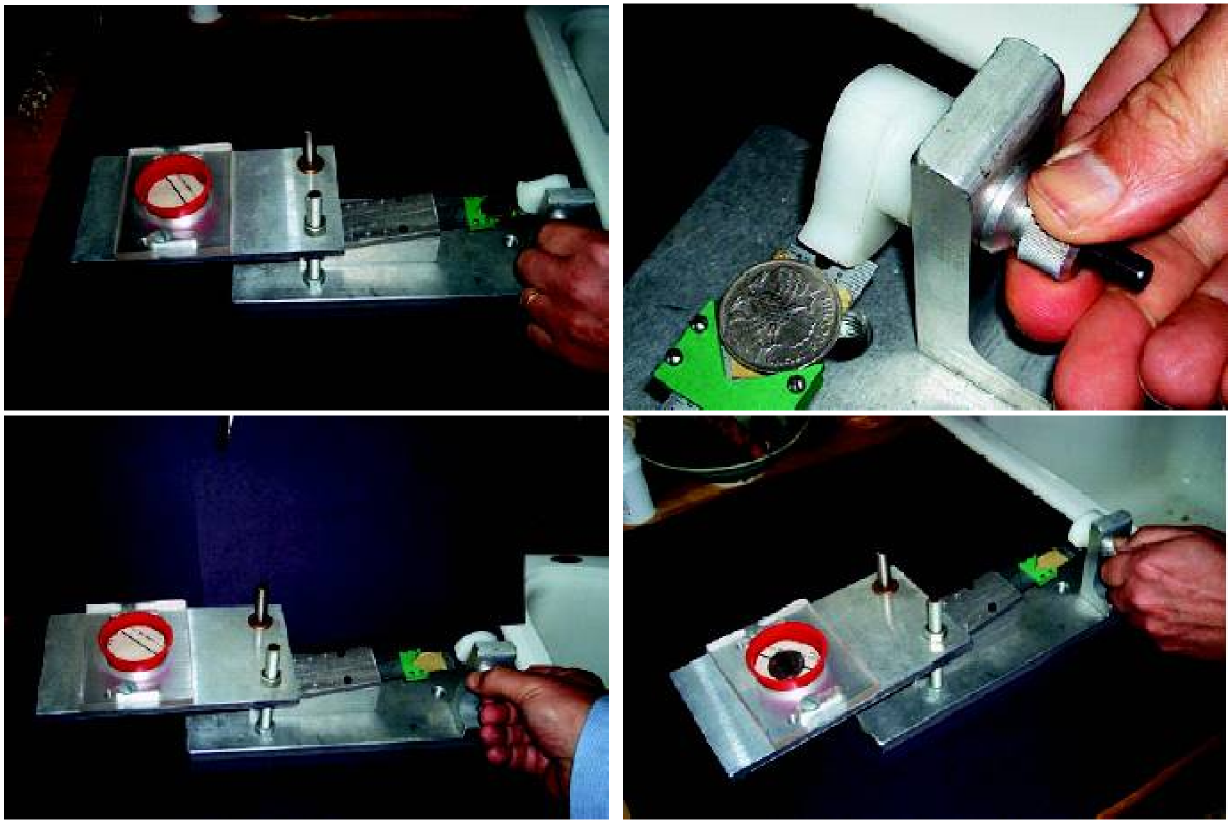
- Free will
- Quantum effects
- The stock market
- Benford's law and tax evasion
- How to win the lottery

Newton's laws and coin tossing

According to Newton's laws of motion*, the knowledge of the initial position, velocity and spin of a coin precisely determines the outcome of a coin toss—*head* or *tail*. So where does randomness enter?

*we neglect quantum and relativistic effects

The perfect coin tosser



from: P. Diaconis et al., Dynamical Bias in the Coin Toss, SIAM Review ‘07

The deterministic universe

"An intelligence which, at a given instant, would know all the forces by which Nature is animated, and the respective situation of all the elements of which it is composed, if furthermore it were vast enough to submit all these data to analysis, would in the same formula encompass the motions of the largest bodies of the universe, and those of the most minute atom: *nothing for it would be uncertain, and the future as well as the past would be present to its eyes*."

Pierre-Simon, Marquis de Laplace, 1814

Laplace (1749-1827)

Generating randomness: a simple model for chaotic dynamics

Consider a machine (a computer) that accepts as an input a number x between 0 and 1, e.g., 0.625, and produces as an output a new number y, again between 0 and 1, according to the rule

$$y = 2x \mod 1$$

where "$\mathrm{mod}\,1$" means that we remove the integer part of $2x$.

For example:

input:	x	0.4	0.7
	$2x$	0.8	1.4
output:	y	0.8	0.4

One may think of x as the analogue of the data describing the coin's initial position, velocity and spin.

We now run this operation several times by feeding the previous output as the new input. After n iterations we thus obtain the output

$$y_n = 2^n x \mod 1$$

When n is large (say $n = 10$), changing the input slightly may have a dramatic effect on the output. We thus have *sensitive dependence on initial conditions*, a characteristic feature of chaotic systems.

For example:

input:	x	0.30	0.31
output:	y_{10}	0.20	0.44

Detecting randomness

We have constructed a system with sensitive dependence on initial data, but how random is it? To see how it compares with a fair coin toss, let us say that we have *heads* if the output y_n is between 0 and 0.5, and *tails* when it is between 0.5 and 1.

The following mathematical theorem tells us that, given any ever-so-small inaccuracy in the initial data, and after a sufficiently large number of iterations, our system produces a fair coin toss.

Let x be uniformly distributed between 0.2999999 and 0.3000001 then the probability of heads is

$$\mathrm{Prob}(0 < y_n < 0.5) \approx 0.5$$

with an error of size 2^{-n}.

. . . in summary:

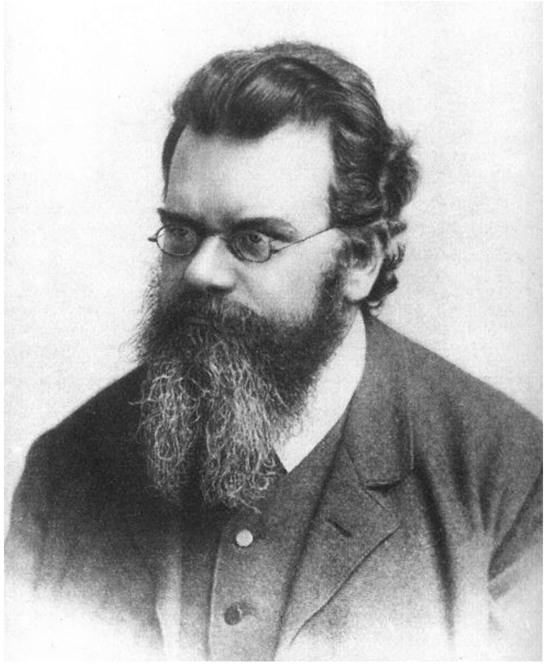
In chaotic systems, a tiny amount of uncertainty in the initial data produces almost perfect randomness after a very short time.

That is, we have an *exponential amplification of randomness*.

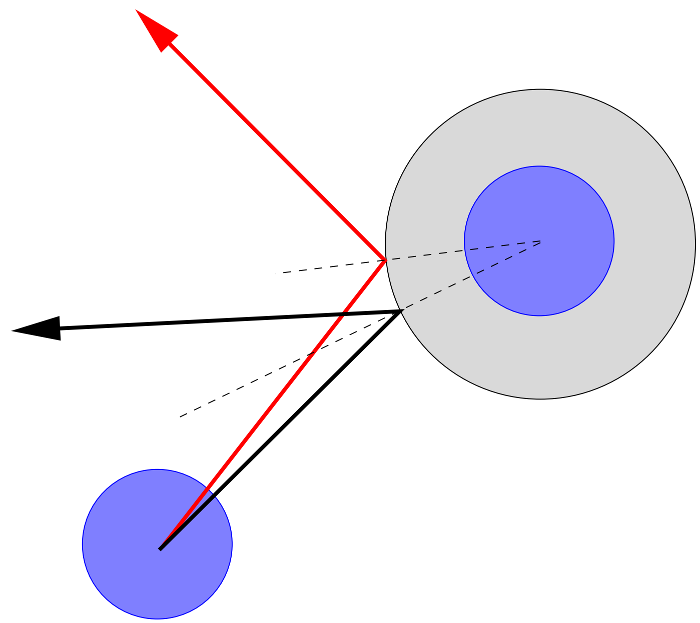
Boltzmann’s statistical mechanics

Boltzmann proposed to explain the motion of a gas cloud by using the dynamics of microscopic particles—atoms and molecules, whose existence was highly disputed during Boltzmann’s lifetime.

In his 1872 paper, Boltzmann derived the famous *Boltzmann equation*, assuming that the dynamics of the colliding gas molecules is chaotic.



Ludwig Boltzmann (1844-1906)



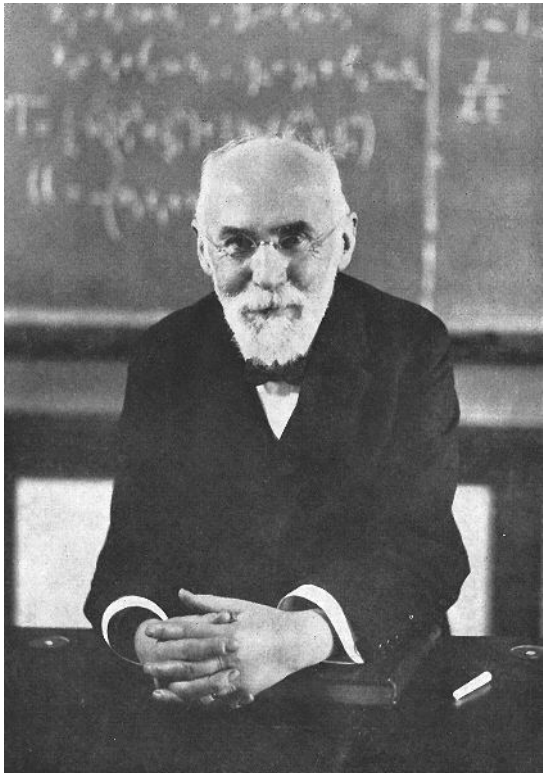
The Boltzmann gas: Sensitive dependence in two-molecule collision.

The first rigorous justification of the Boltzmann equation was given by Oscar Lanford in 1975 for the dynamics over very short time intervals. The problem for the more realistic macroscopic time scales is still wide open.

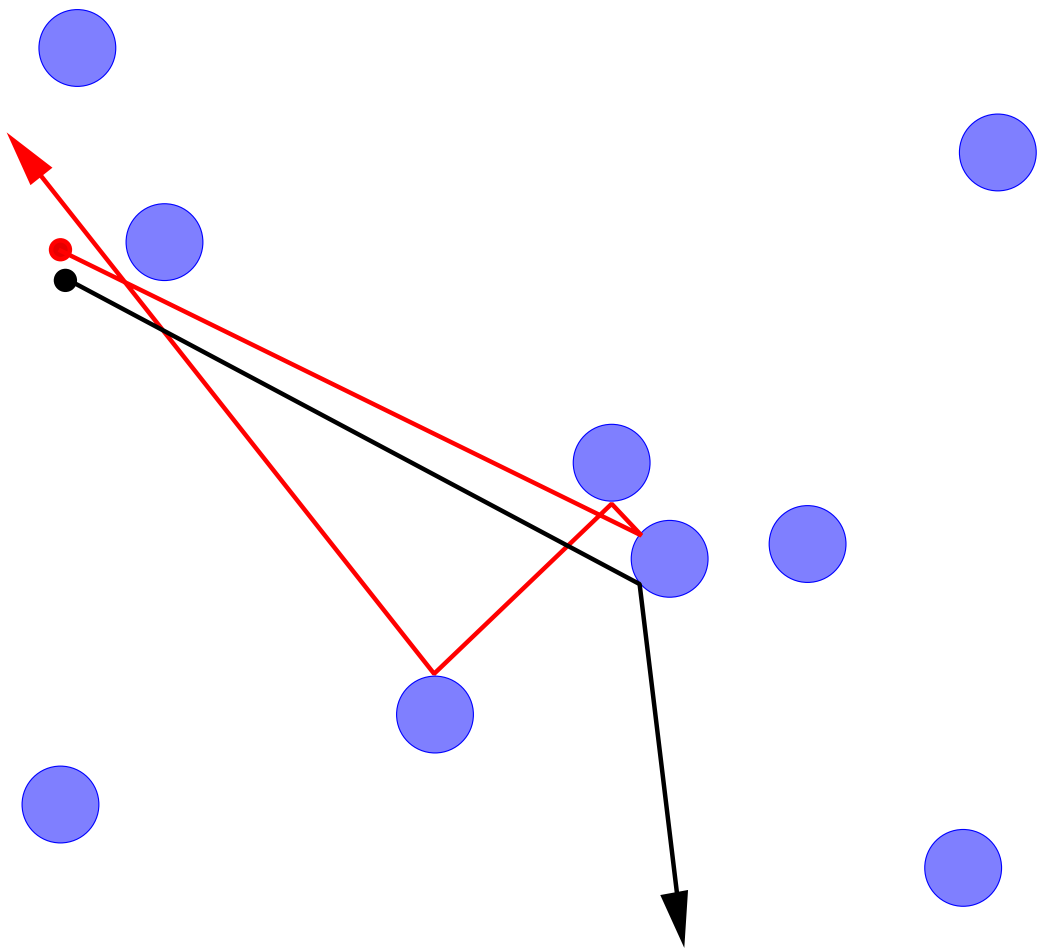
The Lorentz gas

In an attempt to describe the evolution of a dilute electron gas in a metal, Lorentz proposed in 1905 a model, where the heavier atoms are assumed to be fixed, whereas the electrons are interacting with the atoms but not with each other. For simplicity, Lorentz assumed like Boltzmann that the atoms can be modeled by elastic spheres.

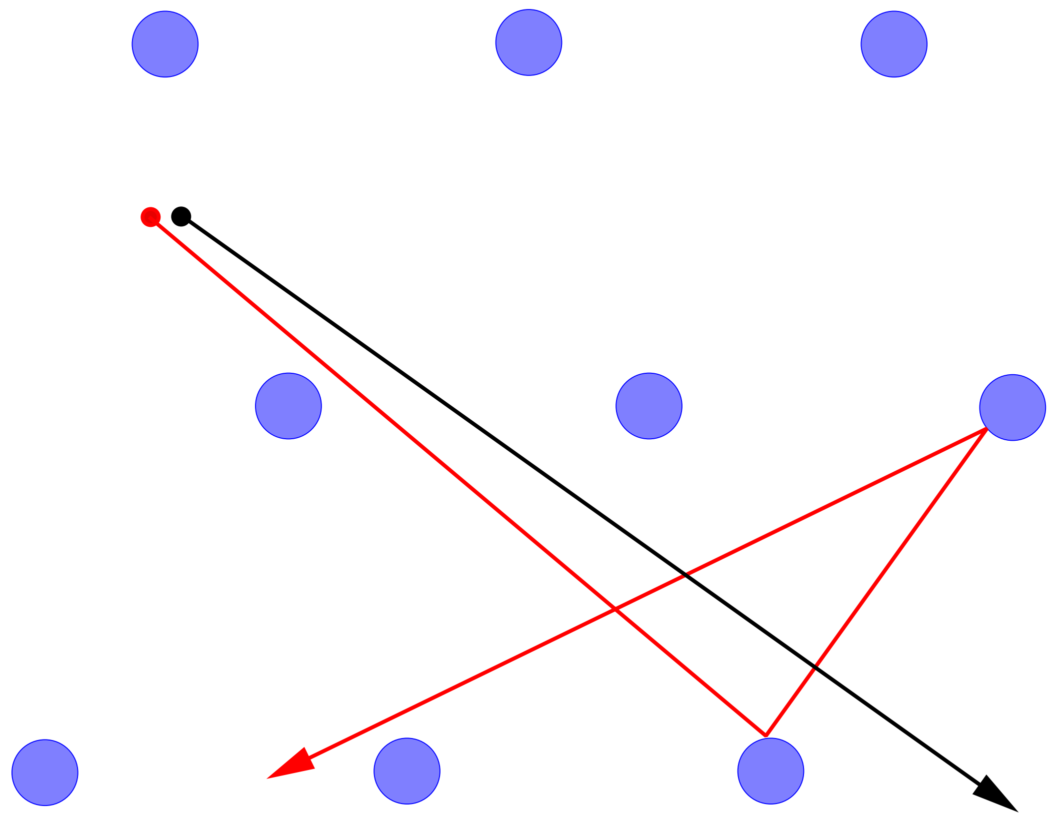
The Lorentz gas is still one of the iconic models for chaotic diffusion, both in a random and periodic configuration of scatterers.



Hendrik Lorentz (1853-1928)



The Lorentz gas with randomly positioned scatterers.

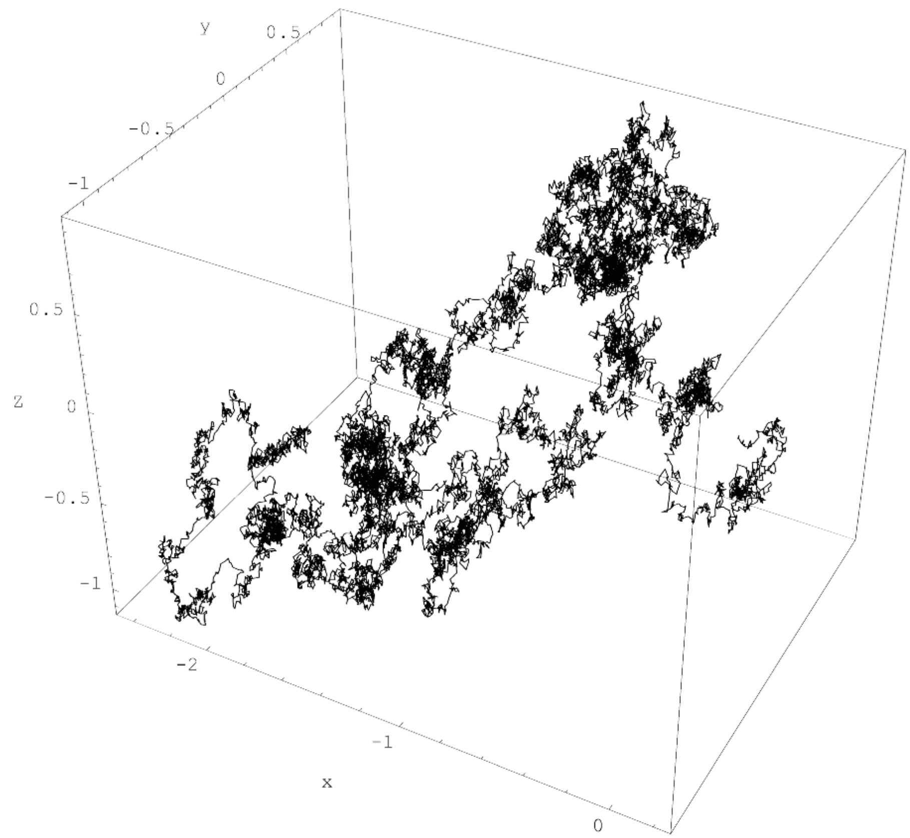


The Lorentz gas with a periodic array of scatterers (crystal).

The periodic Lorentz gas and Brownian motion

Yakov Sinai (Princeton University) is one of the pioneers in understanding the chaotic properties of the periodic Lorentz gas. He proved in 1980, jointly with Leonid Bunimovich, that *in the limit of long times the dynamics appears as random as Brownian motion*.

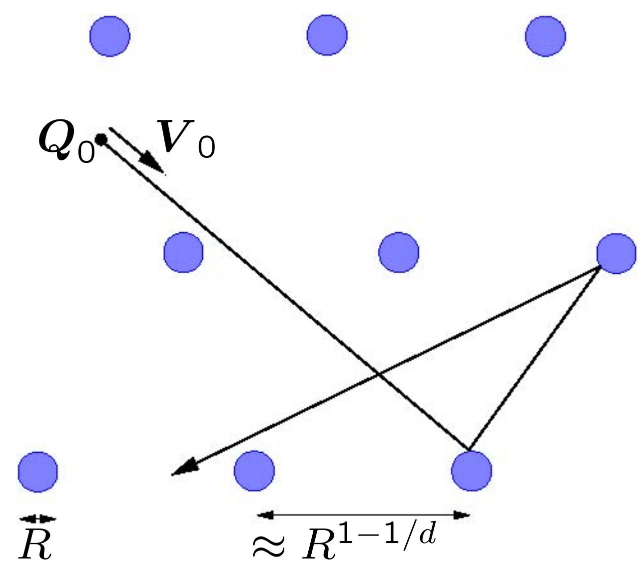
Yakov Sinai

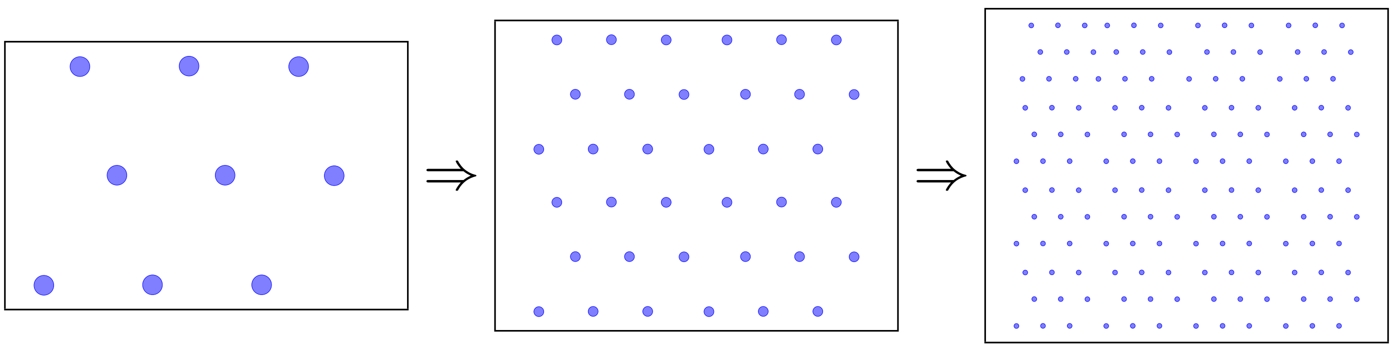


A typical Brownian path in three dimensional space.

The Boltzmann-Grad limit of the Lorentz gas

To prove the laws postulated by Boltzmann and Lorentz, one however needs to consider a different limit, the *Boltzmann-Grad limit*, where the radius R of each scatterer tends to zero, and the distance between the scatterers is rescaled so that the mean free path lengths remains constant.





Visibility in a forest

The first problem in understanding the Lorentz gas in the Boltzmann-Grad limit is concerned with the distribution of the *free path length*, which is the distance an electron travels between consecutive collisions.

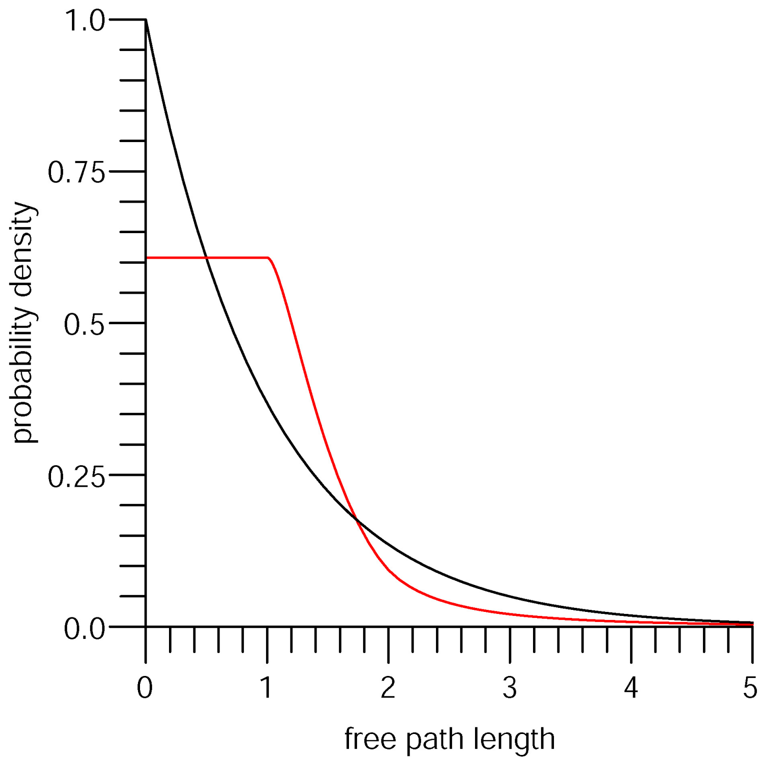
This leads to natural problems in probability theory and number theory, respectively, which, in the two-dimensional case were paraphrased by Pólya as the *problem of visibility in a forest*.

George Pólya (1887-1985)

The distribution of free path lengths

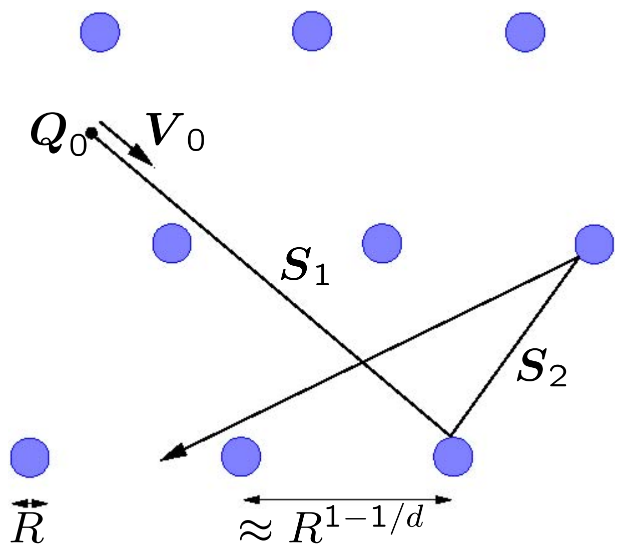
In the case of the Lorentz gas with a random configuration of scatterers, the probability density for finding a free path of length L is $\exp(-L)$ (**black curve**).

The **red curve** represents the distribution for the two-dimensional periodic Lorentz gas (Dahlquist, Nonlinearity 1997; Boca & Zaharescu, Comm. Math. Phys. 2007).



The joint distribution of path segments

Jointly with Andreas Strömbergsson (KTH Stockholm) I have recently developed new techniques that not only allow us to generalize the above results to 3+ dimensions, but also compute the limit distribution of a path $(\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots)$ with random initial data $(\boldsymbol{Q}_0, \boldsymbol{V}_0)$. In particular we prove that the path segments are generated by a Markov process with memory two.



Dynamics of a particle cloud

The result with Strömbergsson allows us to predict the dynamics of a particle cloud in the periodic Lorentz gas.

Remarkably, this dynamics is *not* governed by the linear Boltzmann equation as in the random configuration (Galavotti, 1969), but by a substantially more complicated process.

Future research

- Replace the elastic spheres with more realistic potentials, and thus obtain a model for the chaotic transport of electrons in a crystal

- Consider crystals in electro-magnetic fields; the “free” path segments will then no longer be straight lines

- Investigate crystals with defects

- Compute quantum effects

Highly recommended:

Chance and Chaos
by David Ruelle
(Penguin Books 1993)

Ludwig Boltzmann—The Man Who Trusted Atoms
by Carlo Cercignani
(Oxford University Press 1998)