Kinetic transport in crystals

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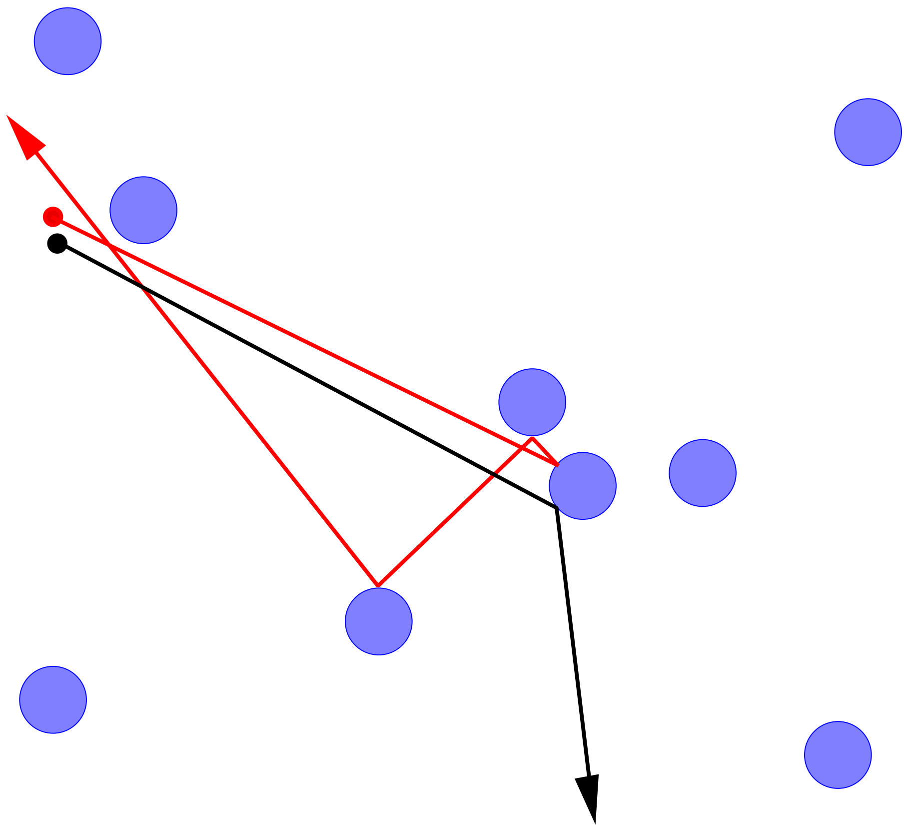
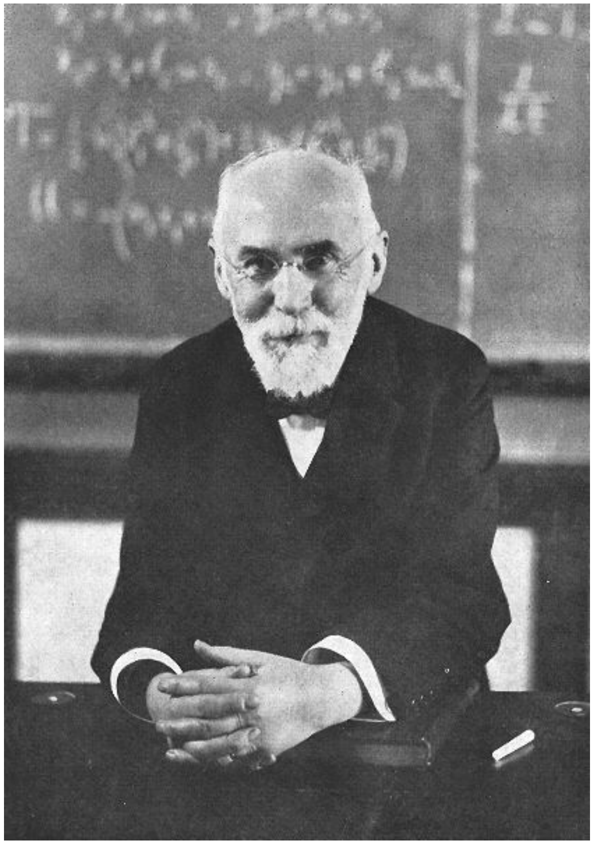
based on joint work with Andreas Strömbergsson (Uppsala)

The Lorentz gas



Arch. Neerl. (1905)



Hendrik Lorentz (1853-1928)

The Boltzmann-Grad limit

- Consider the dynamics in the limit of small scatterer radius ρ

- $\big(\boldsymbol{q}(t), \boldsymbol{v}(t)\big)$ = "microscopic" phase space coordinate at time t

- A dimensional argument shows that, in the limit $\rho \to 0$, the mean free path length (i.e., the average time between consecutive collisions) scales like $\rho^{-(d-1)}$ (= 1/total scattering cross section)

- We thus re-define position and time and use the "macroscopic" coordinates

$$\big(\boldsymbol{Q}(t), \boldsymbol{V}(t)\big) = \big(\rho^{d-1}\boldsymbol{q}(\rho^{-(d-1)}t), \boldsymbol{v}(\rho^{-(d-1)}t)\big)$$

The linear Boltzmann equation

- Time evolution of initial data $(\boldsymbol{Q}, \boldsymbol{V})$:

$$(\boldsymbol{Q}(t), \boldsymbol{V}(t)) = \Phi_\rho^t(\boldsymbol{Q}, \boldsymbol{V})$$

- Time evolution of a particle cloud with initial density $f \in \mathrm{L}^1$:

$$f_t = \mathrm{L}_\rho^t f, \qquad [L_\rho^t f](\boldsymbol{Q}, \boldsymbol{V}) := f\Big(\Phi_\rho^{-t}(\boldsymbol{Q}, \boldsymbol{V})\Big)$$

In his 1905 paper Lorentz suggested that f_t is governed, as $\rho \to 0$, by the linear Boltzmann equation:

$$\left[\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{Q}}\right] f_t(\boldsymbol{Q}, \boldsymbol{V}) = \int_{\mathrm{S}_1^{d-1}} \Big[f_t(\boldsymbol{Q}, \boldsymbol{V}_0) - f_t(\boldsymbol{Q}, \boldsymbol{V})\Big] \sigma(\boldsymbol{V}_0, \boldsymbol{V}) d\boldsymbol{V}_0$$

where the collision kernel $\sigma(\boldsymbol{V}_0, \boldsymbol{V})$ is the cross section of the individual scatterer. E.g.: $\sigma(\boldsymbol{V}_0, \boldsymbol{V}) = \frac{1}{4}\|\boldsymbol{V}_0 - \boldsymbol{V}\|^{3-d}$ for specular reflection at a hard sphere

The linear Boltzmann equation—rigorous proofs

- Galavotti (Phys Rev 1969 & report 1972): Poisson distributed hard-sphere scatterers

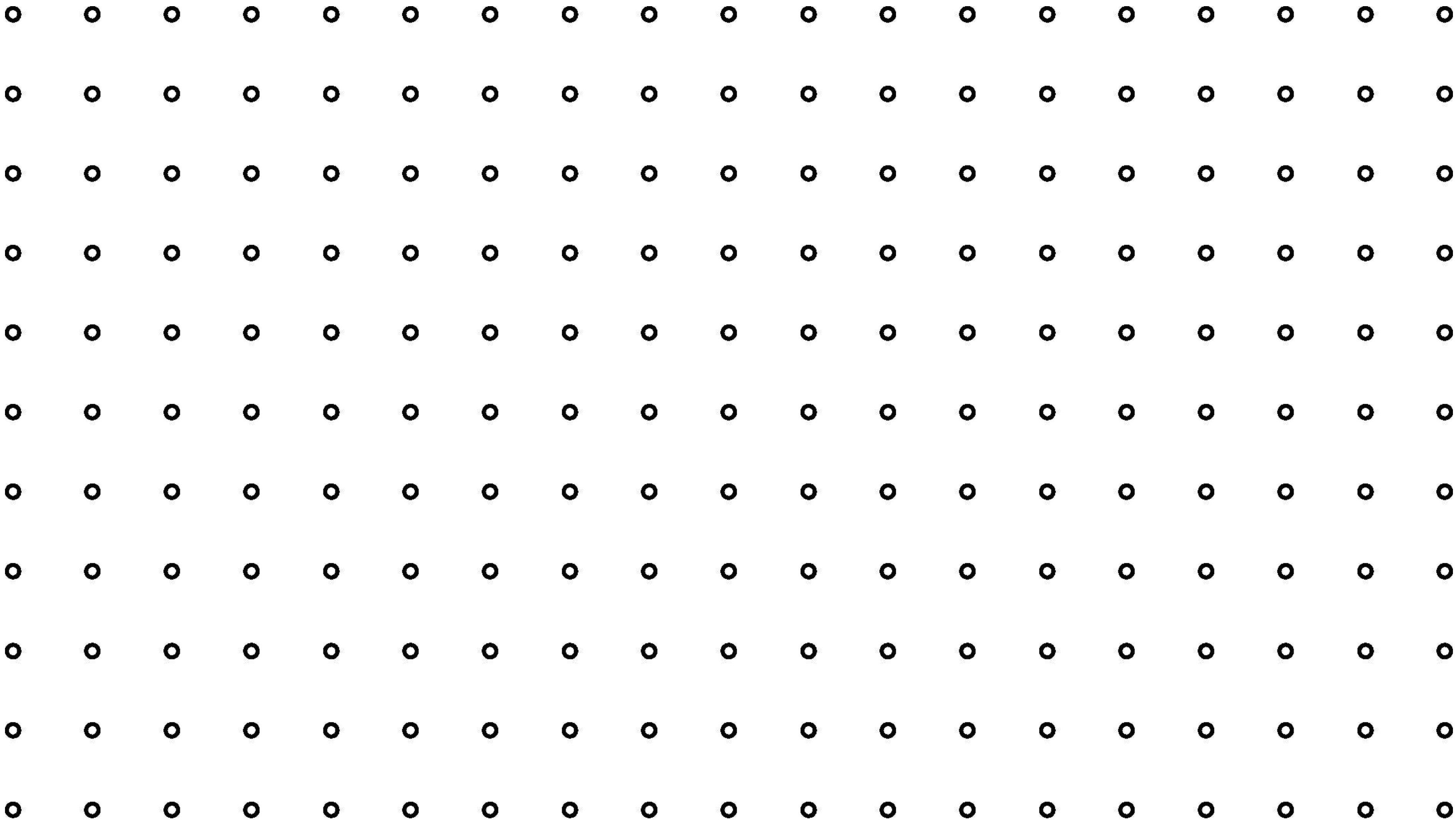
- Spohn (Comm Math Phys 1978): extension to more general random scatterer configurations and potentials

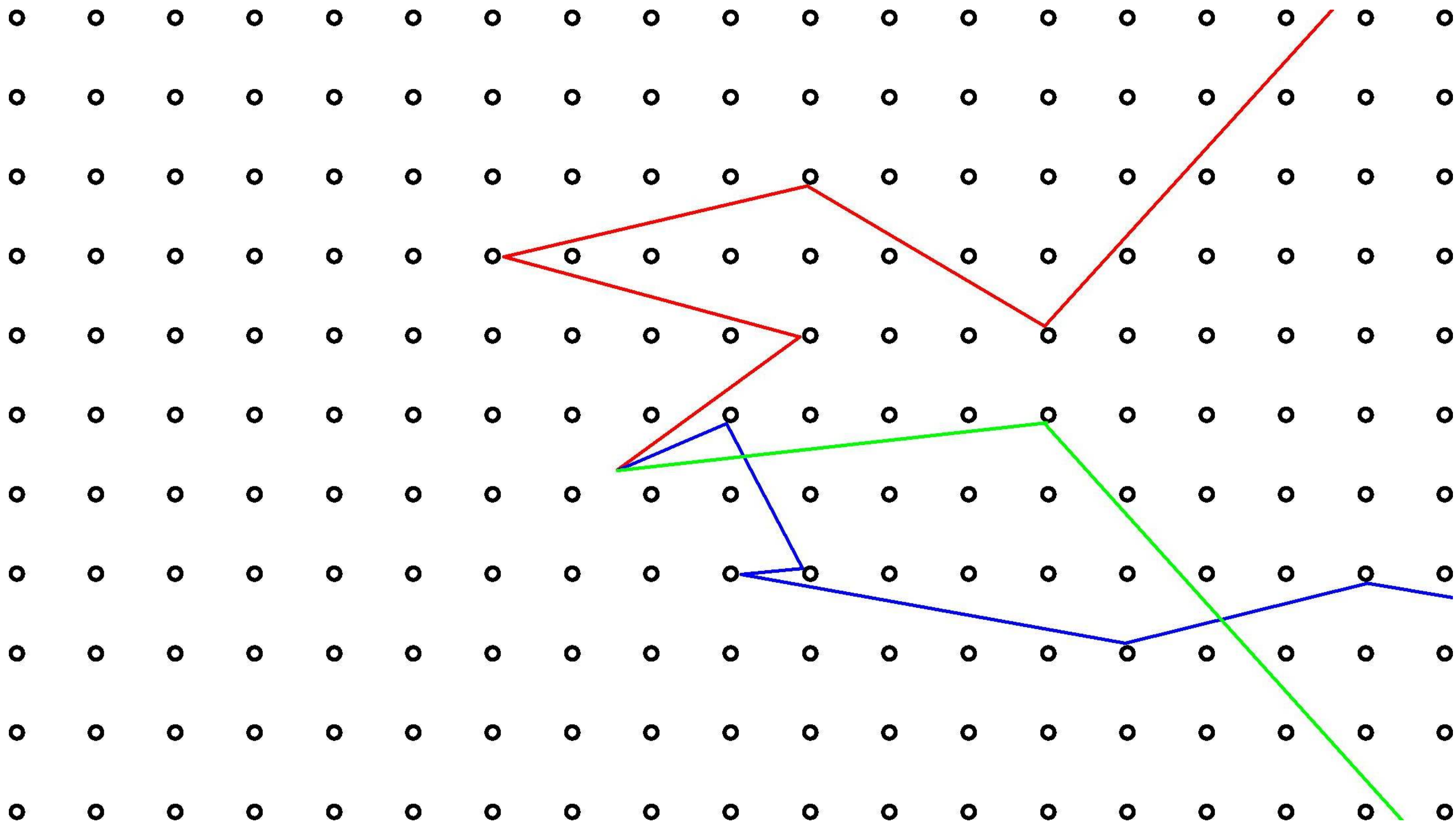
- Boldrighini, Bunimovich and Sinai (J Stat Phys 1983): prove convergence for almost every scatterer configuration

The quantum linear Boltzmann equation

- Spohn (J Stat Phys 1977): Gaussian random potentials, small times
- Erdös and Yau (Contemp Math 1998, Comm Pure Appl Math 2000): General random potentials, weak coupling limit
- Eng and Erdös (Rev Math Phys 2005): Low density limit

The periodic Lorentz gas





Chaotic diffusion for *fixed* scatterer radius ρ

- Bunimovich and Sinai (Comm Math Phys 1980/81): In the case of finite horizon* and in dimension $d = 2$, the dynamics is diffusive in the limit of large times t, and satisfies a central limit theorem with normalization $\sqrt{t}$.

- Melbourne and Nicol (Comm Math Phys 2005): Invariance principles for $d = 2$ and finite horizon.

- Bleher (J Stat Phys 1992), Szasz-Varju (2007): Central limit theorem for infinite horizon; the normalization is now $\sqrt{t \log t}$ (due to the free flight corridors).

- Central limit theorem still unproven in higher dimensions; cf. Chernov (J Stat Phys 1994), Balint-Toth (2007).

*"Finite horizon" means that the scatterers are configured so that the path length between consecutive collisions is bounded.

The Boltzmann-Grad limit

- *Recall:* We are interested in the dynamics in the limit of small scatterer radius

- $(\boldsymbol{q}(t), \boldsymbol{v}(t))$ = “microscopic” phase space coordinate at time t

- Re-define position and time and use the “macroscopic” coordinates

$$(\boldsymbol{Q}(t), \boldsymbol{V}(t)) = (\rho^{d-1}\boldsymbol{q}(\rho^{-(d-1)}t), \boldsymbol{v}(\rho^{-(d-1)}t))$$

A limiting random process

A cloud of particles with initial density $f(\boldsymbol{Q}, \boldsymbol{V})$ evolves in time t to

$$f_t(\boldsymbol{Q}, \boldsymbol{V}) = [L_\rho^t f](\boldsymbol{Q}, \boldsymbol{V}) = f\Big(\Phi_\rho^{-t}(\boldsymbol{Q}, \boldsymbol{V})\Big).$$

Theorem A. For every $t > 0$ there exists a linear operator $L^t : \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d)) \to \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$, such that for every $f \in \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$ and any set $\mathcal{A} \subset \mathrm{T}^1(\mathbb{R}^d)$ with boundary of Lebesgue measure zero,

$$\lim_{\rho\to 0} \int_{\mathcal{A}} [L_\rho^t f](\boldsymbol{Q}, \boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V} = \int_{\mathcal{A}} [L^t f](\boldsymbol{Q}, \boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V}.$$

The operator L^t thus describes the macroscopic diffusion of the Lorentz gas in the Boltzmann-Grad limit $\rho \to 0$.

Note: The family $\{L^t\}_{t\geq 0}$ does *not* form a semigroup.

A generalization of the linear Boltzmann equation

In the case of the periodic Lorentz gas L^t does not form a semigroup, and hence in particular the linear Boltzmann equation does not hold. This problem is resolved by considering extended phase space coordinates $(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+)$ where

$$(\boldsymbol{Q}, \boldsymbol{V}) \in \mathrm{T}^1(\mathbb{R}^d) \text{ — usual position and momentum}$$
$$\xi \in \mathbb{R}_+ \text{ — flight time until the next scatterer}$$
$$\boldsymbol{V}_+ \in \mathrm{S}_1^{d-1} \text{ — velocity after the next hit}$$

We prove the following generalization of the linear Boltzmann equation in the extended phase space:

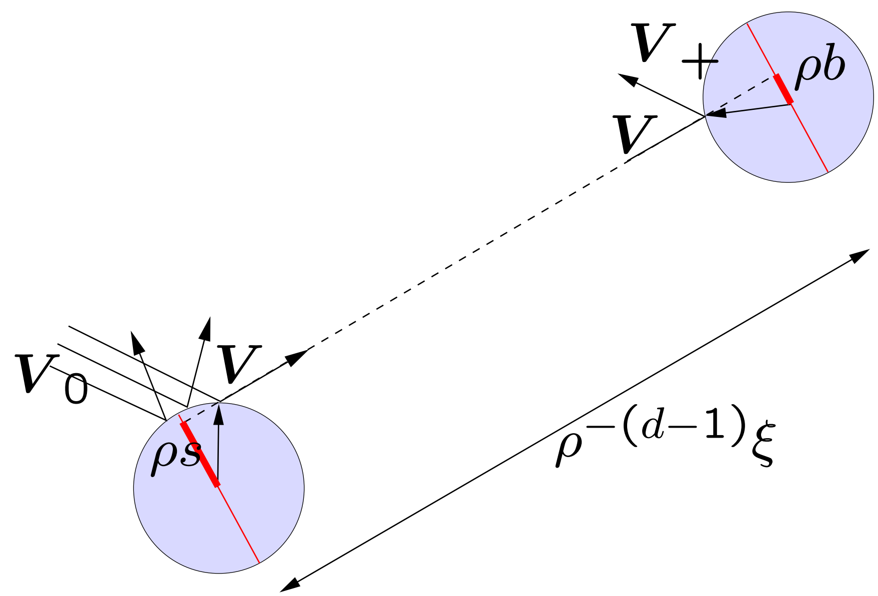
$$\left[\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{Q}} - \frac{\partial}{\partial \xi}\right] f_t(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+) = \int_{\mathrm{S}_1^{d-1}} f_t(\boldsymbol{Q}, \boldsymbol{V}_0, 0, \boldsymbol{V}) p_{\mathbf{0}}(\boldsymbol{V}_0, \boldsymbol{V}, \xi, \boldsymbol{V}_+) d\boldsymbol{V}_0$$

with a new collision kernel $p_{\mathbf{0}}(\boldsymbol{V}_0, \boldsymbol{V}, \xi, \boldsymbol{V}_+)$, given by ...

The collision kernel

$$p_{\mathbf{0}}(\boldsymbol{V}_0, \boldsymbol{V}, \xi, \boldsymbol{V}_+) = \sigma(\boldsymbol{V}, \boldsymbol{V}_+)\, \Phi_{\mathbf{0}}\Big(\xi, b(\boldsymbol{V}, \boldsymbol{V}_+), -s(\boldsymbol{V}, \boldsymbol{V}_0)\Big)$$

- $\sigma(\boldsymbol{V}, \boldsymbol{V}_+)$ the differential cross section
- $\Phi_{\mathbf{0}}\Big(\xi, b(\boldsymbol{V}, \boldsymbol{V}_+), -s(\boldsymbol{V}, \boldsymbol{V}_0)\Big)$ the transition probability to exit with parameter $s(\boldsymbol{V}, \boldsymbol{V}_0)$ and hit the next scatterer after time ξ with impact parameter $b(\boldsymbol{V}, \boldsymbol{V}_+)$



$\boldsymbol{V}_+$
ρb
$\boldsymbol{V}$
$\boldsymbol{V}_0$
$\boldsymbol{V}$
ρs
$\rho^{-(d-1)}\xi$

The function Φ_0

...yields the probability to exit a scatterer with parameter s and hit the next scatterer with impact parameter b after time ξ.

In dimension $d = 2$ (JM & Strömbergsson, Nonlinearity 2008):

$$\Phi_0(\xi, w, z) = \frac{6}{\pi^2} \Upsilon\left(1 + \frac{\xi^{-1} - \max(|w|, |z|) - 1}{|w+z|}\right)$$

$$\Upsilon(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 \le x, \end{cases}$$

cf. also Caglioti & Golse (C.R. Acad. Sci. 2008) and Ustinov (2008).

Our formulas for dimension $d > 2$ are not as explicit and substantially more involved.

The operators L^t in Theorem A can be defined by the relation

$$[L^t g](\boldsymbol{Q}, \boldsymbol{V}) := \int_0^\infty \int_{\mathsf{S}_1^{d-1}} f_t(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+) \, d\boldsymbol{V}_+ \, d\xi$$

where $f_t(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+)$ is a solution of the generalized linear Boltzmann equation subject to the initial condition

$$\lim_{t \to 0} f_t(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+) = g(\boldsymbol{Q}, \boldsymbol{V}) p(\boldsymbol{V}, \xi, \boldsymbol{V}_+)$$

with

$$p(\boldsymbol{V}, \xi, \boldsymbol{V}_+) := \int_\xi^\infty \int_{\mathsf{S}_1^{d-1}} \sigma(\boldsymbol{V}_0, \boldsymbol{V}) p_0(\boldsymbol{V}_0, \boldsymbol{V}, \xi, \boldsymbol{V}_+) \, d\boldsymbol{V}_0 \, d\xi;$$

the latter is a stationary solution of the generalized linear Boltzmann equation.

Why "a generalization" of the linear Boltzmann equation?

$$\left[\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla_{\boldsymbol{Q}} - \frac{\partial}{\partial \xi}\right] f_t(\boldsymbol{Q},\boldsymbol{V},\xi,\boldsymbol{V}_+) = \int_{\mathrm{S}_1^{d-1}} f_t(\boldsymbol{Q},\boldsymbol{V}_0,0,\boldsymbol{V})\,p_{\mathbf{0}}(\boldsymbol{V}_0,\boldsymbol{V},\xi,\boldsymbol{V}_+)\,d\boldsymbol{V}_0$$

Substituting in the above the transition density for the random (rather than periodic) scatterer configuration

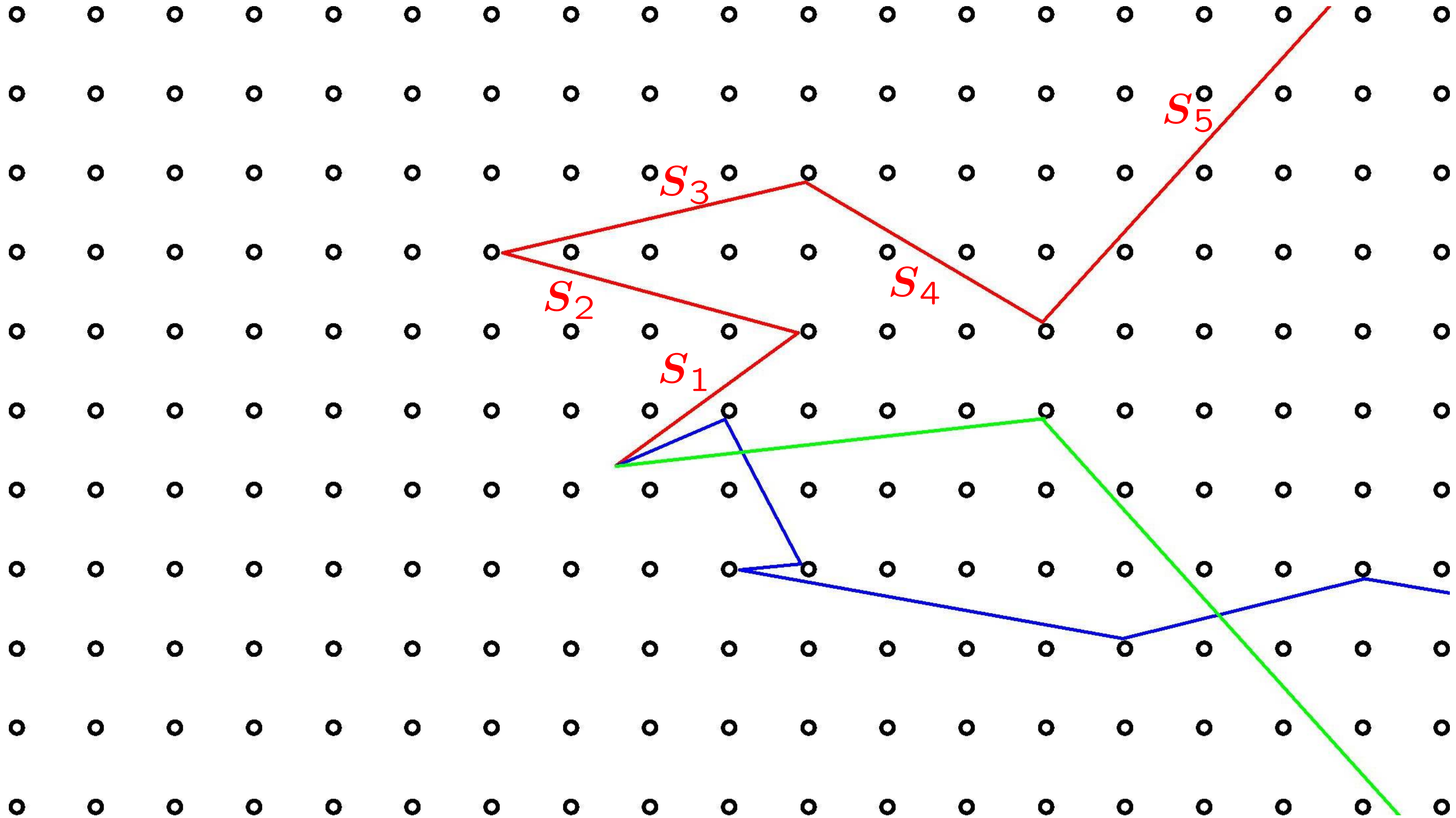
$$p_{\mathbf{0}}(\boldsymbol{V}_0,\boldsymbol{V},\xi,\boldsymbol{V}_+) = \sigma(\boldsymbol{V},\boldsymbol{V}_+)\mathrm{e}^{-\mathrm{vol}(\mathcal{B}_1^{d-1})\,\xi}$$

$$f_t(\boldsymbol{Q},\boldsymbol{V},\xi,\boldsymbol{V}_+) = g_t(\boldsymbol{Q},\boldsymbol{V})\sigma(\boldsymbol{V},\boldsymbol{V}_+)\mathrm{e}^{-\mathrm{vol}(\mathcal{B}_1^{d-1})\,\xi}$$

yields the classical linear Boltzmann equation for $g_t(\boldsymbol{Q},\boldsymbol{V})$.

The key theorem:

Joint distribution of path segments



Joint distribution for path segments

The following theorem proves the existence of a Markov process that describes the dynamics of the Lorentz gas in the Boltzmann-Grad limit.

Theorem B. Fix an a.c. Borel probability measure Λ on $\mathrm{T}^1(\mathbb{R}^d)$. Then, for each $n \in \mathbb{N}$ there exists a probability density $\Psi_{n,\Lambda}$ on $\mathbb{R}^{nd}$ such that, for any set $\mathcal{A} \subset \mathbb{R}^{nd}$ with boundary of Lebesgue measure zero,

$$\lim_{\rho\to 0} \Lambda\Big(\Big\{(\boldsymbol{Q}_0, \boldsymbol{V}_0) \in \mathrm{T}^1(\mathbb{R}^d) : (\boldsymbol{S}_1, \ldots, \boldsymbol{S}_n) \in \mathcal{A}\Big\}\Big) = \int_{\mathcal{A}} \Psi_{n,\Lambda}(\boldsymbol{S}'_1, \ldots, \boldsymbol{S}'_n)\, d\boldsymbol{S}'_1 \cdots d\boldsymbol{S}'_n,$$

and, for $n \geq 3$,

$$\Psi_{n,\Lambda}(\boldsymbol{S}_1, \ldots, \boldsymbol{S}_n) = \Psi_{2,\Lambda}(\boldsymbol{S}_1, \boldsymbol{S}_2) \prod_{j=3}^{n} \Psi(\boldsymbol{S}_{j-2}, \boldsymbol{S}_{j-1}, \boldsymbol{S}_j),$$

where Ψ is a continuous probability density independent of Λ (and the lattice).

Theorem A follows from Theorem B by standard probabilistic arguments.

First step: The distribution of free path lengths

Previous studies

- Polya (Arch Math Phys 1918): "Visibility in a forest" ($d = 2$)
- Dahlquist (Nonlinearity 1997); Boca, Cobeli, Zaharescu (CMP 2000); Caglioti, Golse (CMP 2003); Boca, Gologan, Zaharescu (CMP 2003); Boca, Zaharescu (CMP 2007): Limit distributions for the free path lengths for various sets of initial data ($d = 2$)
- Dumas, Dumas, Golse (J Stat Phys 1997): Asymptotics of mean free path lengths ($d \geq 2$)
- Bourgain, Golse, Wennberg (CMP 1998); Golse, Wennberg (CMP 2000): bounds on possible weak limits ($d \geq 2$)

See also Golse's ICM review (Madrid 2006).


Polya's forest

Lattices

- $\mathcal{L} \subset \mathbb{R}^d$—euclidean lattice of covolume one

- recall $\mathcal{L} = \mathbb{Z}^d M$ for some $M \in \mathsf{SL}(d, \mathbb{R})$, therefore the homogeneous space $X_1 = \mathsf{SL}(d, \mathbb{Z}) \backslash \mathsf{SL}(d, \mathbb{R})$ parametrizes the space of lattices of covolume one

- μ_1—right-$\mathsf{SL}(d, \mathbb{R})$ invariant prob measure on X_1 (Haar)

Affine lattices

- $\mathsf{ASL}(d,\mathbb{R}) = \mathsf{SL}(d,\mathbb{R}) \ltimes \mathbb{R}^d$—the semidirect product group with multiplication law

  $$(M,\boldsymbol{x})(M',\boldsymbol{x}') = (MM', \boldsymbol{x}M' + \boldsymbol{x}').$$

  An action of $\mathsf{ASL}(d,\mathbb{R})$ on $\mathbb{R}^d$ can be defined as

  $$\boldsymbol{y} \mapsto \boldsymbol{y}(M,\boldsymbol{x}) := \boldsymbol{y}M + \boldsymbol{x}.$$

- the space of affine lattices is then represented by $X = \mathsf{ASL}(d,\mathbb{Z})\backslash\mathsf{ASL}(d,\mathbb{R})$ where $\mathsf{ASL}(d,\mathbb{Z}) = \mathsf{SL}(d,\mathbb{Z}) \ltimes \mathbb{Z}^d$, i.e.,

  $$\mathcal{L}_{\boldsymbol{\alpha}} := (\mathbb{Z}^d + \boldsymbol{\alpha})M = \mathbb{Z}^d(1,\boldsymbol{\alpha})(M,\boldsymbol{0})$$

- μ—right-$\mathsf{ASL}(d,\mathbb{R})$ invariant prob measure on X

Let us denote by $\tau_1 = \tau(\boldsymbol{q}, \boldsymbol{v})$ the free path length corresponding to the initial condition $(\boldsymbol{q}, \boldsymbol{v})$. Recall that $\rho^{d-1}\tau_1 = \|\boldsymbol{S}_1\|$.

Theorem C. Fix a lattice $\mathcal{L}$ and the initial position $\boldsymbol{q}$. Let λ be any a.c. Borel probability measure on S_1^{d-1}. Then, for every $\xi > 0$, the limit

$$F_{\mathcal{L},\boldsymbol{q}}(\xi) := \lim_{\rho\to 0} \lambda(\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \rho^{d-1}\tau_1 \le \xi\})$$

exists, is continuous in ξ and independent of λ. Furthermore

$$F_{\mathcal{L},\boldsymbol{q}}(\xi) = \begin{cases} \mu_1(\{M \in X_1 : \mathbb{Z}^d M \cap \mathcal{Z}(\xi) \neq \emptyset\}) & \text{if } \boldsymbol{q} \in \mathcal{L} \\ \mu(\{(M, \boldsymbol{x}) \in X : (\mathbb{Z}^d M + \boldsymbol{x}) \cap \mathcal{Z}(\xi) \neq \emptyset\}) & \text{if } \boldsymbol{q} \notin \mathbb{Q}\mathcal{L}. \end{cases}$$

with the cylinder

$$\mathcal{Z}(\xi) = \Big\{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 < x_1 < \xi, \|(x_2, \ldots, x_d)\| < 1\Big\}.$$

Remarks

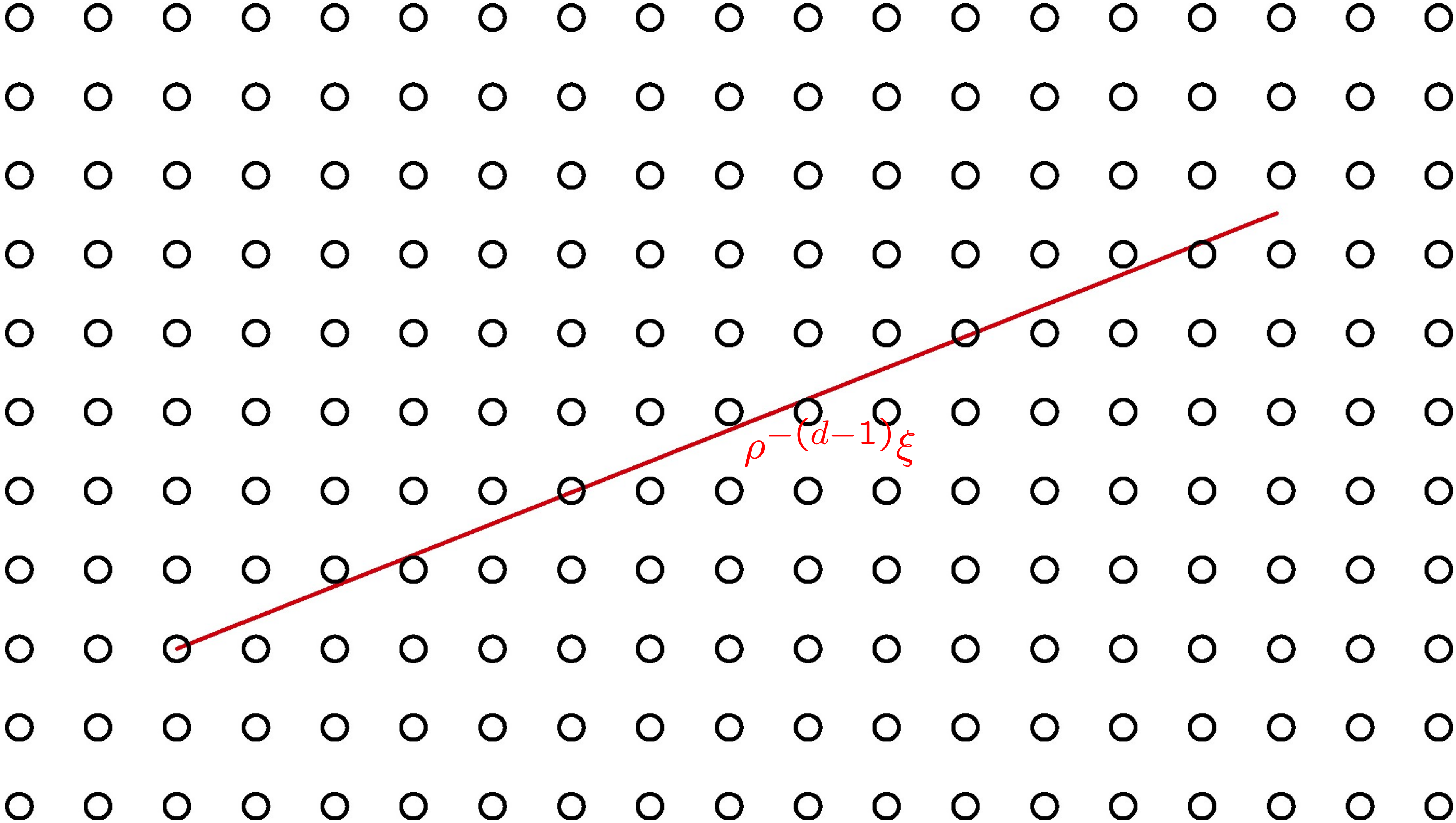
- There are similar formulas for all $\boldsymbol{q} \in \mathbb{Q}\mathcal{L}$.

- Note that in the case $\boldsymbol{q} \notin \mathbb{Q}\mathcal{L}$ the limit $F_{\mathcal{L},\boldsymbol{q}}(\xi) =: F(\xi)$ is independent of $\boldsymbol{q}$ and $\mathcal{L}$; in the case $\boldsymbol{q} \in \mathcal{L}$ the limit $F_{\mathcal{L},\boldsymbol{q}}(\xi) =: F_{\boldsymbol{0}}(\xi)$ is independent of $\mathcal{L}$.

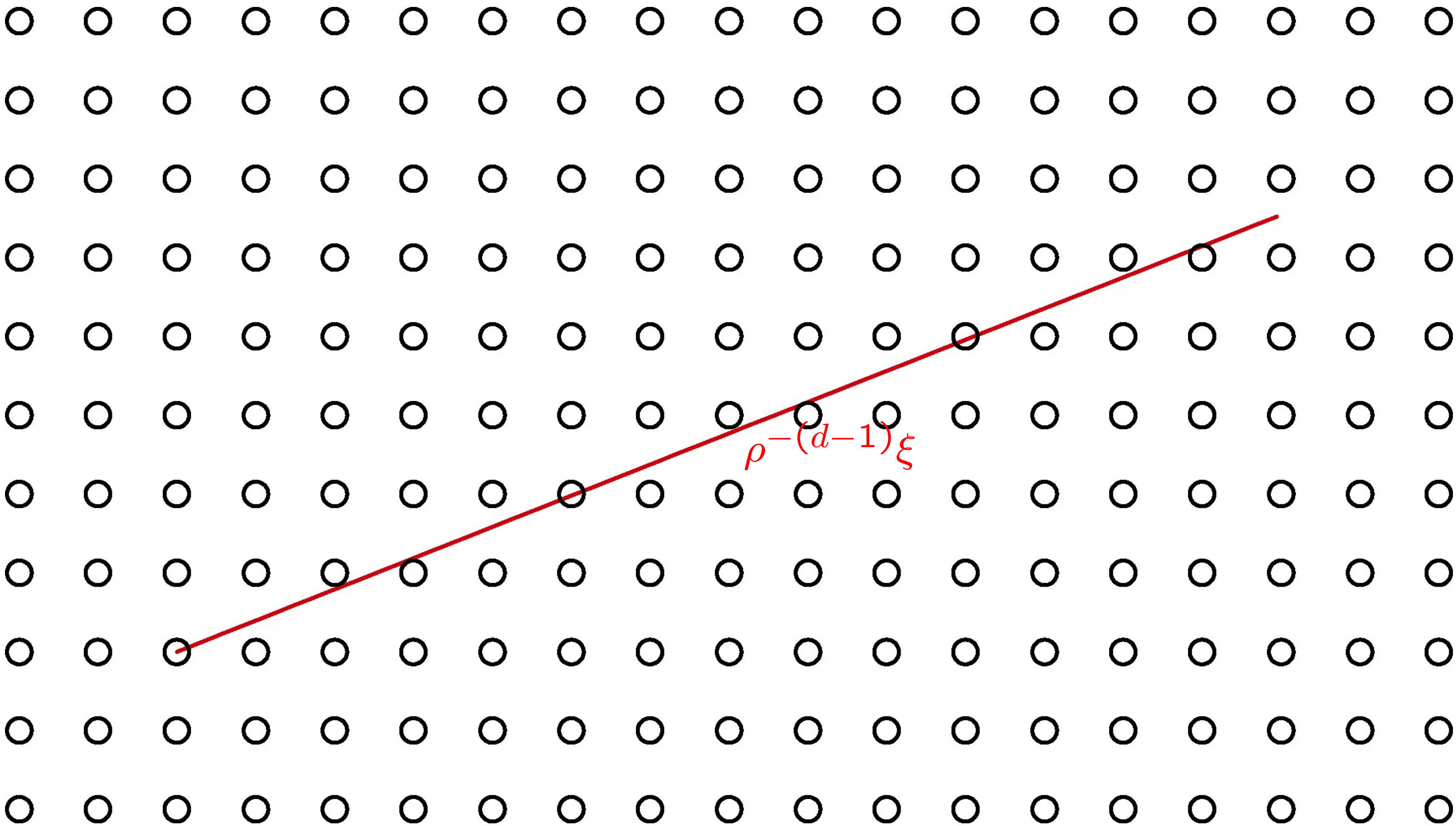
- Instead of rays emerging from the origin we can also deal with the family of rays starting at the point $\rho\boldsymbol{\beta}(\boldsymbol{v})$ in direction $\boldsymbol{v}$. This set-up is important for the joint distribution for the first n path segments in the Lorentz gas.

Outline of proof of Theorem C

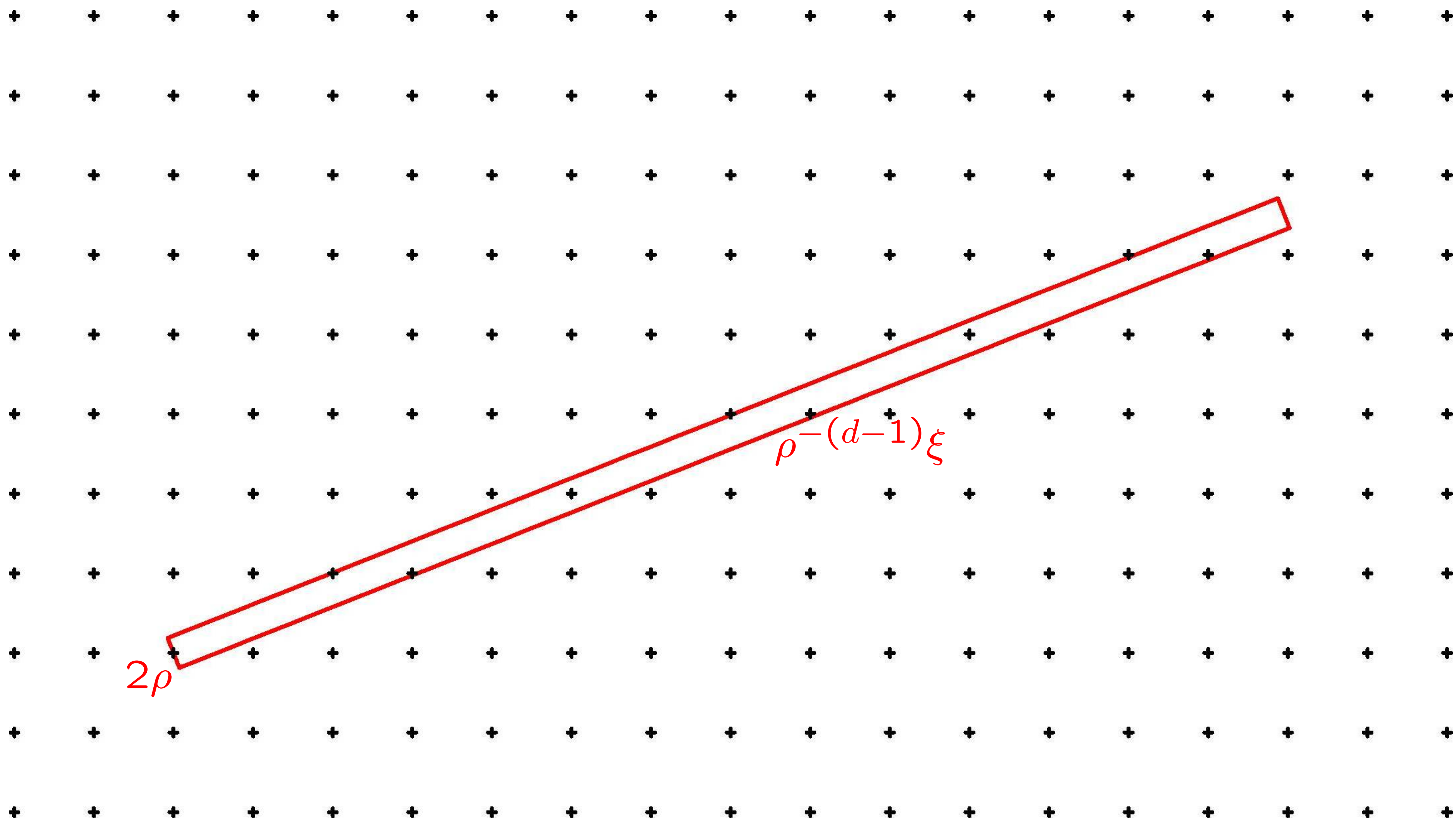
(in the case $\boldsymbol{q} \in \mathcal{L} = \mathbb{Z}^d$)



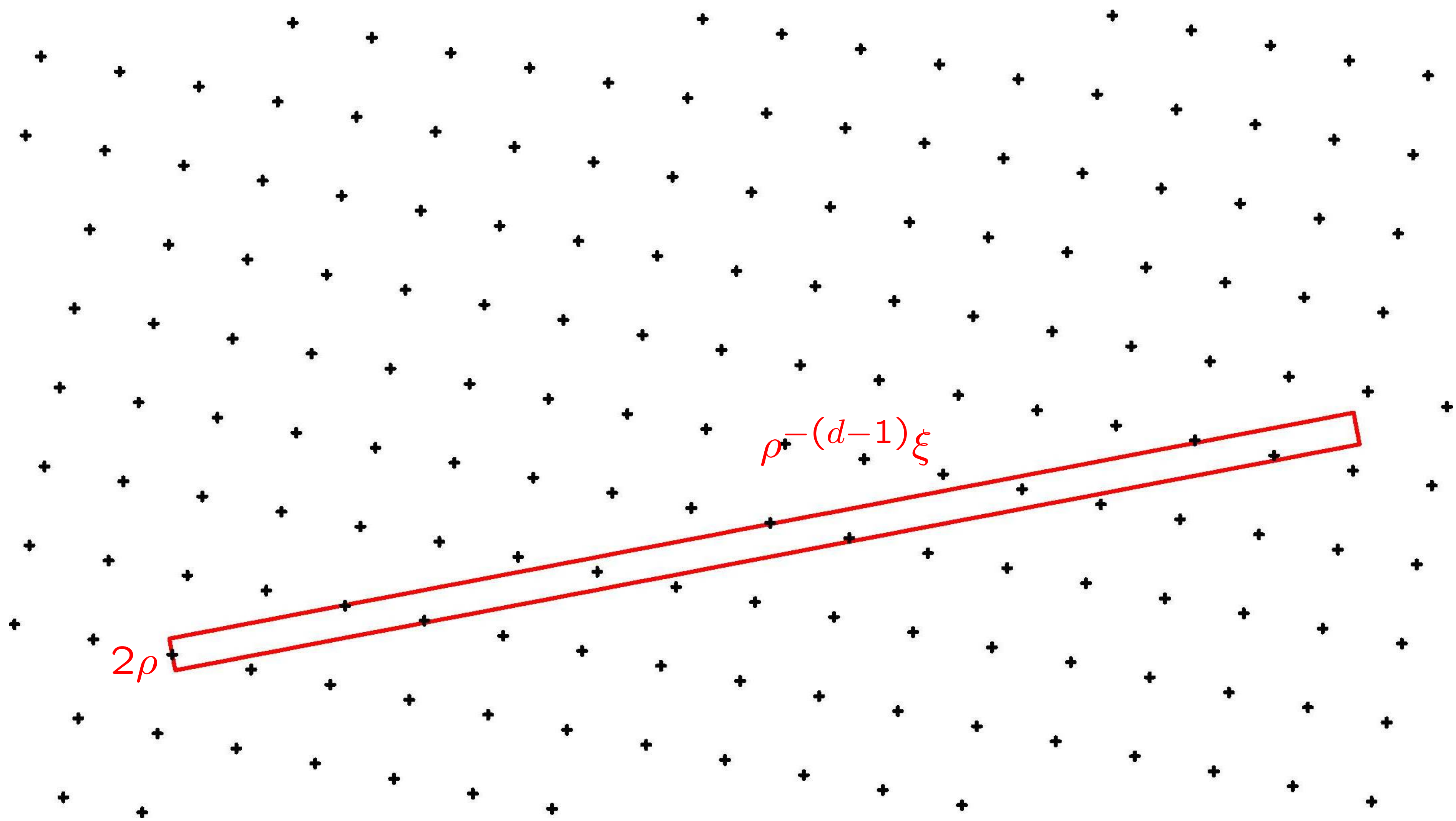
$$\lambda\Big(\Big\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \rho^{d-1}\tau_1 \leq \xi\Big\}\Big) = \ldots$$



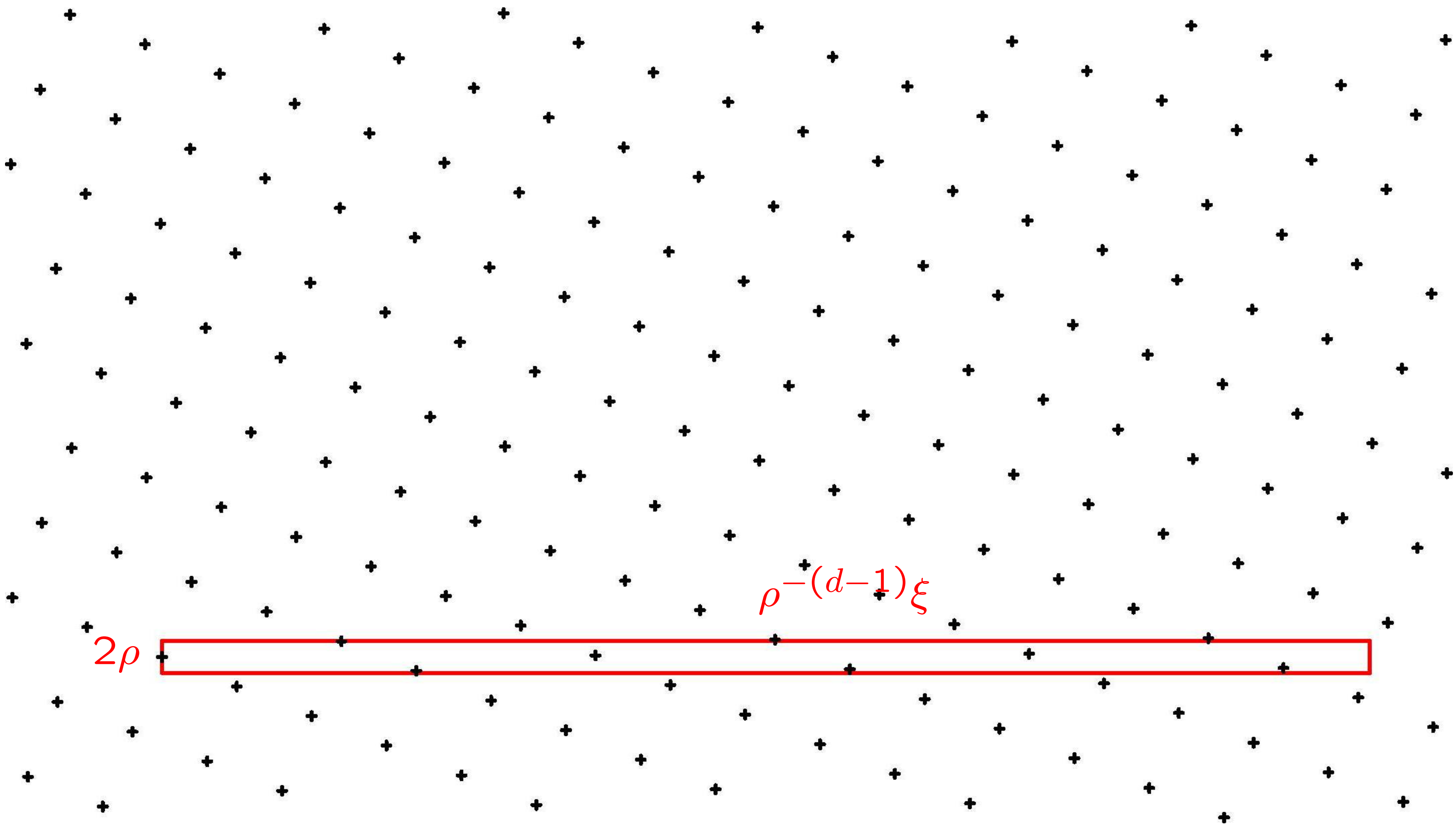
$$= \lambda\Big(\Big\{ \boldsymbol{v} \in \mathsf{S}_1^{d-1} : \text{ at least one scatterer intersects } \mathsf{ray}(\boldsymbol{v}, \rho^{-(d-1)}\xi) \Big\}\Big)$$



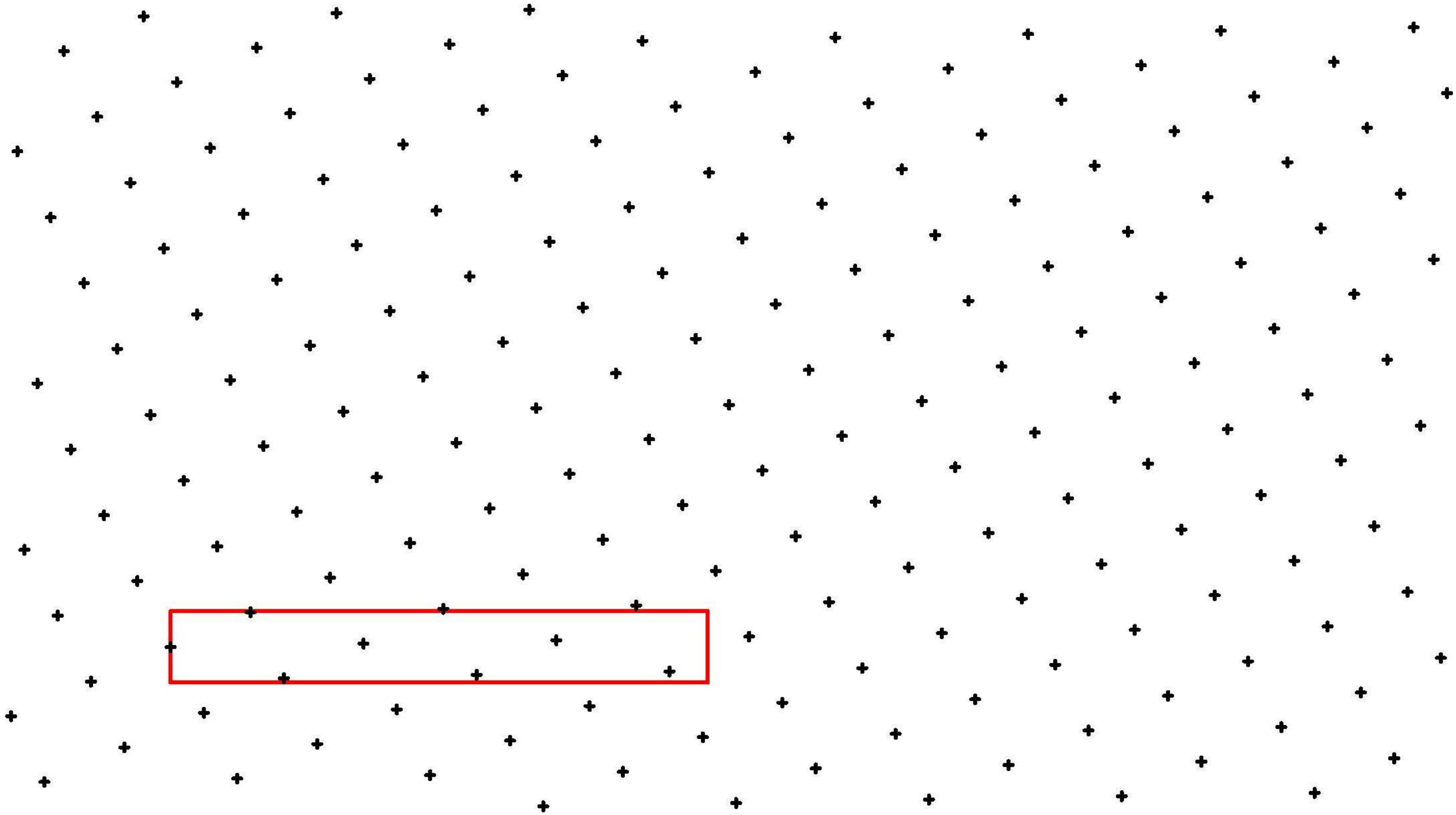
$$\approx \lambda\Big(\Big\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \mathbb{Z}^d \cap \mathcal{Z}(\boldsymbol{v}, \rho^{-(d-1)}\xi, \rho) \neq \emptyset\Big\}\Big)$$



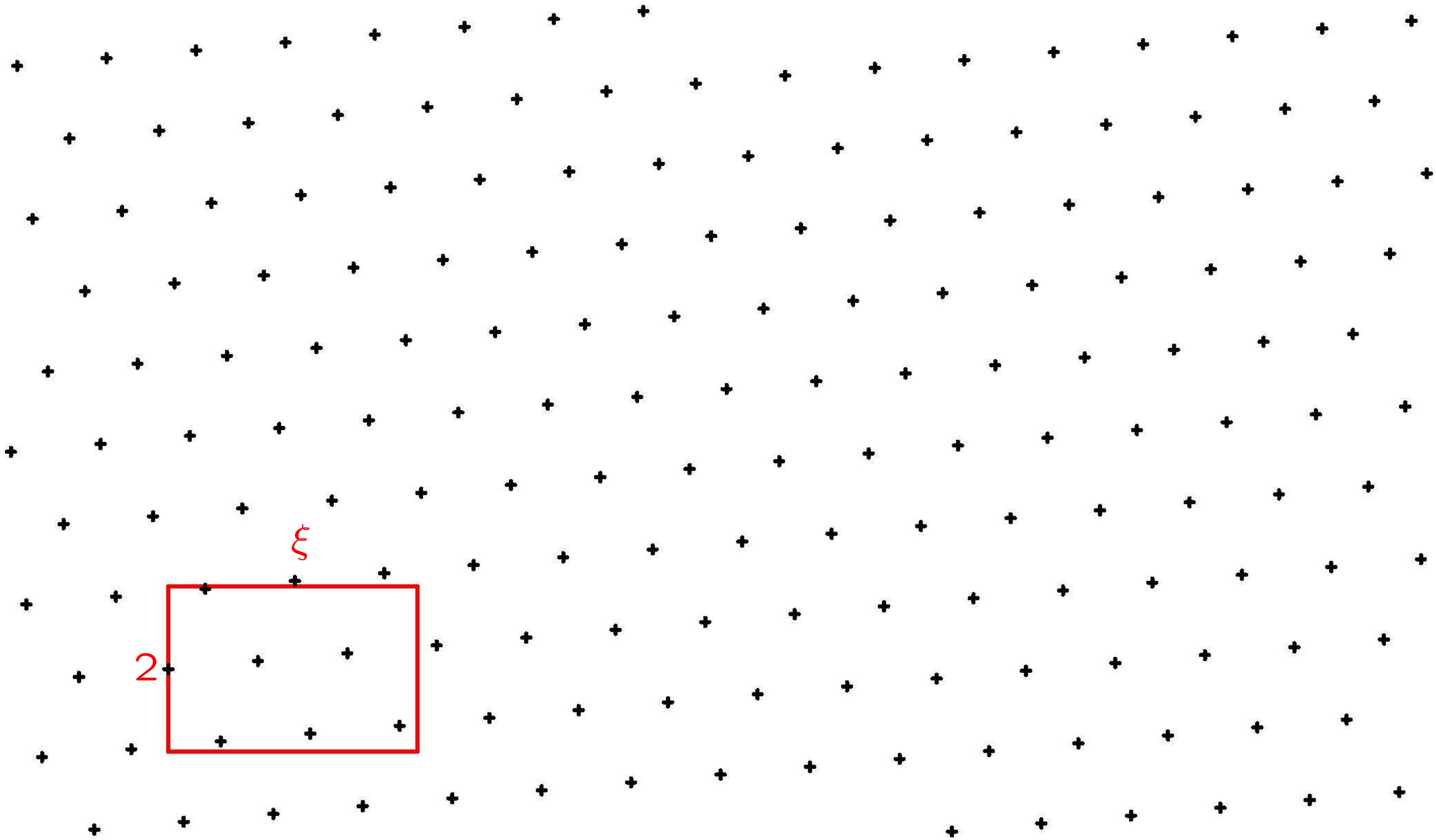
$\big($Rotate by $K(\boldsymbol{v}) \in \mathrm{SO}(d)$ such that $\boldsymbol{v} \mapsto \boldsymbol{e}_1\big)$



$$\lambda\Big(\Big\{\boldsymbol{v}\in \mathsf{S}_1^{d-1} : \mathbb{Z}^d K(\boldsymbol{v}) \cap \mathcal{Z}(\boldsymbol{e}_1, \rho^{-(d-1)}\xi, \rho) \neq \emptyset\Big\}\Big)$$



$\Big($Apply $D_\rho = \mathsf{diag}(\rho^{d-1}, \rho^{-1}, \ldots, \rho^{-1}) \in \mathsf{SL}(d, \mathbb{R})\Big)$



$$\lambda\Big(\Big\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \mathbb{Z}^d K(\boldsymbol{v}) D_\rho \cap \mathcal{Z}(\boldsymbol{e}_1, \xi, 1) \neq \emptyset\Big\}\Big)$$

The following Theorem shows that in the limit $\rho \to 0$ the lattice

$$\mathbb{Z}^d K(\boldsymbol{v}) \begin{pmatrix} \rho^{d-1} & \boldsymbol{0} \\ {}^t\boldsymbol{0} & \rho^{-1}1 \end{pmatrix}$$

behaves like a random lattice with respect to Haar measure μ_1.

Define a flow on $X_1 = \mathrm{SL}(d,\mathbb{Z})\backslash \mathrm{SL}(d,\mathbb{R})$ via right translation by

$$\Phi^t = \begin{pmatrix} \mathrm{e}^{-(d-1)t} & \boldsymbol{0} \\ {}^t\boldsymbol{0} & \mathrm{e}^t 1 \end{pmatrix}, \qquad t = \log 1/\rho.$$

Theorem D. Fix any $M_0 \in \mathrm{SL}(d,\mathbb{R})$. Let λ be an a.c. Borel probability measure on S_1^{d-1}. Then, for every bounded continuous function $f : X_1 \to \mathbb{R}$,

$$\lim_{t\to\infty} \int_{\mathrm{S}_1^{d-1}} f(M_0 K(\boldsymbol{v})\Phi^t)\, d\lambda(\boldsymbol{v}) = \int_{X_1} f(M)\, d\mu_1(M).$$

Theorem D is a direct consequence of the mixing property for the flow Φ^t.

This concludes the proof of Theorem C when $\boldsymbol{q} \in \mathcal{L} = \mathbb{Z}^d M_0$.

The generalization of Theorem D required for the full proof of Theorem C uses Ratner's classification of ergodic measures invariant under a unipotent flow. We exploit a close variant of a theorem by N. Shah (Proc. Ind. Acad. Sci. 1996) on the uniform distribution of translates of unipotent orbits.

The central argument in the proof of Theorem B (joint distribution of path segments) follows a similar route, but is significantly more involved.

Asymptotics

Asymptotics of the limiting distribution for $q \notin \mathbb{Q}\mathcal{L}$

Recall:

$$F(\xi) := \lim_{\rho \to 0} \lambda(\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \rho^{d-1}\tau_1 \le \xi\})$$
$$= \mu(\{(M, \boldsymbol{x}) \in X : (\mathbb{Z}^d M + \boldsymbol{x}) \cap \mathcal{Z}(\xi) \neq \emptyset\}).$$

$$F(\xi) = 1 - \frac{\pi^{\frac{d-1}{2}}}{2^d d\, \Gamma(\frac{d+3}{2})\, \zeta(d)}\, \xi^{-1} + O\Big(\xi^{-1-\frac{2}{d}}\Big) \qquad \text{as } \xi \to \infty$$

$$F(\xi) = \mathrm{vol}(\mathcal{B}_1^{d-1})\, \xi + O\Big(\xi^2\Big) \qquad \text{as } \xi \to 0$$

with $\mathrm{vol}(\mathcal{B}_1^{d-1}) = \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)}$.

Note: for a random scatterer configuration $F(\xi) = 1 - \mathrm{e}^{-\mathrm{vol}(\mathcal{B}_1^{d-1})\xi}$.

Asymptotics of the limiting distribution for $q \in \mathcal{L}$

Recall:

$$F_{\mathbf{0}}(\xi) := \lim_{\rho \to 0} \lambda(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} : \rho^{d-1}\tau_1 \leq \xi\})$$
$$= \mu_1(\{M \in X_1 : \mathbb{Z}^d M \cap \mathcal{Z}(\xi) \neq \emptyset\}).$$

$F_{\mathbf{0}}(\xi) = 1$	for ξ sufficiently large
$F_{\mathbf{0}}(\xi) = \dfrac{\mathrm{vol}(\mathcal{B}_1^{d-1})}{\zeta(d)}\,\xi + O(\xi^2)$	as $\xi \to 0$.

Note: for a random scatterer configuration $F_{\mathbf{0}}(\xi) = F(\xi) = 1 - \mathrm{e}^{-\mathrm{vol}(\mathcal{B}_1^{d-1})\xi}$.

$1/\zeta(d)$ is the relative density of primitive lattice points (i.e., the lattice points visible from the origin).

Conclusions

- We have seen that the dynamics of the periodic Lorentz gas converges, in the Boltzmann-Grad limit, to a random flight process that is Markov with memory two.

- The distribution of the free path lengths has polynomial tails, in stark contrast to the random scatterer configuration, where the distribution is exponential.

- The corresponding evolution equation is a generalized Boltzmann equation with a collision kernel that is independent of the choice of lattice.

- The proof exploits the dynamics on the space of (affine) lattices, and the transition probabilities of the limit process are related to natural measures on these homogeneous spaces.

Outlook

- Long-time dynamics of the limit process? Intermediate scaling limits?
- Other scatterer configurations: Random defects, quasicrystals, electron-phonon interactions?
- Long-range potentials? Electro-magnetic fields?
- Quantum analogue of the generalized linear Boltzmann equation?

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