Energy level statistics, lattice point problems and ergodic theory

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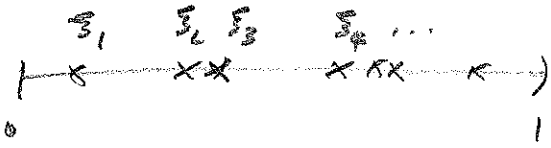
Contents

1. Randomness of point sequences

1.1. ... on the circle S^1



Let χ = characteristic fct of $[-\tfrac{1}{2}, \tfrac{1}{2}] \subset \mathbb{R}$

Then $\chi_\ell(x) = \sum_{m \in \mathbb{Z}} \chi\left(\frac{x - x_0 + m}{\ell}\right)$ is the characteristic fct. of the interval $[x_0 - \frac{\ell}{2}, x_0 + \frac{\ell}{2}] \subset S^1$.
$(\ell < 1)$

Consider the triangular array $\xi_{Nj} \in S^1$
$j = 1, 2, \ldots, N$
$N = 1, 2, \ldots \to \infty$

$$
\begin{array}{lllll}
\xi_{11} & & & & \\
\xi_{21} & \xi_{22} & & & \\
\xi_{31} & \xi_{32} & \xi_{33} & & \\
\vdots & & & \ddots & \\
\xi_{N1} & \xi_{N2} & \xi_{N3} & \cdots & \xi_{NN} \\
\downarrow & & & & \ddots
\end{array}
$$

(in the following we suppress the first index and write $\xi_j = \xi_{Nj}$).

Assume uniform distribution mod 1 holds,

i.e. $\forall\ [a,b] \qquad (b-a<1)$

$$\lim_{N\to\infty} \frac{1}{N} \#\{ j \le N : \xi_j \in [a,b] \bmod 1 \} = b-a$$

This is equivalent to the statement

$$\begin{cases} \text{For } S_N(\ell) := \sum_{j=1}^{N} \chi_\ell(\xi_j) \\ \lim_{N\to\infty} \frac{1}{N} S_N(\ell) = \ell \qquad \forall x_0 \in S^1. \end{cases}$$

The aim is to characterize the different degrees of "randomness" in the deterministic sequence ξ_j by their distribution in small intervals with random center x_0.

Convenient to measure interval lengths on the scale of mean spacing: Set

$$\ell = \frac{L}{N}$$

1.2. Variance

<u>Def.</u> $\langle \ldots \rangle := \int_0^1 \ldots \, dx_0$

Expect. value $\langle S_N(\ell) \rangle = L$.

Variance

$$\Sigma_N^2(\ell) = \langle [S_N(\ell) - L]^2 \rangle$$

$$= \langle S_N(\ell)^2 \rangle - L^2$$

$$\sum_{j,k=1}^{N} \sum_{m,n} \int_0^1 \chi\left(\frac{\xi_j - x_0 + m}{\ell}\right) \chi\left(\frac{\xi_k - x_0 + n}{\ell}\right) dx_0$$

$$= \sum_{j,k=1}^{N} \sum_{n} \int_{-\infty}^{\infty} \chi\left(\frac{\xi_j - x_0 + m}{\ell}\right) \chi\left(\frac{\xi_k - x_0}{\ell}\right) dx_0$$

$$= \sum_{j,k} \sum_{m} \ell \, \Delta\left(\frac{\xi_j - \xi_k + m}{\ell}\right)$$

$$\Delta(x) = \int_{-\infty}^{\infty} \chi(x + x_0)\, \chi(x_0)\, dx_0$$

$$= \max\{1 - |x|, 0\}$$

$j = k$ term: (Δ has compact support $\subset [-1,1]$)

$$\ell \sum_{j=1}^{N} \sum_{m} \Delta\left(\frac{m}{\ell}\right) \underset{\ell > 1}{=} L\,\Delta(0) = L$$

So

$$\boxed{\Sigma_N^2(L) \underset{\substack{\ell \to 0 \\ N \to \infty}}{=} L - L^2 + L R_N^2(\ell, \Delta)}$$

with the 2-pt correlation function

$$R_N^2(\ell, \Psi) = \frac{1}{N} \sum_{j \neq k = 1}^{N} \sum_{m \in \mathbb{Z}} \Psi\left(\frac{\xi_j - \xi_k + m}{\ell}\right)$$

Note by Poisson summation $\left[\sum_m f(m) = \sum_n \hat{f}(n)\right]$

$$R_N^2(\ell, \Psi) = \frac{L}{N^2} \sum_{j \neq k = 1}^{N} \sum_{n \in \mathbb{Z}} \hat{\Psi}\left(\frac{Ln}{N}\right) e(n(\xi_j - \xi_k))$$

$$e(x) = \exp(2\pi i x)$$

Here Ψ can be any fct with abs. conv. Fourier series. (E.g. $\Psi = \Delta$.)

1.3 Variance for IID

s.t. $A \le C B$

Suppose ξ_j are indep. random variables uniformly distributed on $[0,1] \simeq S^1$.

Then

$$\mathbb{E}\, R_N^2(\ell, f) = \frac{N(N-1)}{N^2} L \underset{\substack{\ell \to 0 \\ N \to \infty \\ L \gg 1}}{\sim} L$$

(n=0 term)

$$\mathbb{E}\left[R_N^2 - L\right]^2 = \frac{L^2}{N^4} \sum_{\substack{j \neq k \\ j' \neq k'}} \sum_{\substack{n \neq 0 \\ n' \neq 0}} \hat{f}\left(\frac{Ln}{N}\right) \overline{\hat{f}\left(\frac{Ln'}{N}\right)}$$

$$\mathbb{E}\left[e\left(n(\xi_j - \xi_k) - n'(\xi_{j'} - \xi_{k'})\right)\right]$$

$$\begin{cases} = 1 & \text{if } n = n',\ j = j',\ k = k' \\ & \text{or } n = -n',\ j = k',\ k = j' \\ = 0 & \text{otherwise} \end{cases}$$

$$= \frac{L^2}{N^4} \overbrace{\sum_{j \neq k} 1}^{O(N^2)} \underbrace{\sum_{n \neq 0} \left[\hat{f}\left(\frac{Ln}{N}\right) \overline{\hat{f}\left(+\frac{Ln}{N}\right)} + \hat{f}\left(\frac{Ln}{N}\right) \hat{f}\left(-\frac{Ln}{N}\right) \right]}_{O\left(\frac{N}{L}\right)}$$

$$\ll \frac{L}{N}$$

So $\mathbb{E}\left[R_N^2 - L\right]^2 \ll \frac{L}{N}$.

Chebyshev's ineq. implies $\forall\, \varepsilon_N$

$$\text{meas}\left\{ \xi \in \mathbb{T}^N : \left| R_N^2 - L \right| > \varepsilon_N \right\} \ll \frac{L}{\varepsilon_N^2 N}$$

If $\frac{L}{N} \to 0$ choose $\varepsilon_N \to 0$ s.t. $\frac{L}{\varepsilon_N^2 N} \to 0$

$$\Rightarrow \qquad R_N^2 - L \longrightarrow 0 \quad \text{a.s.}$$

$$\Rightarrow \qquad \frac{1}{L} \Sigma_N^2(L) \longrightarrow 1 \quad \text{a.s.}$$

a.s. = almost surely refers here to the product measure (Haar measure) $\prod_{N=1}^{\infty} \prod_{j=1}^{N} \xi_{Nj}$ on $\mathbb{T}^\infty$ (i.e. we assume in partic. ξ_{Nj} and $\xi_{N+1,j}$ are chosen independently).

It is possible (but slightly more involved) to generalize this to the case when $\xi_{Nj} = \xi_{N+1,j}$, i.e. when $\xi_{Nj} = \xi_j$ is N-independent.

1.4 Limit thms for IID

Regime I: $\boxed{L \to \infty}$ $\left(L/N \to 0 \right)$

$$\text{meas}\left\{ x_0 \in [0,1] : \frac{S_N(\ell) - L}{\sqrt{\Sigma_N^2(\ell)}} > R \right\} \xrightarrow{\text{a.s.}} \frac{1}{\sqrt{2\pi}} \int_R^\infty e^{-t^2/2}\, dt$$

$\left(\text{since } \Sigma^2 \to \infty \right)$ CLT

Regime II: $\boxed{L = \text{const}}$ (the most interesting)

$$E_N(k, L) = \text{meas}\left\{ x_0 \in [0,1] : S_N(\ell) = k \right\} \xrightarrow{\text{a.s.}} \frac{L^k}{k!} e^{-L}$$

Poisson distr

$\left(\text{since } \Sigma^2 \sim L \right)$

These limit thms are slightly non-standard. Usually one takes meas with respect to the IID ξ_j'. Here, one (generic) realization of $\xi_1, \xi_2, \dots$ is fixed.

1.5 Gap distribution

Assume now the ξ_j are ordered, $\xi_{j+1} \geq \xi_j$.
Consider the gaps between ξ_j,

$$s_j = N(\xi_{j+1} - \xi_j) \quad , \quad j = 1, \dots, N$$

where $\xi_{N+1} = \xi_1 + 1$.

We say $\{ \xi_j \}$ has a limiting gap distribution $P(s)$ if for all bounded continuous $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$

$$\int g(s) P_N(s)\, ds \xrightarrow[N\to\infty]{} \int g(s) P(s)\, ds$$

with $P_N(s) = \frac{1}{N} \sum_{j=1}^{N} \delta(s - s_j)$.

<u>Thm</u> The following statements are equivalent.

(A) $\{ \xi_j \}$ has a ~~limiting~~ continuous limiting gap distribution $P(s)$.

(B) $\operatorname{meas} \{ x_0 \in [0,1) : S_N(\ell) = 0 \} \xrightarrow[N\to\infty]{} E(0,L)$

for all <u>fixed</u> $L > 0$.

The density of $E(0,L)$ is related to $P(s)$ by

$$\frac{d^2 E(0,L)}{dL^2} = P(s) \quad , \quad L > 0. \quad (*)$$

<u>Example</u>: For IID, $E(0,L) = e^{-L}$, $P(s) = e^{-s}$

Proof "$\Longrightarrow$"

$$\operatorname{meas}\left\{ x_0 \in [0,1) : \#\left\{ j \le N : \xi_j \in \left[x_0, x_0 + \tfrac{L}{N}\right] + \mathbb{Z} \right\} = 0 \right\}$$

$$= \sum_{j=1}^{N} \operatorname{meas}\left\{ x_0 \in [\xi_j, \xi_{j+1}) : \#\{ \quad \} = 0 \right\}$$

$$= \sum_{j=1}^{N} \left(\xi_{j+1} - \xi_j - \frac{L}{N} \right) \chi_{[L,\infty)} \left(N(\xi_{j+1} - \xi_j) \right)$$

$$= \underbrace{\sum_{j=1}^{N} (\xi_{j+1} - \xi_j)}_{=1} - \sum_{j=1}^{N} (\xi_{j+1} - \xi_j) \chi_{[0,L)} \left(N(\xi_{j+1} - \xi_j) \right)$$

$$- \frac{L}{N} \sum_{j=1}^{N} \chi_{[L,\infty)} \left(N(\xi_{j+1} - \xi_j) \right)$$

$$= 1 - \frac{1}{N} \sum_{j=1}^{N} g(s_j)$$

where $g(x) = \begin{cases} x & \text{if } x \in [0, L] \\ L & \text{if } x \in [L, \infty) \end{cases}$

is bounded continuous.

By assumption

$$\xrightarrow[N\to\infty]{} 1 - \int_0^\infty g(s)\, P(s)\, ds .$$

$$= 1 - \int_0^L s\, P(s)\, ds - L \int_L^\infty P(s)\, ds$$

(by def.)

$$= E(0,L) \Longrightarrow (*) .$$

Proof "$\Leftarrow$"

Lemma

The sequence of prob measures $\{P_N(s)\,ds\}$ on $\mathbb{R}$ is tight.

Proof $K>0$

$$\int_K^\infty P_N(s)\,ds = \frac{1}{N}\#\{j \le N : s_j > K\}$$

$$\le \frac{1}{K}\frac{1}{N}\sum_{j\le N} s_j\, \chi_{[K,\infty)}(s_j)$$

$$\le \frac{1}{K}\frac{1}{N}\sum_{j\le N} s_j$$

$$= \frac{1}{K}\sum_{j\le N}(\xi_{j+1}-\xi_j)$$

$$= \frac{1}{K}$$

$\blacksquare$

Proof "$\Leftarrow$" cont'd

Since $\{ P_N(s)\, ds \}$ is tight, it's relatively compact (Helly-Prokharov Thm) and therefore every subsequence contains a convergent subsequence.

Suppose $\forall g$ bounded cont.

$$\int g(s)\, P_{N_i}(s)\, ds \underset{i\to\infty}{\Longrightarrow} \int g(s)\, P(s)\, ds$$

From the argument in "$\Rightarrow$" we find that $P(s)$ determines $E(0,L)$. (via (*))

If different subsequences have different limits this will lead to different $E(0,L)$ — a contradiction.

Hence the limit $P(s)$ of any convergent subsequence is unique and thus every subsequence must converge.

$\blacksquare$

2. $m\alpha$ mod 1 (Marel-Sinai; Bleher, JM)

use cont'd fractions

this method generalizes to higher dimensions

2.1 Set-up

Take $\xi_m = m\alpha \pmod 1$, fix L.

(The Poisson scaling limit.)

$$S_N(\ell) = \sum_{m=1}^{N} \sum_{n\in\mathbb{Z}} \chi\left(\frac{N}{L}(m\alpha + n - x_0)\right)$$

$$= \sum_{(m,n)\in\mathbb{Z}^2} \chi_{(0,1]}\left(\frac{m}{N}\right) \chi_{[-\frac{L}{2},\frac{L}{2}]}\left(N(m\alpha+n-x_0)\right)$$

$$= \sum_{(m,n)\in\mathbb{Z}^2} \Psi\left((m,n-x_0)\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{N} & 0\\ 0 & N\end{pmatrix}\right)$$

with $\Psi(x,y) = \chi_{(0,1]}(x)\,\chi_{[-\frac{L}{2},\frac{L}{2}]}(y)$

Define $G(\mathbb{R}) := SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ with multipl. law

$$(M,\xi)(M',\xi') = (MM', \xi M' + \xi')$$

This group has the matrix rep $(M,\xi) \to \begin{pmatrix} M & 0\\ \xi & 1\end{pmatrix} \in SL(3,\mathbb{R})$

Set $F(M,\xi) = \sum_{m\in\mathbb{Z}^2} \Psi(mM + \xi)$

Note, with Ψ as above, this sum is always finite. Furthermore

$$S_N(\ell) = F(M,\xi) \quad \text{for} \quad M = \begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{N} & \\ & N\end{pmatrix}$$

$$\xi = (0,-x_0)M$$

Prop 2.1 $F(\hat{\gamma} g) = F(g) \quad \forall \hat{\gamma} \in \Gamma = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$.

Proof: $\hat{\gamma} = (\gamma, n) \qquad \gamma \in SL(2,\mathbb{Z})$

$n \in \mathbb{Z}^2$

$= (\gamma, 0)(1, n)$

i) $F((1,n)(M,\xi)) = F(M, nM+\xi)$

$$= \sum_m \Psi((m+n)M + \xi)$$

$= F(M,\xi)$ ✓

ii) $F((\gamma,0)(M,\xi)) = F(\gamma M, \xi)$

$$= \sum_m \Psi(m\gamma M + \xi)$$

$$= \sum_m \Psi(mM + \xi) \quad ✓$$

since $\gamma \mathbb{Z}^2 = \mathbb{Z}^2$ $\square$

$\Rightarrow$ F lives on the homogeneous space $\Gamma \backslash G$

$G = G(\mathbb{R})$

$\Gamma = G(\mathbb{Z})$

Remark:

We may write

$$F(g) = \sum_{\gamma \in \pi(\Gamma) \backslash \Gamma} \Psi(\pi(\gamma \xi))$$

where $\pi : G \longrightarrow \mathbb{R}^2$

$$(M, \xi) \mapsto \xi$$

From this invariance under Γ is seen directly.

2.2. Geometry of $\Gamma \backslash G$

<u>Aim:</u> Find a good parametrization of $\Gamma \backslash G$

$$= SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2 \backslash SL(2,\mathbb{R}) \ltimes \mathbb{R}^2 .$$

Iwasawa decomposition for $SL(2,\mathbb{R})$:

$$M = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\phi/2 & +\sin\phi/2 \\ -\sin\phi/2 & \cos\phi/2 \end{pmatrix}$$

$$\tau := u + iv \quad \in \mathbb{H} = \{ \tau \in \mathbb{C} : \operatorname{Im}\tau > 0 \}$$

$$\phi \in [0, 4\pi)$$

This yields a 1-1 map $SL(2,\mathbb{R}) \to \mathbb{H} \times [0,4\pi)$

Action of $SL(2,\mathbb{R})$ on $\mathbb{H} \times [0,4\pi)$ becomes

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, \phi) = \left(\frac{a\tau + b}{c\tau + d} , \phi - 2\arg(c\tau + d) \right).$$

$$PSL(2,\mathbb{R}) = SL(2,\mathbb{R}) / \{\pm 1\} \cong \mathbb{H} \times [0, 2\pi)$$

can be identified with the unit tangent bundle of $\mathbb{H}$, and thus $SL(2,\mathbb{Z}) \backslash SL(2,\mathbb{R})$ with the unit tangent bundle of $SL(2,\mathbb{Z}) \backslash \mathbb{H}$

For a general $g \in G$ write

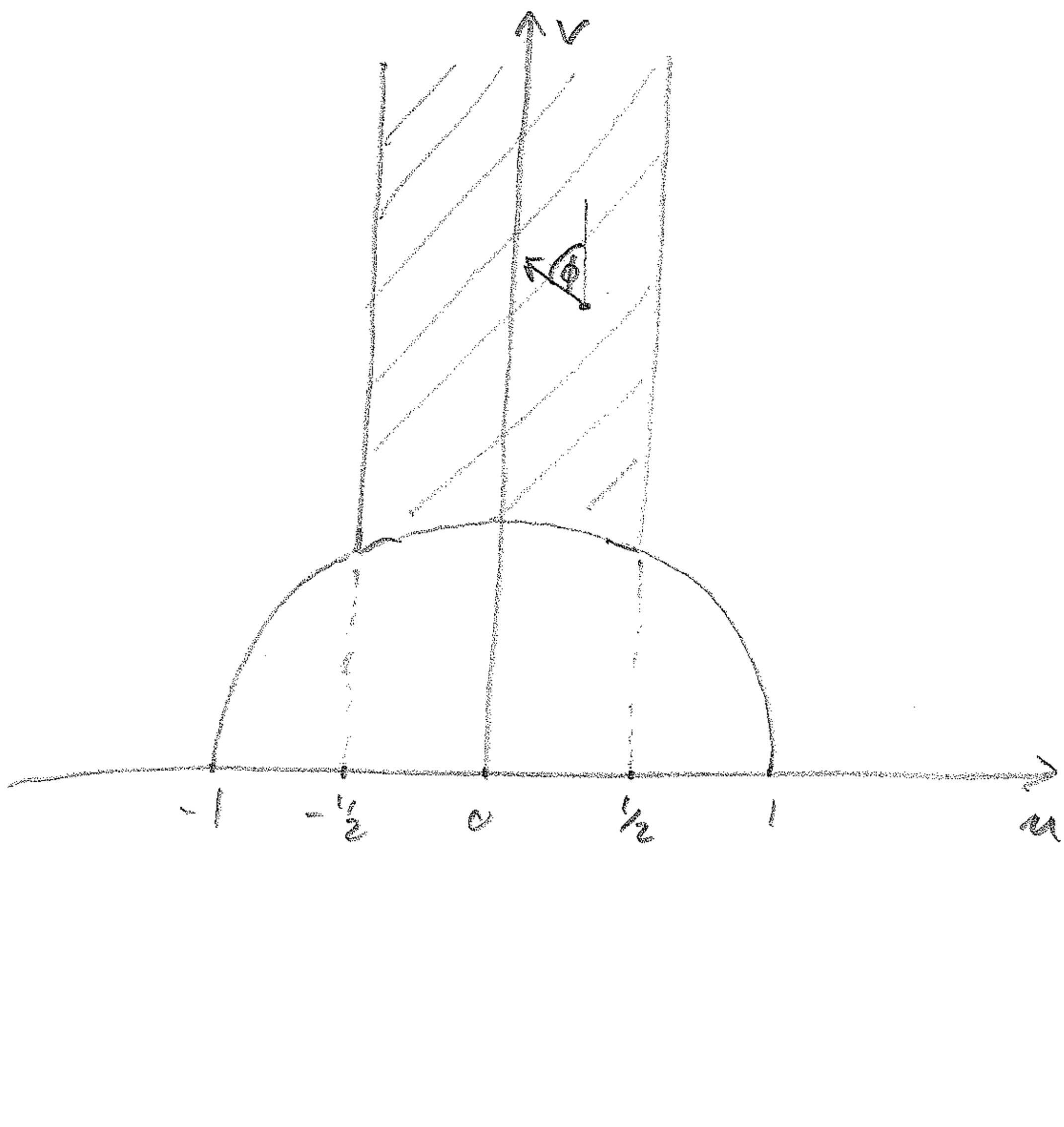
$$g = [1, \xi] [M, 0]$$

from which it is evident that ξ can be parametrized by $\mathbb{T}^2 = \mathbb{Z}^2 \backslash \mathbb{R}^2$.

We conclude that $\Gamma \backslash G$ can be parametrized

by $$\underset{\tau}{\underset{\uparrow}{\mathcal{F}}} \times \underset{\phi}{\underset{\uparrow}{[0,2\pi)}} \times \underset{\xi}{\underset{\uparrow}{[0,1)^2}}$$

$\mathcal{F}$ - fundamental region of $SL(2,\mathbb{Z})$ on $\mathbb{H}$:



2.3 Dynamics

We have

$$S_N(\ell) = F(g_0 \Phi^t)$$

where • $g_0 = \left(1, (0, -x_0)\right)\left(\left(\begin{smallmatrix}1 & \alpha \\ 0 & 1\end{smallmatrix}\right), 0\right)$

$= \left(\left(\begin{smallmatrix}1 & \alpha \\ 0 & 1\end{smallmatrix}\right), (0, -x_0)\right)$

(initial condition)

• $\Phi^t = \left(\begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2}\end{pmatrix}, 0\right)$

(flow acting on $\Gamma \backslash G$ by right translation

• $t = 2 \log N$.

We will now work out contracting / expanding directions around the orbit $M_0 \Phi^t$.

Local parametrization of $G(\mathbb{R})$:

$$g = \overbrace{\left(\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, (0, y) \right)}^{=: N(\alpha, y)} \left(\begin{pmatrix} e^{-s/2} & 0 \\ 0 & e^{s/2} \end{pmatrix}, 0 \right)$$

$$\left(\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}, (x, 0) \right)$$

Let $d(g, g')$ be a <u>left</u> invariant Riemannian metric on G (and thus on $\Gamma \backslash G$).

Choose $g := (\alpha, s, \beta, x, y)$
$g' := (\alpha', s', \beta', x', y')$
two nearby points.

Since $g \Phi^t = \Phi^t (e^t \alpha, s, e^{-t} \beta, e^{-t/2} x, e^{t/2} y)$

we have

$d(g \Phi^t, g' \Phi^t) =$

$$= d\left((e^t \alpha, s, e^{-t} \beta, e^{-t/2} x, e^{+t/2} y), (e^t \alpha', s', e^{-t} \beta', e^{-t/2} x', e^{t/2} y') \right) \quad (*)$$

So x, y — expanding

β, y — contracting

s — flow direction (neutral)

2.4 Mixing & uniform distribution

<u>Thm 2.2</u> If f is bounded continuous $\Gamma\backslash G \to \mathbb{R}$ then

$$\lim_{t\to\infty} \iint_{\mathbb{T}^2} f(N(x,y)\,\Phi^t)\,dx\,dy$$

$$= \frac{1}{\mu(\Gamma\backslash G)} \int_{\Gamma\backslash G} f\,d\mu$$

Haar measure

<u>Proof</u>:

Mixing* states that $\forall f, h \in L^2(\Gamma\backslash G)$

$$\lim_{t\to\infty} \int_{\Gamma\backslash G} f(g\,\Phi^t)\,h(g)\,d\mu(g)$$

$$\longrightarrow \frac{1}{\mu(\Gamma\backslash G)} \int f\,d\mu \int h\,d\mu$$

Choose f as above, and h the characteristic fct of the set

* proved by Moore for semisimple Lie gps. The result can be extended to $\Gamma\backslash G$ considered here.

$$S_\varepsilon = \{ (\alpha, s, \beta, x, y) : \alpha, y \in [0,1]$$
$$s, \beta, x \in [-\varepsilon, \varepsilon] \}$$

By uniform continuity of f and eq (*) on p. 2-4, given any $\delta > 0$

$$\sup_{\substack{g \in S_\varepsilon \\ t > 0}} | f(g\phi^t) - f(N(\alpha, y)\Phi^t) | < \delta$$

$\blacksquare$

The theorem also follows from Ratner's theorem, as we shall see later.

Theorem 2.2 (p. 2-5) can be extended to hold for piecewise continuous bounded functions f, by approximating f by bounded continuous $f_\pm$, s.t.

$$f_- \le f \le f_+$$

$$\int (f_+ - f_-)\, d\mu < \varepsilon .$$

Such $f_\pm$ can be found $\forall\, \varepsilon > 0$.

<u>Thm 2.3</u> Fix $l > 0$.

$$\operatorname{meas}\left((\varphi, x_0) \in \mathbb{T}^2 : \ S_N(l) = k \right) \xrightarrow[N \to \infty]{} \frac{1}{\mu(\Gamma \backslash G)}\, \mu\left(g : \ F(g) = k \right).$$

<u>Proof</u> Apply Thm 2.2 (p. 2-5) to the characteristic function of the set $F(g) = k$.

$\blacksquare$

Remark 1 There is no such limit for α fixed, x_0 random.

However, Ratner's theorem can be applied to show that for α random x_0 fixed a limit exist. This limit is the same for all $x_0 \notin \mathbb{Q}$ and coincides with the r.h.s of the above theorem.

Remark 2

Sequences we believe behave like iid with respect to the above statistical measures are

- $n^d \alpha$ mod 1 (under diophantine conditions on α) cf. results by Sinai, Rudnick & Sarnak & Zaharescu
- $2^n \alpha$ mod 1, established for almost all α by Rudnick & Zaharescu.

3. $\sqrt{4n}$ mod 1 (Elkies & McMullen)

The sequence $\sqrt{4n}$ $(n=1,2,3,\dots)$ is uniformly distributed mod 1.

As in Sect. 2 we keep L fixed (the scaling limit where one expects Poisson for iid).

$$S_N(\ell) = \sum_{n=1}^{N} \sum_{m\in\mathbb{Z}} \chi\left(\frac{N}{L}\left(\sqrt{4n} - x_0 + m\right)\right)$$

$$\left(x_0 - m - \frac{L}{2N}\right)^2 \le 4n \le \left(x_0 - m + \frac{L}{2N}\right)^2$$

$$-\frac{L}{N}(x_0 - m) \le 4n - (x_0-m)^2 - \left(\frac{L}{2N}\right)^2 \le \frac{L}{N}(x_0-m)$$

also

$$\left|\sqrt{4n} - x_0 + m\right| \le \frac{L}{2N}$$

ignore these for simplicity*

$$S_N(\ell) = \sum_{m,n\in\mathbb{Z}} \chi_{(0,1]}\left(\frac{x_0 - m \pm \frac{L}{2N}}{\sqrt{4N}}\right) \times \chi\left(\frac{\sqrt{N}\left[4n - (x_0-m)^2 - \left(\frac{L}{2N}\right)^2\right]}{\frac{1}{\sqrt{N}} L (x_0 - m)}\right)$$

---

*These terms can be handled by standard approximation arguments.

Simplest case: $q = 1$ [Subst. $(m,n) \to (-m,-n)$]

$$S_N(\ell) = \sum_{m,n\in\mathbb{Z}} \chi_{[0,1]}\left(\frac{x_0+m}{\sqrt{N}}\right) \chi_{[-\frac{1}{2},\frac{1}{2}]}\left(\frac{\sqrt{N}\,(n+x_0^2+2x_0 m)}{\frac{1}{\sqrt{N}}(x_0+m)}\right)$$

(we have substituted $n \to n+m^2$)

Consider the subgroup $\{N_1(x) : x\in\mathbb{R}\}$, with

$$N_1(x) = \left[\begin{pmatrix}1 & 2x\\ 0 & 1\end{pmatrix}, (x, x^2)\right]$$

$$N_2(y) = \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, (0, y)\right]$$

Note $N_1(x)\,N_2(y) = N_2(y)\,N_1(x)$

$$N(x,y) = N_1(x)\,N_2(y)$$

Take $g = N_1(x)\,\Phi^t \qquad x = x_0,\ t = \log N$

$$= \left[\begin{pmatrix}1 & 2x\\ 0 & 1\end{pmatrix}, (x, x^2)\right]\left[\begin{pmatrix}\frac{1}{\sqrt{N}} & 0\\ 0 & \sqrt{N}\end{pmatrix}, 0\right]$$

$$= \Bigg[\underbrace{\begin{pmatrix}\frac{1}{\sqrt{N}} & 2x\sqrt{N}\\ 0 & \sqrt{N}\end{pmatrix}}_{=M}, \underbrace{\left(\frac{x}{\sqrt{N}}, \sqrt{N}\,x^2\right)}_{=\xi}\Bigg]$$

$$(m,n)\,M + \xi = \left(\frac{m+x}{\sqrt{N}}, \sqrt{N}\,(2mx+n+x^2)\right)$$

Hence $S_N(\ell) = \sum_{m,n} \Psi((m,n)M+\xi)$, cf. Sec 2,

with $\Psi(x,y) = \chi_{[0,1]}(x)\,\chi_{[-\frac{1}{2},\frac{1}{2}]}\left(\frac{y}{x}\right)$.

$$F(M, \xi) = \sum_{m,n} \Psi((m,n) M + \xi)$$

has the same properties as those discussed in Sec 2. (Note: Ψ has compact support.)

Instead of Thm 2.2 we now apply the following equidistribution theorem which is a consequence of Ratner's theorem (cf. Shah, Elkies & McMullen)

<u>Thm 3.1</u> $\forall f$ bounded continuous $\Gamma\backslash G \to \mathbb{R}$

$$\lim_{t\to\infty} \int_0^1 F(N_1(x_0)\,\Phi^t)\,dx_0 = \frac{1}{\mu(\Gamma\backslash G)} \int_{\Gamma\backslash G} F\,d\mu$$

From this we conclude (as in Sec 3.2, approximating piecewise constant fcts by bounded continuous functions)

<u>Thm 3.2</u> Fix $L > 0$.

$$\operatorname{meas}(x_0 \in [0,1] : S_N(\ell) = k) \xrightarrow[N\to\infty]{} \frac{1}{\mu(\Gamma\backslash G)}\, \mu(g : F(g) = k)$$

The above analysis can be extended to $\tau \in \mathbb{Q}$, if Γ has to be replaced by a suitable congruence subgroup.

4 Energy level statistics

Δ – Laplace Beltrami op on compact Riemannian manifold $\mathcal{M}$

Spectrum

$$\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots$$

Weyl's law $(\lambda \to \infty)$

$$\#\{ j : \lambda_j < \lambda \} = \sigma_d \, \lambda^{d/2} + O(\lambda^{d/2 - 1})$$

Rescaling $X_j = \sigma_d \, \lambda_j^{d/2}$, $\sigma_d = \dfrac{\mathrm{vol}(B_d)\,\mathrm{vol}(\mathcal{M})}{(2\pi)^d}$

yields a sequence of mean density 1:

$$N(X) := \#\{ j : X_j < X \} \sim X \,, \quad X \to \infty.$$

Study statistical properties of the array (cf. Sec 1) $\xi_{Nj} \in [0,1)$ def. by

$$\xi_{Nj} = \frac{1}{X} X_j \,, \quad N = N(X)$$

<u>Remark 1</u> One could also (and in fact this is more standard) simply view the X_j as random variables on $\mathbb{R}$ and ask for the distribution of

$$N(X, L) = N(X + \tfrac{L}{2}) - N(X - \tfrac{L}{2})$$

(the number of X_j in the interval $X + [-\frac{L}{2}, \frac{L}{2}]$)

with X random in $[T, 2T]$ (say)

$T \to \infty$.

<u>Remark 2</u> (Semiclassics)

Let $E_j(\hbar)$ be the eigenvalues of the Hamiltonian $H(\hbar)$ in the interval

$$[E, E + c\hbar]$$

classical energy (fixed)

(this choice is natural from a semiclassics point of view) In the case discussed above

$$H(\hbar) = -\hbar^2 \Delta$$

$$E_j(\hbar) = \hbar^2 \lambda_j \qquad E = \hbar^2 \lambda$$

$$E_j(\hbar) \in [E, E + c\hbar) \iff \lambda_j \in [\lambda, \lambda + \underset{\substack{\| \\ 1 \text{ for simplicity}}}{\tfrac{c}{\sqrt{E}}} \sqrt{\lambda})$$

Hence (again from a semiclassical point of view) it is natural to set

$$\xi_{Nj} = \frac{\lambda_j - \lambda}{\sqrt{\lambda}}$$

where $N = N(\lambda) = \#\{j : \lambda_j \in [\lambda, \lambda + \sqrt{\lambda}]\}$

Note that for the above example

$$\#\{j : \xi_{Nj} \in [a,b]\} = \#\{j : \lambda_j \in \lambda + [a\sqrt{\lambda}, b\sqrt{\lambda}]\}$$

$$= \sigma_d \, \lambda^{d/2}\left[\left(1+\frac{b}{\sqrt{\lambda}}\right)^{d/2} - \left(1+\frac{a}{\sqrt{\lambda}}\right)^{d/2}\right] + O(\lambda^{\frac{d-1}{2}})$$

$$= \sigma_d \, \lambda^{\frac{d-1}{2}} (b-a) + O(\lambda^{\frac{d}{2}-1})$$

$$\sim (b-a) \, N(\lambda)$$

hence ξ_{Nj} are uniformly distributed in $[0,1)$, as assumed in Sec. 1.

<u>Ex</u> The eigenvalues of a 2dim harmonic oscillators are $E_{m,n} = \hbar(m\omega + n\omega')$ Set w.l.o.g $\omega' = 1$. Then the statistics of the $E_{m,n} \in [E, E+\hbar]$ are those discussed in Sec. 2.

In the following we will consider the Hamiltonian

$$H = (i\partial_x - \alpha)^2 + (i\partial_y - \beta)^2$$

acting on functions on $\mathbb{T}^2 \simeq [0, 2\pi)^2$

Eigenfunctions : $\varphi_{m,n}(x,y) = e^{i(mx+ny)}$

Eigenvalues : $\lambda_{m,n} = (m-\alpha)^2 + (n-\beta)^2$

where $m, n \in \mathbb{Z}$.

The question of eigenvalues in random intervals translates now into a question of lattice points in (thin) annuli.

5. Lattice points in shifted circles

5.1 Poisson summation formula

Thm 5.1 For $f \in S(\mathbb{R}^d)$

$$\sum_{m \in \mathbb{Z}^d} f(m) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n)$$

Proof: Calculate Fourier coeff. of the period. fct. $\varphi(\alpha) = \sum_m f(m+\alpha)$:

$$\hat{\varphi}_n := \int_0^1 \varphi(\alpha)\, e(-n\alpha)\, d\alpha$$

$$= \int_0^1 \sum_m f(m+\alpha)\, e(-n\alpha)\, d\alpha$$

$$= \sum_m \int_m^{m+1} f(\alpha)\, e(-n\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} f(\alpha)\, e(-n\alpha)\, d\alpha$$

$$=: \hat{f}(n)$$

$$\varphi(0) = \sum_n \hat{\varphi}_n \Rightarrow \text{Thm.} \qquad \blacksquare$$

5.2 Hardy-Voronoi formula

<u>Def</u> $\lambda_{m,n} = (m-\alpha)^2 + (n-\beta)^2$.

<u>Thm 5.2</u> For $h \in C_0^\infty(\mathbb{R})$

J_0 - Bessel fct

$$\sum_{m,n} h(\lambda_{mn}) = \pi \sum_{n=0}^{\infty} r_{\alpha,\beta}(n) \int_0^\infty h(x) J_0(2\pi\sqrt{nx})\,dx$$

$$r_{\alpha,\beta}(n) = \sum_{\substack{k^2+l^2=n \\ k,l\in\mathbb{Z}}} e(k\alpha + l\beta)$$

<u>Proof</u> Apply Poisson summation with

$$f(x,y) = h\left((x-\alpha)^2 + (y-\beta)^2\right)$$

$\blacksquare$

Ignoring convergence issues, put $h = \chi_{[0,\lambda]}$

$$N(\lambda) = \pi\lambda + \sqrt{\lambda}\sum_{n=1}^{\infty} \frac{r_{\alpha,\beta}(n)}{\sqrt{n}} J_1(2\pi\sqrt{n\lambda})$$

J_1 Bessel fct

5.3 Asymptotics

$$J_1(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{3\pi}{4}\right) \qquad (z \to \infty)$$

$$N(\lambda) \sim \pi\lambda + \frac{\lambda^{1/4}}{\pi} \sum_{n=1}^{\infty} \frac{r_{\alpha,\beta}(n)}{n^{3/4}} \cos\left(2\pi\sqrt{n\lambda} - \frac{3\pi}{4}\right)$$

$$(\lambda \to \infty)$$

5.4 Almost periodic functions (cf. Heath-Brown, Bleher)

Let
$$\varphi(t) = \sum_{\substack{r=1 \\ r \text{ sq-free}}}^{\infty} \frac{1}{r^{3/4}} \varphi_r(\sqrt{r}\, t)$$

with
$$\varphi_r(\xi) = \sum_{k=1}^{\infty} \frac{r_{\alpha\beta}(k^2 r)}{k^{3/2}} \cos\left(2\pi k \xi - \frac{3\pi}{4}\right)$$

Note that

$$\frac{N(\lambda) - \pi\lambda}{\lambda^{1/4}} = \frac{1}{\pi} \varphi(\sqrt{\lambda})$$

$\varphi(t)$ is almost periodic in the sense that given $\varepsilon > 0$ $\exists$ N such that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T |\varphi(t) - \varphi_N(t)|^2 < \varepsilon \qquad (*)$$

where
$$\varphi_N(t) = \sum_{\substack{r=1 \\ \text{sq-free}}}^{N} \frac{1}{r^{3/4}} \varphi_r(\sqrt{r}\,t).$$

and φ_r are continuous fcts on $\mathbb{T}$.

$(*)$ implies that the value distribution of φ and φ_N, for t unif distributed in $[0,T]$ $(T\to\infty)$ are arbitrarily close for N suff large.

5.5. Value distribution of $\varphi_N(t)$.

<u>Lemma 5.3</u> For any bounded continuous $g: \mathbb{R} \to \mathbb{R}$

$$\frac{1}{T}\int_0^T g(\varphi_N(t))\,dt \xrightarrow[T\to\infty]{} \int_{\mathbb{T}^N} g\Big(\sum_{r=1}^{N} \frac{1}{r^{3/4}} \varphi_r(\xi_r)\Big) d\xi_1 \cdots d\xi_N$$

<u>Proof</u> Use Weyl's theorem on equidistribution of straight lines with irrational slope on $\mathbb{T}^N$. Note $\{\sqrt{r} : r \text{ sq. free}\}$ are linearly independent over $\mathbb{Q}$.

Using (*) one concludes (cf. Heath-Brown, Bleher)

<u>Thm 5.4</u>

The value distribution of $\varphi(t)$, $t \in [0,T]$ uniformly* distributed, is, as $T \to \infty$, given by the random variable

$$\sum_{\substack{r=1 \\ \text{sq. free}}}^{\infty} \frac{1}{r^{3/4}} \varphi_r(\xi_r)$$

$\xi_1, \xi_2, \ldots$ iid on $\mathbb{T}$!

<u>Remark</u>

The above statements are much less understood in dimension > 2.

* may be replaced by other suitable distributions

6. ... thin annuli

Consider the number of lattice points of an annulus with

minor radius $\quad t - \delta$

outer $\quad$ " $\quad t + \delta$

$$\frac{\delta}{t} \to 0 .$$

$$\frac{N((t+\delta)^2) - N((t-\delta)^2) - 4\pi t \delta}{t^{1/2}}$$

$$\sim \frac{1}{\pi} \varphi(t+\delta) - \frac{1}{\pi}\varphi(t-\delta)$$

$$= \frac{1}{\pi} \sum_{\substack{r=1 \\ \text{sq. free}}}^{\infty} \frac{1}{r^{3/4}} \quad \tilde{\varphi}_r(\sqrt{r}\, t, \sqrt{r}\, \delta)$$

$$\tilde{\varphi}_r(\xi, \eta) = -2 \sum_{k=1}^{\infty} \frac{r_{Q,2}(k^2 r)}{k^{3/2}} \sin\left(2\pi k \xi - \frac{3\pi}{4}\right) \sin(2\pi k \eta)$$

In view of Sec. 4 we are interested in the quantity

$$N(x,L) = N(x+\tfrac{L}{2}) - N(x-\tfrac{L}{2})$$

(ignore factors of π arising from the rescaling in the following).

$$\begin{aligned} X + \tfrac{L}{2} &= (t+\delta)^2 \\ X - \tfrac{L}{2} &= (t-\delta)^2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} X &= t^2+\delta^2 \sim t^2 \\ L &= 4t\delta \end{aligned}$$

<u>Regime I</u> $\quad \frac{L}{\sqrt{X}} \to \infty \quad \left(\Leftrightarrow \quad \delta = \delta(t) \to \infty\right)$

Limiting distribution of $\dfrac{N(X,L)}{X^{1/4}}$

is given by

$$\frac{1}{\pi} \sum_{\substack{r=1 \\ \text{sq. free}}}^{\infty} \frac{1}{r^{3/4}} \tilde{\varphi}_r(\xi_r, \eta_r)$$

with $(\xi_1,\eta_1),(\xi_2,\eta_2),\dots$ iid on $\mathbb{T}^2$ (distr. with respect to Haar measure)

<u>Proof</u>: Similar to Thm 5.4, use equidistribution thm on torus $\mathbb{T}^{2N}$.

Regime II $\quad \frac{L}{\sqrt{X}} \to 4\delta \quad (\delta \text{ fixed})$

.... given by

$$\frac{1}{\pi} \sum_{\substack{r=1 \\ \text{sq. free}}}^{\infty} \frac{1}{r^{3/4}} \tilde{f}_r(\xi_r, \sqrt{r}\,\delta)$$

Regime III $\quad \frac{L}{\sqrt{X}} \to 0 \,, \; L \to \infty$

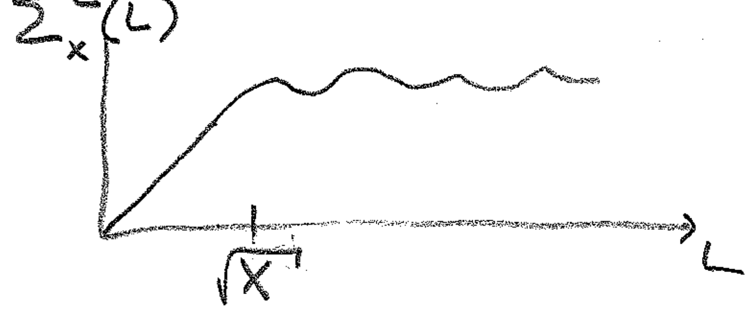
$(\Leftrightarrow \; \delta \to 0 \,, \; t\delta \to \infty)$

Expect (cf. Section 1) a CLT

for $\quad \dfrac{N(X,L)}{L^{1/2}}$

under diophantine condition on (α, β).

(Notice the different normalization in regime I; the variance "saturates" in the transition regime II:



Regime IV

Expect Poisson distribution of $N(X,L)$, under diophantine conditions on (α, β).

Theta functions (cf. hand-out)

Theta functions provide an alternative approach to the problem of lattice points in thin annuli.

Consider the pair correlation function

$$R_2(X) = \frac{1}{X}\sum_{\substack{x_i \neq x_j \\ \leq X}} \psi(x_i - x_j)$$

$\psi \in C_0^\infty(\mathbb{R})$ (for simplicity)

$$= \frac{1}{X}\sum_{x_i, x_j \leq X} \psi(x_i - x_j) \; \underbrace{+ \frac{1}{X}\sum_{x_i \leq X} \psi(0)}_{\xrightarrow{X \to \infty}\ \psi(0)}$$

Now

$$\frac{1}{X}\sum_{x_i, x_j \leq X} \psi(x_i - x_j) = \int \left| \frac{1}{\sqrt{X}} \sum_{x_j \leq X} e^{2\pi i x_j u} \right|^2 \hat{\psi}(u)\,du$$

with

$$\hat{\psi}(u) = \int \psi(x)\, e^{2\pi i x u}\, dx$$

In the case when

$$\{X_j\} = \{ \|m-\alpha\|^2 : m \in \mathbb{Z}^2 \} \qquad \alpha \in \mathbb{R}^2 \text{ fixed}$$

the exponential sum squared

$$\left| \frac{1}{\sqrt{X}} \sum_{X_j \leq X} e^{2\pi i X_j u} \right|^2$$

can be identified with a $|$theta function$|^2$

on $\Gamma \backslash G$ $\qquad G = SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$

$\qquad\qquad\qquad \Gamma = \text{"} SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2 \text{"}$

cf. Θ hand-out.

u parametrizes a unipotent orbit

which is expanding as $X \to \infty$.

7. Ratner's theorem

G — Lie group

Γ — lattice in G *

$\mathcal{U} = \{u^t\}$

one-parameter unipotent subgroup

7.1 Ratner's theorems

Thm 1 (Ratner's orbit closure thm)

For every $g \in G$ $\exists$ connected closed subgp $H \subset G$ such that

- $\mathcal{U} \subset H$
- $\Gamma g H \subset \Gamma \backslash G$ is closed and has finite right-H-invariant volume (i.e. $g^{-1}\Gamma g \cap H$ is a lattice in H)
- $\Gamma g \mathcal{U}$ is dense in $\Gamma g H$

* For Thm 2 it is only necessary to assume Γ is discrete.

<u>Thm 2</u> (Ratner's measure classification)

For every ergodic U-invariant probability measure ν $\exists$ $g \in G$ and a closed connected subgroup H such that

- ν is right-H-invariant
- ν is supported on $\Gamma g H$

(this determines ν uniquely)

<u>Remark 1</u>

A linear group (i.e. matrix group) is unipotent if all eigenvalues are 1.

<u>Remark 2</u>

Thm 2 gives a complete classification of ergodic U-invariant probability measures. ν is in fact the normalized Haar measure on $\Gamma_H \backslash H$, $\Gamma_H = g^{-1}\Gamma g \cap H$.

7.2 Application to uniform distribution

Aim understand limit of averages along unipotent orbits:

bounded continuous $\Gamma\backslash G \to \mathbb{R}$

$$\frac{1}{T_i}\int_0^{T_i} f(g_i u^t)\, dt \xrightarrow[i\to\infty]{} ?$$

$T_i \to \infty$, Γg_i seq. of points in $\Gamma\backslash G$

<u>Strategy</u>

- Define prob measures ν_i by
$$\nu_i(f) = \frac{1}{T_i}\int_0^{T_i} f(g_i u^t)\, dt .$$
- Show that the sequence $\{\nu_i\}$ is tight (trivial if $\Gamma\backslash G$ is compact).
Then (Helly-Prokharov Thm.) $\{\nu_i\}$ is relatively compact, i.e. every subsequence contains a convergent subsequence, s.t. $\nu_{i_k} \xrightarrow[w]{} \tilde{\nu}$.

- It is easy to see (since $T_i \to \infty$ the unipotent trajectory is expanding) that $\tilde{\nu}$ must be U-invariant.
- Identify $\tilde{\nu}$ by finding the appropriate subgroup $H \subset G$. Note: $\tilde{\nu}$ is in general not ergodic; hence a decomposition into ergodic components is necessary

$$\tilde{\nu} = \int \nu \, dP(\nu)$$

where ν is ergodic U-invariant.

<u>Remark 3</u>

In the application to quadratic form problems an additional difficulty is that the relevant test functions f are <u>un</u>bounded continuous.

8. Further Reading

Sections 1-3:

J. Marklof, Energy level statistics, lattice point problems and almost modular functions www.maths.bris.ac.uk/~majm

" The n-point correlations between values of linear forms, ETDS 20 (2000) 1127

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N. Elkies & C. McMullen, Gaps in $\sqrt{n}$ mod 1 and ergodic theory, Duke Math J 123 (2004) 95-139

Sections 4-7

P.M. Bleher, Trace formula for quantum integrable systems, lattice point problem, and small divisors, IMA Vol 109, Springer 1999, pp. 1-38

J. Marklof, Pair correlation densities of inhomog. quadratic forms, Annals of Math 158 (2003) 419-471

D.W. Morris, Ratner's Theorems on unipotent flows, Chicago lectures in Maths series (2005).