QUANTUM MAPS*

JENS MARKLOF

Contents

1. Set-up

Let $\mathcal{M}$ be a d-dimensional compact smooth manifold, and μ a probability measure on $\mathcal{M}$ which is absolutely continuous with respect to Lebesgue measure. We consider invertible C^∞ maps $\Phi : \mathcal{M} \to \mathcal{M}$ which preserve μ, and assume that the fixed points of each iterate Φ^n form a set of measure zero. This fixed point set is furthermore closed, since Φ is continuous, and therefore, due to the compactness of $\mathcal{M}$, it has Minkowski content zero (cf. appendix).

Let $M_N(\mathbb{C})$ be the space of $N \times N$ matrices with complex coefficients. For a given infinite subset (*index set*) $\mathcal{I} \subset \mathbb{N}$, we say two sequences of matrices,

$$\boldsymbol{A} := \{A_N\}_{N\in\mathcal{I}}, \qquad \boldsymbol{B} := \{B_N\}_{N\in\mathcal{I}}, \tag{1.1}$$

are *semiclassically equivalent*, if

$$\|A_N - B_N\| \to 0 \tag{1.2}$$

as $N \in \mathcal{I}$ tends to infinity, where $\|\cdot\|$ denotes the usual operator norm

$$\|A\| := \sup_{\psi\in\mathbb{C}^N-\{0\}} \frac{\|A\psi\|}{\|\psi\|}. \tag{1.3}$$

We denote this equivalence relation by

$$\boldsymbol{A} \sim \boldsymbol{B}. \tag{1.4}$$

Lemma 1.1. *If $\boldsymbol{A} \sim \boldsymbol{B}$ then $\operatorname{Tr} A_N = \operatorname{Tr} B_N + o(N)$.*

Proof. We have

$$\frac{1}{N}|\operatorname{Tr} A_N - \operatorname{Tr} B_N| \le \|A_N - B_N\| \to 0. \tag{1.5}$$

□

Let us define the product of two matrix sequences by $\boldsymbol{AB} = \{A_N B_N\}_{N\in\mathcal{I}}$, the inverse of $\boldsymbol{A}$ by $\boldsymbol{A}^{-1} = \{A_N^{-1}\}_{N\in\mathcal{I}}$, and its hermitian conjugate by $\boldsymbol{A}^\dagger = \{A_N^\dagger\}_{N\in\mathcal{I}}$.

Axiom 1.1 (The correspondence principle for quantum observables)**.** Fix a measure μ as above. For some index set $\mathcal{I} \subset \mathbb{N}$, there is a sequence $\mathbf{Op} := \{\mathrm{Op}_N\}_{N\in\mathcal{I}}$ of linear maps,

$$\mathrm{Op}_N : C^\infty(\mathcal{M}) \to M_N(\mathbb{C}), \qquad a \mapsto \mathrm{Op}_N(a),$$

so that

(a) for all $a \in C^\infty(\mathcal{M})$,

$$\mathbf{Op}(\overline{a}) \sim \mathbf{Op}(a)^\dagger;$$

(b) for all $a_1, a_2 \in C^\infty(\mathcal{M})$,

$$\mathbf{Op}(a_1)\mathbf{Op}(a_2) \sim \mathbf{Op}(a_1 a_2);$$

(c) for all $a \in C^\infty(\mathcal{M})$,

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} \mathrm{Op}_N(a) = \int_{\mathcal{M}} a \, d\mu.$$

Note that (b), (c) imply for any $m \in \mathbb{N}$

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} \mathrm{Op}_N(1)^m = 1, \tag{1.6}$$

and hence a density-1-subsequence of eigenvalues of $\mathrm{Op}_N(1)$ are close to one. Most quantization schemes of course satisfy $\mathrm{Op}_N(1) = 1_N$.

In standard quantization recipes (such as the one discussed in Section 5.1) one in addition has the property that

$$\mathrm{Op}_N(a_1)\,\mathrm{Op}_N(a_2) - \mathrm{Op}_N(a_2)\,\mathrm{Op}_N(a_1) \sim \frac{1}{2\pi \mathrm{i} N}\, \mathrm{Op}_N(\{a_1, a_2\}) \tag{1.7}$$

where $\{\ ,\ \}$ is the Poisson bracket. This assumption is however not necessary for any of the results proved in this paper. The axioms (a)–(c) in fact apply to examples without quantum mechanical significance. One interesting case arises in the discretization of maps, where one can choose observables with the property

$$\mathbf{Op}(a_1)\mathbf{Op}(a_2) = \mathbf{Op}(a_1 a_2) = \mathbf{Op}(a_2)\mathbf{Op}(a_1). \tag{1.8}$$

Axiom 1.2 (The correspondence principle for quantum maps)**.** There is a sequence of unitary matrices $\boldsymbol{U}(\Phi) := \{U_N(\Phi)\}_{N\in\mathcal{I}}$ such that for any $a \in C^\infty(\mathcal{M})$ we have

$$\boldsymbol{U}(\Phi)^{-1}\mathbf{Op}(a)\boldsymbol{U}(\Phi) \sim \mathbf{Op}(a \circ \Phi).$$

2. Trace asymptotics and Weyl's law

The following proposition is the key tool to understand the distribution of eigenvalues of $U_N(\Phi)$.

Proposition 2.1 (Trace asymptotics)**.** *For $n \neq 0$*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} U_N(\Phi)^n = 0. \tag{2.1}$$

Proof. Given any $\epsilon > 0$, we can find an integer R and a partition of unity on $\mathcal{M}$ by mollified characteristic functions (see appendix),

$$1 = \widetilde{\chi}_{\mathrm{fix}}(\xi) + \sum_{r=1}^{R} \widetilde{\chi}_r(\xi) \quad \forall \xi \in \mathcal{M} \tag{2.2}$$

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*This hand-out is an adaptation of the paper [J. Marklof and S. O'Keefe, Weyl's law and quantum ergodicity for maps with divided phase space, with an appendix by S. Zelditch, *Nonlinearity* **18** (2005) 277-304] to smooth maps. [Note that in Axiom 2.2 of this paper 'continuity of Φ' should be replaced by 'continuity of Φ^{-1}'.]

For more background and recent developments see M. Degli Esposti and S. Graffi (Eds.), *The mathematical aspects of quantum maps* (Springer, Berlin, 2003).

where the support of $\widetilde{\chi}_{\rm fix}$ contains a small open neighbourhood of the fixed points of Φ^n, and $\int \widetilde{\chi}_{\rm fix} d\mu < \epsilon$. The support of $\widetilde{\chi}_r$, with $r = 1, \ldots, R$, is chosen small enough, so that $\operatorname{supp} \widetilde{\chi}_r \cap \Phi^n(\operatorname{supp} \widetilde{\chi}_r) = \emptyset$ for all $\xi \in \mathcal{M}$. This is possible since (by continuity of Φ^n) there is a sufficiently small radius $\eta = \eta(\epsilon)$ such that for all balls $\mathcal{B}_\eta \subset \mathcal{K}$ we have $\mathcal{B}_\eta \cap \Phi^n(\mathcal{B}_\eta) = \emptyset$.

By the linearity of Op, we have

$$\operatorname{Tr} U_N(\Phi)^n = \operatorname{Tr}[U_N(\Phi)^n \operatorname{Op}_N(\widetilde{\chi}_{\rm fix})] + \sum_{r=1}^{R} \operatorname{Tr}[U_N(\Phi)^n \operatorname{Op}_N(\widetilde{\chi}_r)] + o(N). \tag{2.3}$$

We begin with the first term on the right hand side:

$$\operatorname{Tr}[U_N(\Phi)^n \operatorname{Op}_N(\widetilde{\chi}_{\rm fix})] = \operatorname{Tr}[U_N(\Phi)^n \operatorname{Op}_N^{\rm sym}(\widetilde{\chi}_{\rm fix})] + o_\epsilon(N), \tag{2.4}$$

with the symmetrized $\operatorname{Op}_N^{\rm sym}(\widetilde{\chi}_{\rm fix})$ as defined in (B.1). Suppose ψ_j and $\mu_j \geq 0$ are the (normalized) eigenstates and eigenvalues of $\operatorname{Op}_N^{\rm sym}(\widetilde{\chi}_{\rm fix})$. Then

$$|\operatorname{Tr}[U_N(\Phi)^n \operatorname{Op}_N^{\rm sym}(\widetilde{\chi}_{\rm fix})]| = \left| \sum_{j=1}^{N} \mu_j \langle \psi_j, U_N(\Phi)^n \psi_j \rangle \right| \tag{2.5}$$

$$\leq \sum_{j=1}^{N} \mu_j \tag{2.6}$$

$$= \operatorname{Tr} \operatorname{Op}_N^{\rm sym}(\widetilde{\chi}_{\rm fix}) \tag{2.7}$$

$$= \operatorname{Tr} \operatorname{Op}_N(\widetilde{\chi}_{\rm fix}) + o_\epsilon(N) \tag{2.8}$$

$$= N O(\epsilon) + o_\epsilon(N). \tag{2.9}$$

For the last term in the sum (2.3) we have

$$\boldsymbol{U}(\Phi)^n \mathbf{Op}(\widetilde{\chi}_r) \sim \boldsymbol{U}(\Phi)^n \mathbf{Op}(\widetilde{\chi}_r^{1/2}) \mathbf{Op}(\widetilde{\chi}_r^{1/2}) \sim \mathbf{Op}(\widetilde{\chi}_r^{1/2} \circ \Phi^{-n}) \boldsymbol{U}(\Phi)^n \mathbf{Op}(\widetilde{\chi}_r^{1/2}) \tag{2.10}$$

so

$$\operatorname{Tr}[U_N(\Phi)^n \operatorname{Op}_N(\widetilde{\chi}_r)] \tag{2.11}$$

$$= \operatorname{Tr}[\operatorname{Op}_N(\widetilde{\chi}_r^{1/2} \circ \Phi^{-n}) U_N(\Phi)^n \operatorname{Op}_N(\widetilde{\chi}_r^{1/2})] + o_\epsilon(N) \tag{2.12}$$

$$= \operatorname{Tr}[\operatorname{Op}_N(\widetilde{\chi}_r^{1/2}) \operatorname{Op}_N(\widetilde{\chi}_r^{1/2} \circ \Phi^{-n}) U_N(\Phi)^n] + o_\epsilon(N) \tag{2.13}$$

$$= \operatorname{Tr}[\operatorname{Op}_N(\widetilde{\chi}_r^{1/2} \cdot \widetilde{\chi}_r^{1/2} \circ \Phi^{-n}) U_N(\Phi)^n] + o_\epsilon(N) \tag{2.14}$$

$$= o_\epsilon(N) \tag{2.15}$$

since $\widetilde{\chi}_r^{1/2} \cdot \widetilde{\chi}_r^{1/2} \circ \Phi^{-n} = 0$ by assumption. Therefore

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} U_N(\Phi)^n = O(\epsilon), \tag{2.16}$$

which holds for every arbitrarily small $\epsilon > 0$. This concludes the proof. □

Theorem 2.2 (Weyl's law)**.** *For any $a \in \mathrm{C}^\infty(\mathcal{M})$ and for every continuous function $h : \mathbb{S}^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{C}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h(\theta_j) = \int_{\mathbb{S}^1} h(\theta) d\theta. \tag{2.17}$$

Proof. Let us first assume that the test function h has only finitely many non-zero Fourier coefficients, i.e.,

$$h(\theta) = \sum_{n\in\mathbb{Z}} \widehat{h}(n) e(n\theta) \tag{2.18}$$

is a finite sum. We then have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h(\theta_j) = \lim_{N\to\infty} \frac{1}{N} \sum_{n\in\mathbb{Z}} \widehat{h}(n) \operatorname{Tr} U_N(\Phi)^n = \widehat{h}(0) = \int h(\theta)\, d\theta. \tag{2.19}$$

We now extend this result to test functions $h \in \mathrm{C}^1(\mathbb{S}^1)$. Let

$$h_K(\theta) = \sum_{\substack{n\in\mathbb{Z} \\ |n|\leq K}} \widehat{h}(n) e(n\theta) \tag{2.20}$$

be the truncated Fourier series. Since $h \in \mathrm{C}^1(\mathbb{S}^1)$, its Fourier series converges absolutely and uniformly and hence, for any $\epsilon > 0$, there is a K such that $h_K(\theta) - \epsilon \leq h(\theta) \leq h_K(\theta) + \epsilon$ for all $\theta \in \mathbb{S}^1$. By (2.19), the limits of the left and right hand side of

$$\frac{1}{N} \sum_{j=1}^{N} h_K(\theta_j) - \epsilon \leq \frac{1}{N} \sum_{j=1}^{N} h(\theta_j) \leq \frac{1}{N} \sum_{j=1}^{N} h_K(\theta_j) + \epsilon \tag{2.21}$$

exist and differ by less than 2ϵ, hence (2.19) holds also for the current h. The extension of (2.19) to h in $\mathrm{C}(\mathbb{S}^1)$ is achieved by the same argument, i.e., by approximating h pointwise by functions $h_\epsilon \in \mathrm{C}^1(\mathbb{S}^1)$ so that $h_\epsilon(\theta) - \epsilon \leq h(\theta) \leq h_\epsilon(\theta) + \epsilon$. □

3. Generalized Weyl's law

Proposition 3.1 (Generalized trace asymptotics)**.** *For every $a \in \mathrm{C}^\infty(\mathcal{M})$ and $n \neq 0$,*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}[\operatorname{Op}_N(a) U_N(\Phi)^n] = 0. \tag{3.1}$$

Proof. By linearity of the relation (3.1) we may assume without loss of generality that a is real and $\min_\xi a(\xi) \geq 0$. This implies that $a^{1/2} \in \mathrm{C}^\infty(\mathcal{M})$. Analogously to the proof of Proposition 2.1, we have

$$\operatorname{Tr}[\operatorname{Op}_N(a) U_N(\Phi)^n] = \operatorname{Tr}[U_N(\Phi)^n \operatorname{Op}_N(\widetilde{\chi}_{\rm fix} \cdot a)] + \sum_{r=1}^{R} \operatorname{Tr}[U_N(\Phi)^n \operatorname{Op}_N(\widetilde{\chi}_r \cdot a)]. \tag{3.2}$$

The proof is concluded in the same way as the proof of Proposition 2.1, with all mollified characteristic functions $\widetilde{\chi}$ replaced by $\widetilde{\chi} \cdot a$. □

Theorem 3.2 (Generalized Weyl's law)**.** *Let $\varphi_j \in \mathbb{C}^N$ ($j = 1, \ldots, N$) be an orthonormal basis of eigenstates of $U_N(\Phi)$, with corresponding eigenphases $\theta_j \in \mathbb{S}^1$. Then, for every $a \in \mathrm{C}^\infty(\mathcal{M})$ and every continuous function $h : \mathbb{S}^1 \to \mathbb{C}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h(\theta_j) \langle \operatorname{Op}_N(a) \varphi_j, \varphi_j \rangle = \int_{\mathcal{M}} a\, d\mu \int_{\mathbb{S}^1} h(\theta)\, d\theta. \tag{3.3}$$

Proof. We may assume again without loss of generality that a is real and $\min_\xi a(\xi) \geq 0$. In view of Proposition 3.1 and the proof of Theorem 2.2 we have for every h_K with finite Fourier expansion (as in (2.18))

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} h_K(\theta_j) \langle \operatorname{Op}_N(a) \varphi_j, \varphi_j \rangle = \int_{\mathcal{M}} a\, d\mu \int_0^1 h_K(\theta)\, d\theta. \tag{3.4}$$

For any $h \geq 0$ we have

$$\left|\sum_{j=1}^{N} h(\theta_j)\langle \operatorname{Op}_N(a)\varphi_j,\varphi_j\rangle - \sum_{j=1}^{N} h(\theta_j)\|\operatorname{Op}_N(a^{1/2})\varphi_j\|^2\right| \tag{3.5}$$
$$\leq \sup h \left|\operatorname{Tr}[\operatorname{Op}_N(a) - \operatorname{Op}_N(a^{1/2})\operatorname{Op}_N(a^{1/2})^\dagger]\right| = o(N)\sup h.$$

Hence (3.4) is equivalent to

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N} h_K(\theta_j)\|\operatorname{Op}_N(a^{1/2})\varphi_j\|^2 = \int_{\mathcal{M}} a\,d\mu \int_0^1 h_K(\theta)\,d\theta. \tag{3.6}$$

We now use the same approximation argument as in the proof of Theorem 2.2, for $h \in \mathrm{C}^1(\mathbb{S}^1)$. Given any $\epsilon > 0$, there is a K such that $h_K(\theta) - \epsilon \leq h(\theta) \leq h_K(\theta) + \epsilon$ for all $\theta \in \mathbb{S}^1$. The limits of the left and right hand side of

$$\frac{1}{N}\sum_{j=1}^{N}[h_K(\theta_j) - \epsilon]\|\operatorname{Op}_N(a^{1/2})\varphi_j\|^2 \tag{3.7}$$
$$\leq \frac{1}{N}\sum_{j=1}^{N} h(\theta_j)\|\operatorname{Op}_N(a^{1/2})\varphi_j\|^2$$
$$\leq \frac{1}{N}\sum_{j=1}^{N}[h_K(\theta_j) + \epsilon]\|\operatorname{Op}_N(a^{1/2})\varphi_j\|^2$$

differ by less than

$$2\epsilon \sup h_K \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\|\operatorname{Op}_N(a^{1/2})\varphi_j\|^2 \tag{3.8}$$
$$\leq 2\epsilon \sup h_K \lim_{N\to\infty}\frac{1}{N}\operatorname{Tr}[\operatorname{Op}_N(a^{1/2})\operatorname{Op}_N(a^{1/2})^\dagger] \tag{3.9}$$
$$= 2\epsilon \sup h_K \int_{\mathcal{M}} a\,d\mu \tag{3.10}$$

which can be arbitrarily small for $\epsilon \to 0$. Thus

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N} h(\theta_j)\|\operatorname{Op}_N(a^{1/2})\varphi_j\|^2 = \int_{\mathcal{M}} a\,d\mu \int_0^1 h(\theta)\,d\theta. \tag{3.11}$$

A similar approximation argument shows that (3.11) holds also for all continuous h. In view of (3.5), the relation (3.11) is equivalent to (3.3). The assumption $h \geq 0$ can be removed by using the linearity of (3.3) in h. □

4. Quantum ergodicity

Theorem 4.1. *Suppose Φ acts ergodically on $\mathcal{M}$. Let $\varphi_1, \dots, \varphi_N \in \mathbb{C}^N$ be an orthonormal basis of eigenstates of $U_N(\phi)$. Then, for any $a \in \mathrm{C}^\infty(\mathcal{M})$,*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j\in J_N}\left|\langle \operatorname{Op}_N(a)\varphi_j,\varphi_j\rangle - \int_{\mathcal{M}} a\,d\mu\right|^2 = 0. \tag{4.1}$$

Proof. We may assume without loss of generality that $\int_{\mathcal{M}} a\,d\mu = 0$ and $|a| \leq 1$. It is then sufficient to show

$$S_2(a,N) := \frac{1}{N}\sum_{j=1}^{N}|\langle \operatorname{Op}_N(a)\varphi_j,\varphi_j\rangle|^2 \to 0 \tag{4.2}$$

as $N \to \infty$.

Define the ergodic average of a by

$$a^T := \frac{1}{T}\sum_{n=0}^{T-1} a \circ \Phi^n. \tag{4.3}$$

Since φ_j are the eigenfunctions of $U_N(\Phi)$ we have

$$S_2(a,N) = \frac{1}{N}\sum_{j=1}^{N}\left|\frac{1}{T}\sum_{n=0}^{T-1}\langle U_N(\Phi)^{-n}\operatorname{Op}_N(a)U_N(\Phi)^n\varphi_j,\varphi_j\rangle\right|^2 \tag{4.4}$$
$$\leq \frac{1}{N}\sum_{j=1}^{N}\left\|\frac{1}{T}\sum_{n=0}^{T-1}U_N(\Phi)^{-n}\operatorname{Op}_N(a)U_N(\Phi)^n\varphi_j\right\|^2 \tag{4.5}$$
$$= \frac{1}{N}\sum_{j=1}^{N}\left\|\operatorname{Op}_N(a^T)\varphi_j\right\|^2 + o_T(1), \tag{4.6}$$

by Axiom 1.2. Now

$$\frac{1}{N}\sum_{j=1}^{N}\left\|\operatorname{Op}_N(a^T)\varphi_j\right\|^2 = \frac{1}{N}\sum_{j=1}^{N}\langle \operatorname{Op}_N(a^T)^\dagger\operatorname{Op}_N(a^T)\varphi_j,\varphi_j\rangle \tag{4.7}$$
$$= \frac{1}{N}\sum_{j=1}^{N}\langle \operatorname{Op}_N(|a^T|^2)\varphi_j,\varphi_j\rangle + o_T(1) \tag{4.8}$$
$$= \int_{\mathcal{M}}|a^T|^2 d\mu + o_T(1). \tag{4.9}$$

Since Φ acts ergodically on $\mathcal{M}$, we have a mean ergodic theorem for test functions $a \in \mathrm{L}^2(\mathcal{M})$, i.e.,

$$\lim_{T\to\infty}\int_{\mathcal{M}}|a^T|^2 d\mu = 0, \tag{4.10}$$

and hence $\limsup_{N\to\infty} S_2(a,N)$ becomes arbitrarily small for T sufficiently large. □

Corollary 4.2. *There is a set sequence $\boldsymbol{I} := \{I_N\}_{N\in\mathcal{I}}$ with density 1 such that*

$$\langle \operatorname{Op}_N(a)\varphi_j,\varphi_j\rangle \to \int_{\mathcal{M}} a\,d\mu \tag{4.11}$$

for all $j \in I_N$, $N \to \infty$.

Proof. Apply Chebyshev's inequality with the variance given in (4.1). □

5. Examples

In this section we construct a well known example of quantum observables on the two-dimensional torus $\mathcal{M} = \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ satisfying Axiom 1.1 and corresponding examples of quantum maps satisfying Axiom 1.2.

5.1. Quantum tori.

It is convenient to represent a vector $\psi \in \mathbb{C}^N$ as a function $\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. Let us define the translation operators

$$[t_1\psi](Q) = \psi(Q+1) \tag{5.1}$$

and

$$[t_2\psi](Q) = e_N(Q)\psi(Q), \tag{5.2}$$

where $e_N(x) := e(x/N) = \exp(2\pi i x/N)$. One easily checks that

$$t_1^{m_1}t_2^{m_2} = t_2^{m_2}t_1^{m_1}e_N(m_1m_2) \qquad \forall m_1,m_2 \in \mathbb{Z}. \tag{5.3}$$

These relations are known as the *Weyl-Heisenberg commutation relations.* For $\boldsymbol{m} = (m_1, m_2) \in \mathbb{Z}^2$ put

$$T_N(\boldsymbol{m}) = e_N\left(\frac{m_1 m_2}{2}\right) t_2^{m_2} t_1^{m_1}. \tag{5.4}$$

Then

$$T_N(\boldsymbol{m}) T_N(\boldsymbol{n}) = e_N\left(\frac{\omega(\boldsymbol{m},\boldsymbol{n})}{2}\right) T_N(\boldsymbol{m}+\boldsymbol{n}) \tag{5.5}$$

with the symplectic form

$$\omega(\boldsymbol{m},\boldsymbol{n}) = m_1 n_2 - m_2 n_1. \tag{5.6}$$

For any $a \in \mathrm{C}^\infty(\mathbb{T}^2)$, we define the quantum observable

$$\mathrm{Op}_N(a) = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \widehat{a}(\boldsymbol{m}) T_N(\boldsymbol{m}) \tag{5.7}$$

where

$$\widehat{a}(\boldsymbol{m}) = \int_{\mathbb{T}^2} a(\boldsymbol{\xi}) e(-\boldsymbol{\xi}\cdot\boldsymbol{m})\, d\xi \tag{5.8}$$

are the Fourier coefficients of a. The observable $\mathrm{Op}_N(a)$ is also called the *Weyl quantization of* a. Axiom 1.1 (a) is trivially satisfied. Axioms 1.1 (b) and (c) follow from the following lemmas.

Lemma 5.1. *For all* $a_1, a_2 \in \mathrm{C}^\infty(\mathbb{T}^2)$

$$\begin{aligned} &\| \mathrm{Op}_N(a_1)\,\mathrm{Op}_N(a_2) - \mathrm{Op}_N(a_1 a_2)\| \\ &\qquad \le \frac{\pi}{N}\left(\sum_{\boldsymbol{m}\in\mathbb{Z}^2} \|\boldsymbol{m}\| |\widehat{a}_1(\boldsymbol{m})|\right)\left(\sum_{\boldsymbol{n}\in\mathbb{Z}^2} \|\boldsymbol{n}\| |\widehat{a}_2(\boldsymbol{n})|\right). \end{aligned} \tag{5.9}$$

Proof. Using the commutation relations (5.3) we find

$$\mathrm{Op}_N(a_1)\,\mathrm{Op}_N(a_2) \tag{5.10}$$

$$= \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^2} \widehat{a_1}(\boldsymbol{m})\widehat{a_2}(\boldsymbol{n}) T_N(\boldsymbol{m}) T_N(\boldsymbol{n}) \tag{5.11}$$

$$= \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^2} e_N\left(\frac{\omega(\boldsymbol{m},\boldsymbol{n})}{2}\right) \widehat{a_1}(\boldsymbol{m})\widehat{a_2}(\boldsymbol{n}) T_N(\boldsymbol{m}+\boldsymbol{n}) \tag{5.12}$$

$$= \sum_{\boldsymbol{m},\boldsymbol{k}\in\mathbb{Z}^2} e_N\left(\frac{\omega(\boldsymbol{m},\boldsymbol{k})}{2}\right) \widehat{a_1}(\boldsymbol{m})\widehat{a_2}(\boldsymbol{k}-\boldsymbol{m}) T_N(\boldsymbol{k}) \tag{5.13}$$

with $\boldsymbol{k} = \boldsymbol{n}+\boldsymbol{m}$, and hence

$$\begin{aligned} &\| \mathrm{Op}_N(a_1)\,\mathrm{Op}_N(a_2) - \mathrm{Op}_N(a_1 a_2)\| \\ &\qquad \le \sum_{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}^2} \left| e_N\left(\frac{\omega(\boldsymbol{m},\boldsymbol{n})}{2}\right) - 1\right| \left|\widehat{a_1}(\boldsymbol{m})\right| \left|\widehat{a_2}(\boldsymbol{n})\right| \end{aligned} \tag{5.14}$$

The lemma now follows from

$$|e(x)-1| \le |2\pi x|, \qquad |\omega(\boldsymbol{m},\boldsymbol{n})| \le \|\boldsymbol{m}\|\,\|\boldsymbol{n}\|. \tag{5.15}$$

□

Lemma 5.2. *For any* $a \in \mathrm{C}^\infty(\mathbb{T}^2)$ *and* $R > 1$

$$\frac{1}{N}\mathrm{Tr}\,\mathrm{Op}_N(a) = \int_{\mathbb{T}^2} a\, d\mu + O_{a,R}(N^{-R}). \tag{5.16}$$

Proof. Note that

$$\mathrm{Tr}\, T_N(\boldsymbol{m}) = \begin{cases} N & \text{if } \boldsymbol{m} = \boldsymbol{0} \bmod N\mathbb{Z}^2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.17}$$

The lemma now follows from the rapid decay of the Fourier coefficients $\widehat{a}(\boldsymbol{m})$ for $\|\boldsymbol{m}\| \to \infty$. □

Note that we have the alternative representation for $\mathrm{Op}_N(a)$,

$$[\mathrm{Op}_N(a)\psi](Q) = \sum_{m\in\mathbb{Z}} \widetilde{a}\left(m, \frac{Q}{N} + \frac{m}{2N}\right)\psi(Q+m) \tag{5.18}$$

where

$$\widetilde{a}(m,q) = \int_{\mathbb{R}/\mathbb{Z}} a(p,q)\, e(-pm)\, dp, \tag{5.19}$$

which is sometimes useful. Note that, for any $R > 1$, there is a constant C_R such that

$$|\widetilde{a}(m,q)| \le C_R (1+|m|)^{-R} \tag{5.20}$$

for all m, q. This fact is proved using integration by parts.

5.2. **Quantum maps.** A *twist map* Ψ_f is a map $\mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$\Psi_f : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p + f(q) \\ q \end{pmatrix} \mod \mathbb{Z}^2 \tag{5.21}$$

where we take $f : \mathbb{R} \to \mathbb{R}$ such that Ψ_f is C^∞. E.g. $f(q) = rq + 2\pi\kappa\cos(2\pi q)$ for constants $r \in \mathbb{Z}$, $\kappa \in \mathbb{R}$. Obviously Lebesgue measure $d\mu = dp\, dq$ is invariant under Ψ_f. A *linked twist map* Φ is now obtained by combining two twist maps, Ψ_{f_1} and Ψ_{f_2}, by setting

$$\Phi = \mathrm{R} \circ \Psi_{f_1} \circ \mathrm{R}^{-1} \circ \Psi_{f_2} \tag{5.22}$$

with the rotation

$$\mathrm{R} : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} q \\ -p \end{pmatrix} \mod \mathbb{Z}^2. \tag{5.23}$$

Since Ψ_{f_1}, Ψ_{f_2} and R preserve μ, so does Φ. More explicitly, we have

$$\mathrm{R} \circ \Psi_f \circ \mathrm{R}^{-1} : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p \\ q - f(p) \end{pmatrix} \mod \mathbb{Z}^2 \tag{5.24}$$

and thus

$$\Phi : \begin{pmatrix} p \\ q \end{pmatrix} \mapsto \begin{pmatrix} p + f_2(q) \\ q - f_1(p + f_2(q)) \end{pmatrix} \mod \mathbb{Z}^2. \tag{5.25}$$

We define the quantization of the twist map Ψ_f by the unitary operator

$$[U_N(\Psi_f)]\psi(Q) = e\left[-NV\left(\frac{Q}{N}\right)\right]\psi(Q) \tag{5.26}$$

where $V : \mathbb{R} \to \mathbb{R}$ is any C^∞ function satisfying $f = -V'$, and $NV((Q+N)/N) = NV(Q/N) \bmod \mathbb{Z}$ in order to have a well-defined phase. In the above example $f(q) = rq + 2\pi\kappa\cos(2\pi q)$ we could e.g. take $V(q) = -\frac{r}{2}q^2 - \kappa\sin(2\pi q)$ with the additional ristriction that r must be even if N is odd.

Proposition 5.3. *For any* $a \in \mathrm{C}^\infty(\mathbb{T}^2)$ *we have*

$$\|U_N(\Psi_f)^{-1}\,\mathrm{Op}_N(a) U_N(\Psi_f) - \mathrm{Op}_N(a\circ\Psi_f)\| = O(N^{-2}) \tag{5.27}$$

where the implied constant depends on a.

Proof. We have

$$\begin{aligned} &[U_N(\Psi_f)^{-1}\,\mathrm{Op}_N(a) U_N(\Psi_f)\psi](Q) = \sum_{m\in\mathbb{Z}} \widetilde{a}\left(m, \frac{Q}{N} + \frac{m}{2N}\right) \\ &\qquad e\left\{-N\left[V\left(\frac{Q+m}{N}\right) - V\left(\frac{Q}{N}\right)\right]\right\}\psi(Q+m), \end{aligned} \tag{5.28}$$

and

$$[\mathrm{Op}_N(a\circ\Psi_f)\psi](Q)=\sum_{m\in\mathbb{Z}}\widetilde{a}\left(m,\frac{Q}{N}+\frac{m}{2N}\right) e\left[mf\left(\frac{Q}{N}+\frac{m}{2N}\right)\right]\psi(Q+m), \tag{5.29}$$

since

$$\widetilde{(a\circ\Psi_f)}(m,q)=e[mf(q)]\,\widetilde{a}(m,q). \tag{5.30}$$

Therefore

$$\|U_N(\Psi_f)^{-1}\,\mathrm{Op}_N(a)U_N(\Psi_f)-\mathrm{Op}_N(a\circ\Psi_f)\| \le \max_q\sum_{m\in\mathbb{Z}}\left|\widetilde{a}\left(m,q+\frac{m}{2N}\right)c_m(q,N)\right| \tag{5.31}$$

with

$$c_m(q,N)=e\left\{-N\left[V\left(q+\frac{m}{N}\right)-V\left(q\right)\right]\right\}-e\left[mf\left(q+\frac{m}{2N}\right)\right]. \tag{5.32}$$

Since $|c_m(q,N)|\le 2$ and $|\widetilde{a}(m,q)|\le(1+|m|)^{-5}$, we have

$$\max_q\sum_{|m|\ge N^{1/2}}\left|\widetilde{a}\left(m,q+\frac{m}{2N}\right)c_m(q,N)\right|\le N^{-2}. \tag{5.33}$$

For $|m|<N^{1/2}$, Taylor expansion around $x=q+\frac{m}{2N}$ yields (the second order terms cancel)

$$V\left(x+\frac{m}{2N}\right)-V\left(x-\frac{m}{2N}\right)=V'\left(x\right)\frac{m}{N}+O\left(\frac{m^3}{N^3}\right) \tag{5.34}$$

$$=-f\left(x\right)\frac{m}{N}+O\left(\frac{m^3}{N^3}\right). \tag{5.35}$$

uniformly for all $|m|<N^{1/2}$ and all q. Hence in this case

$$c_m(q,N)=O\left(\frac{m^3}{N^2}\right) \tag{5.36}$$

and

$$\max_q\sum_{|m|<N^{1/2}}\left|\widetilde{a}\left(m,q+\frac{m}{2N}\right)c_m(q,N)\right| \tag{5.37}$$

$$\le O(N^{-2})\max_q\sum_{m\in\mathbb{Z}}\left|m^3\widetilde{a}\left(m,q+\frac{m}{2N}\right)\right| \tag{5.38}$$

$$=O(N^{-2}). \tag{5.39}$$

□

The discrete Fourier transform $\mathcal{F}_N$ is a unitary operator defined by

$$[\mathcal{F}_N\psi](P)=\frac{1}{\sqrt{N}}\sum_{Q=0}^{N-1}\psi(Q)e_N(-QP). \tag{5.40}$$

Its inverse is given by the formula

$$[\mathcal{F}_N^{-1}\psi](Q)=\frac{1}{\sqrt{N}}\sum_{P=0}^{N-1}\psi(P)e_N(PQ). \tag{5.41}$$

Proposition 5.4. *For any* $a\in\mathrm{C}^\infty(\mathbb{T}^2)$

$$\mathcal{F}_N^{-1}\,\mathrm{Op}_N(a)\mathcal{F}_N=\mathrm{Op}_N(a\circ\mathrm{R}) \tag{5.42}$$

with the rotation R *as in* (5.23).

Proof. This follows from $\mathcal{F}_N^{-1}t_1\mathcal{F}_N=t_2^{-1}$ and $\mathcal{F}_N^{-1}t_2\mathcal{F}_N=t_1$. □

The Fourier transform may therefore be viewed as a quantization of the rotation R which satisfies an *exact* correspondence principle, cf. Axiom 1.2.

The quantization of the linked twist map is now defined by

$$U_N(\Phi)=\mathcal{F}_N\,U_N(\Psi_{f_1})\,\mathcal{F}_N^{-1}\,U_N(\Psi_{f_2}). \tag{5.43}$$

Proposition 5.5. *For any* $a\in\mathrm{C}^\infty(\mathbb{T}^2)$*, we have*

$$\|U_N(\Phi)^{-1}\,\mathrm{Op}_N(a)U_N(\Phi)-\mathrm{Op}_N(a\circ\Phi)\|=O(N^{-2}) \tag{5.44}$$

where the implied constant depends on a*.*

Proof. Apply Propositions 5.3 and 5.4. □

The quantum map $U_N(\Phi)$ thus satisfies Axiom 1.2.

Appendix A. Minkowski content

We fix an atlas of local charts $\phi_j:\mathcal{V}_j\to\mathbb{R}^d$, where the open subsets $\mathcal{V}_j$ cover $\mathcal{M}$. In the following we thus identify subsets $\mathcal{S}$ of $\mathcal{M}$ with subsets Σ of $\mathbb{R}^d$ in the standard way. Let Σ be a subset of $\mathbb{R}^d$, and

$$\Sigma(\epsilon)=\{\xi\in\mathbb{R}^d:d(\xi,\Sigma)\le\epsilon\} \tag{A.1}$$

its closed ϵ-neighbourhood, where $d(\,\cdot\,,\,\cdot\,)$ is the euclidean metric on $\mathbb{R}^d$. The *s-dimensional upper Minkowski content* of Σ is defined as

$$\mathfrak{M}^{*s}(\Sigma):=\limsup_{\epsilon\to0}(2\epsilon)^{s-d}\nu\big(\Sigma(\epsilon)\big), \tag{A.2}$$

where ν is Lebesgue measure. We say Σ *has Minkowski content zero* if $\mathfrak{M}^{*d}(\Sigma)=0$. This is equivalent to saying that for every $\delta>0$ we can cover Σ with equi-radial euclidean balls of total measure less than δ. We say *a subset $\mathcal{S}$ of $\mathcal{M}$ has Minkowski content zero* if each of the sets $\Sigma_j:=\phi_j(\mathcal{S}|_{\mathcal{V}_j})\subset\mathbb{R}^d$ has Minkowski content zero.

Appendix B. Mollified characteristic functions

Consider the characteristic function $\chi_{\mathcal{D}}$ of a domain $\mathcal{D}\subset\mathcal{M}$ with boundary of Minkowski content zero. An *ϵ-mollified characteristic function* $\widetilde{\chi}_{\mathcal{D}}\in\mathrm{C}^\infty(\mathcal{M})$ has values in $[0,1]$ and $\widetilde{\chi}_{\mathcal{D}}(x)=\chi_{\mathcal{D}}(x)$ on a set of x of measure $1-\epsilon$. Since $\mathcal{D}$ has boundary of Minkowski content zero, we can construct such a smoothed function for any $\epsilon>0$. Furthermore we are able to construct ϵ-mollified $\widetilde{\chi}_{\mathcal{D}}$ whose support is either contained in $\mathcal{D}$, or whose support contains $\mathcal{D}$, again for any $\epsilon>0$. Note that if $\widetilde{\chi}_{\mathcal{D}}$ is ϵ-mollified, so is $\widetilde{\chi}_{\mathcal{D}}^n$ for any $n\in\mathbb{N}$ with the same ϵ.

After mollification, we may associate with a characteristic function $\chi_{\mathcal{D}}$ a quantum observable $\mathrm{Op}_N(\widetilde{\chi}_{\mathcal{D}})$. Since $\mathrm{Op}_N(\widetilde{\chi}_{\mathcal{D}})$ is in general not hermitian, it is sometimes more convenient to consider the symmetrized version, the positive definite hermitian matrix

$$\mathrm{Op}_N^{\mathrm{sym}}(\widetilde{\chi}_{\mathcal{D}}):=\mathrm{Op}_N(\widetilde{\chi}_{\mathcal{D}}^{1/2})\,\mathrm{Op}_N(\widetilde{\chi}_{\mathcal{D}}^{1/2})^\dagger. \tag{B.1}$$

Note that $\widetilde{\chi}_{\mathcal{D}}^{1/2}\in\mathrm{C}^\infty(\mathcal{M})$ since $\widetilde{\chi}_{\mathcal{D}}\ge0$. Furthermore, we have

$$\mathbf{Op}^{\mathrm{sym}}(\widetilde{\chi}_{\mathcal{D}})\sim\mathbf{Op}(\widetilde{\chi}_{\mathcal{D}}). \tag{B.2}$$

School of Mathematics, University of Bristol, Bristol BS8 1TW, United Kingdom

j.marklof@bristol.ac.uk