Quantum Lorentz gas in the Boltzmann-Grad limit: random vs periodic

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based on joint papers with
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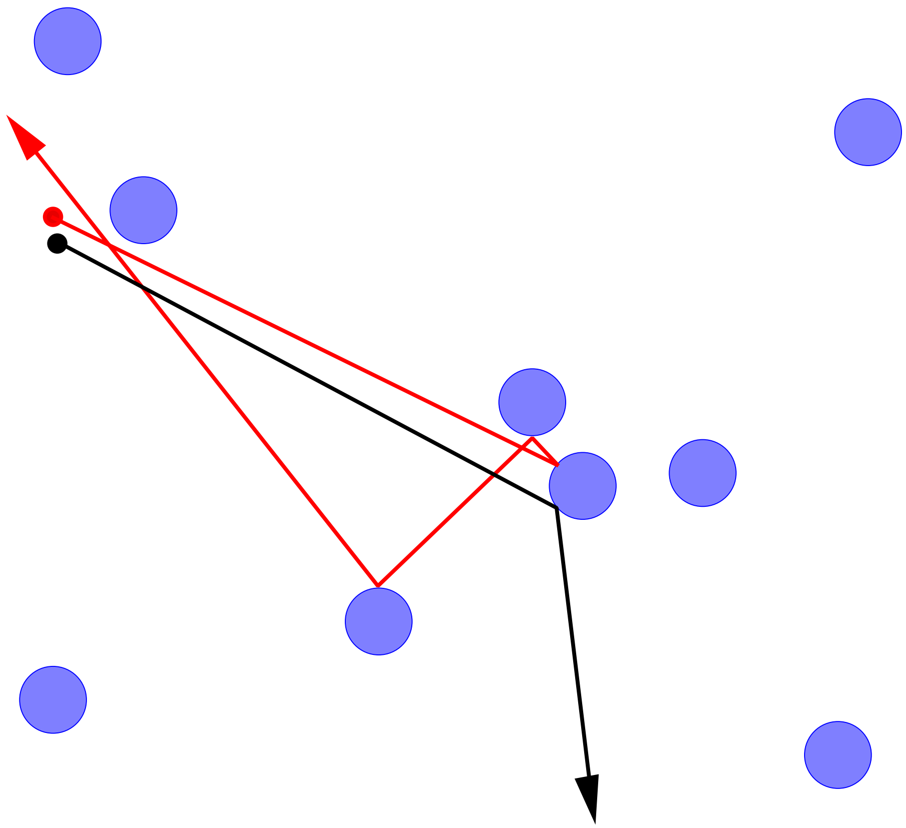
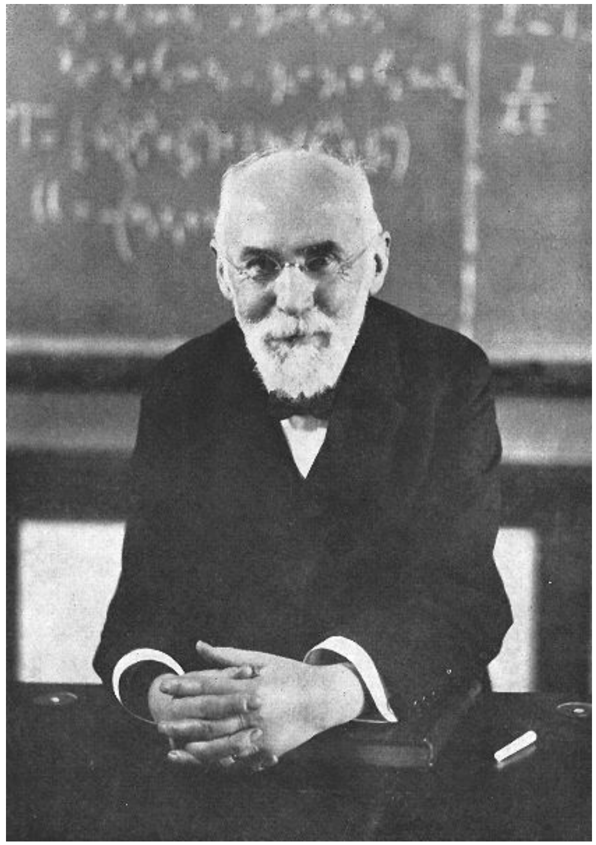
supported by EPSRC grant EP/S024948/1

The Lorentz gas

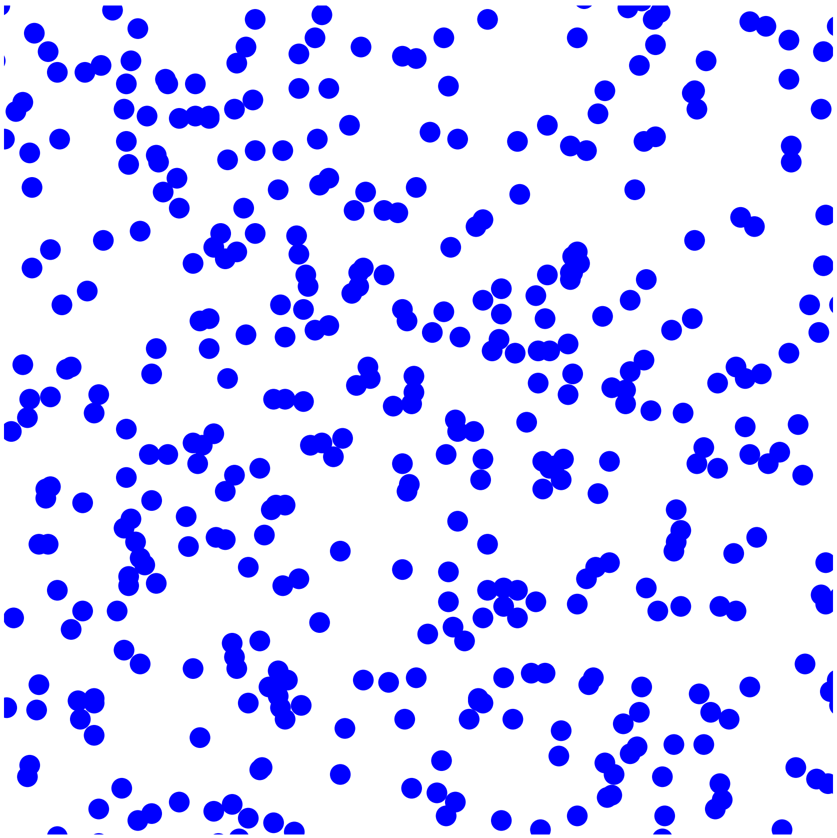


Arch. Neerl. (1905)

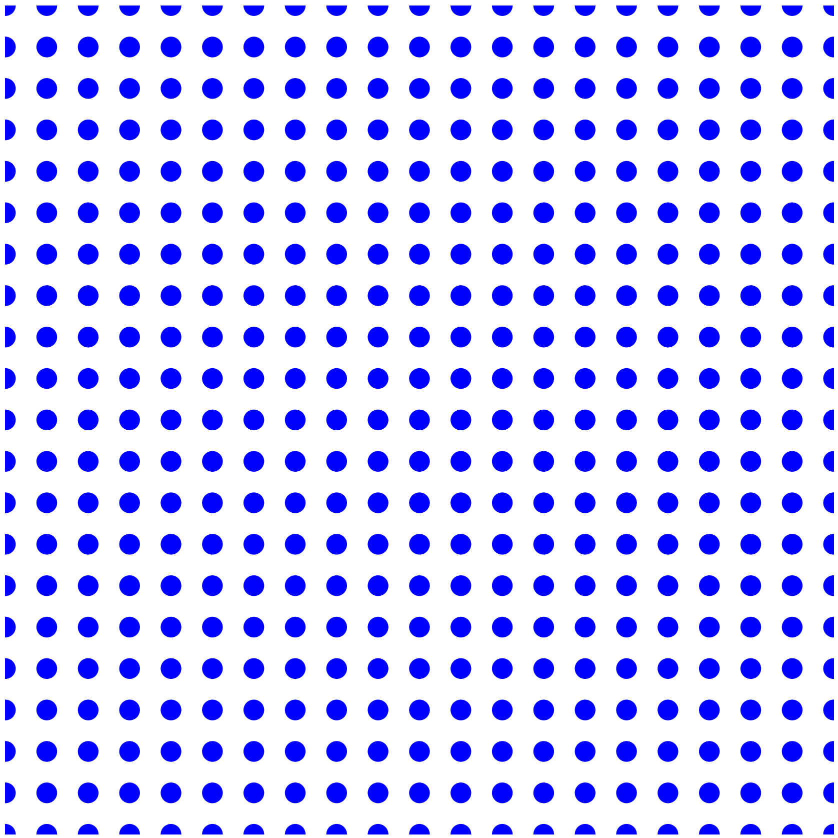


Hendrik Lorentz (1853-1928)

Lorentz gas in the small scatterer limit



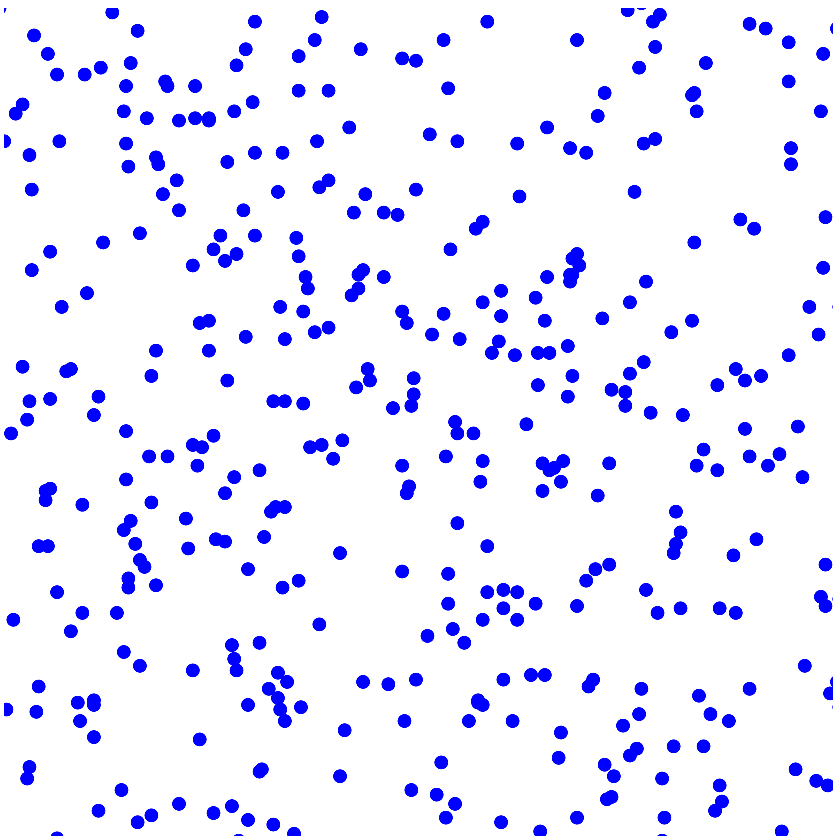
Fixed random scatterer configuration



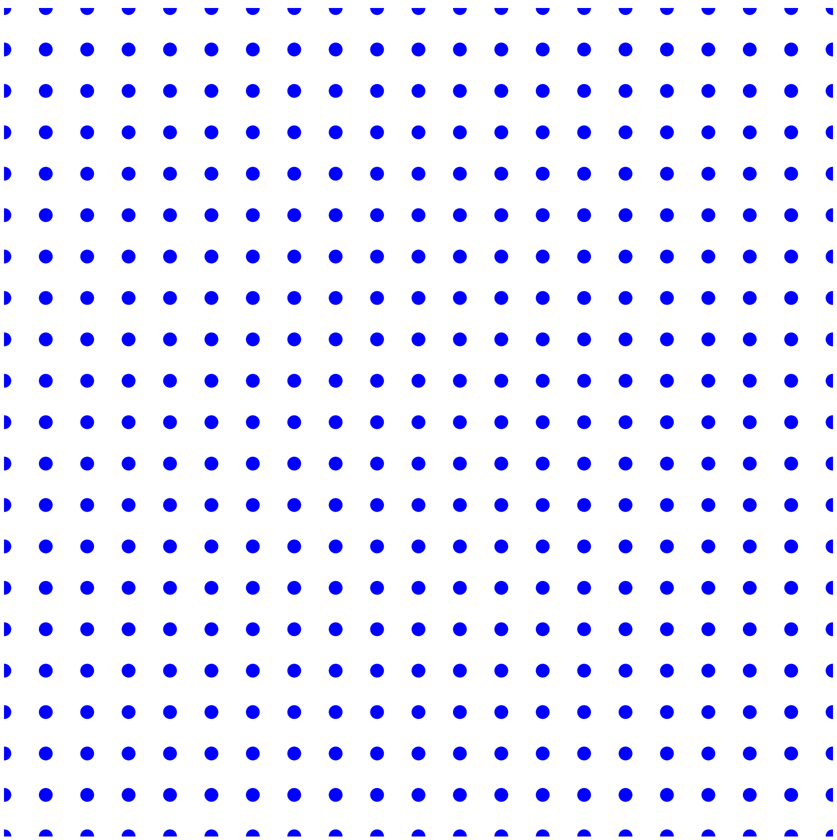
Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $r = 1/4$, mean free path$= \frac{1}{2r} = 2$

Lorentz gas in the small scatterer limit



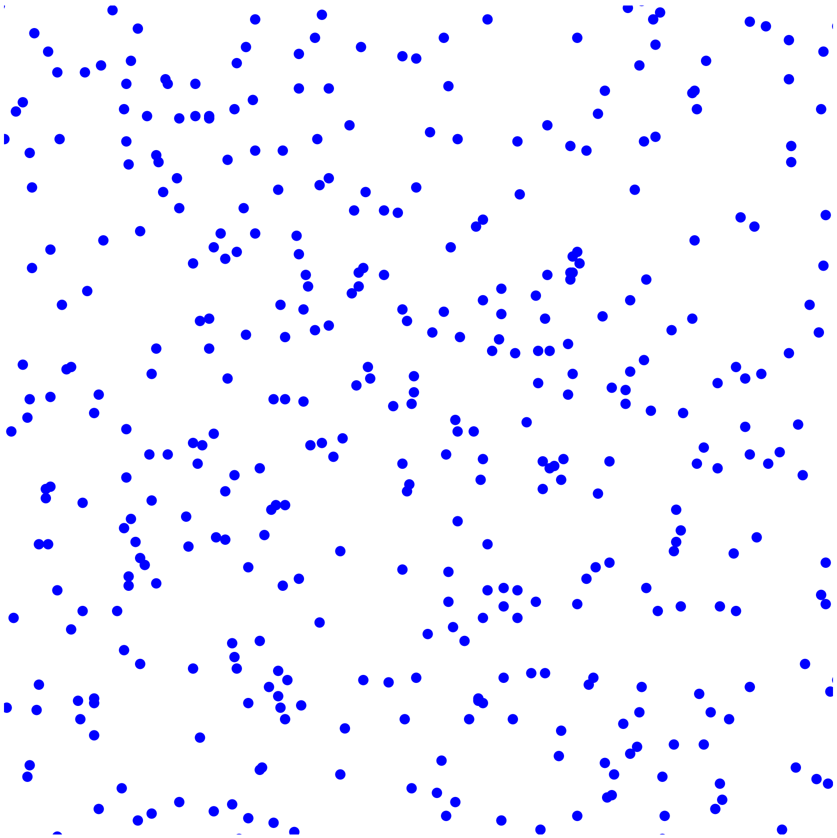
Fixed random scatterer configuration



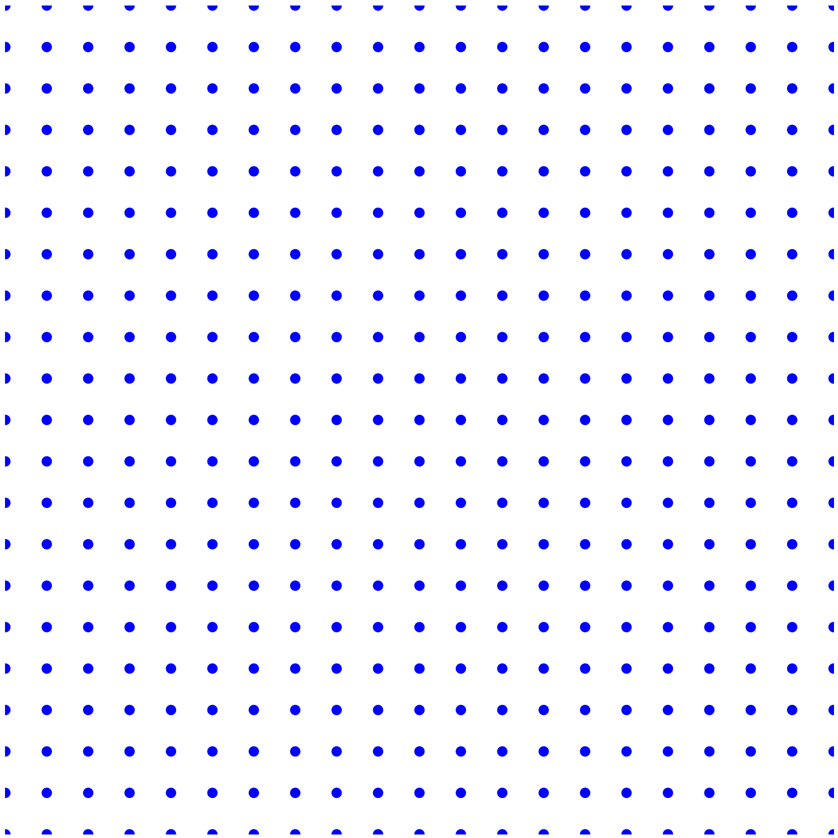
Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $r = 1/6$, mean free path$= \frac{1}{2r} = 3$

Lorentz gas in the small scatterer limit



Fixed random scatterer configuration



Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $r = 1/8$, mean free path$= \frac{1}{2r} = 4$

The Boltzmann-Grad (=low-density) limit

- Consider the dynamics in the limit of small scatterer radius r, dimension $d \geq 2$
- $\big(\boldsymbol{q}(t), \boldsymbol{v}(t)\big)$ = “microscopic” phase space coordinate at time t
- A volume argument shows that for $r \to 0$ the mean free path length (i.e., the average time between consecutive collisions) is asymptotic to

$$\frac{1}{\text{total scattering cross section}} = \frac{1}{r^{d-1} \operatorname{vol} B_1^{d-1}}$$

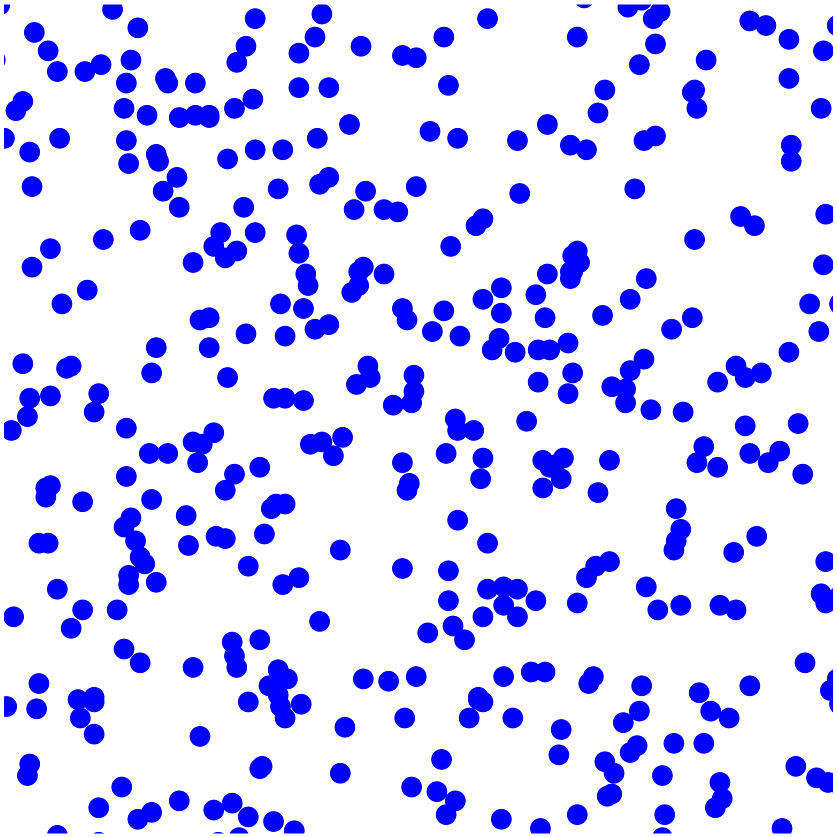
- We thus measure position and time in the “macroscopic” coordinates

$$\big(\boldsymbol{x}(t), \boldsymbol{y}(t)\big) = \big(r^{d-1}\boldsymbol{x}(r^{1-d}t), \boldsymbol{v}(r^{1-d}t)\big)$$

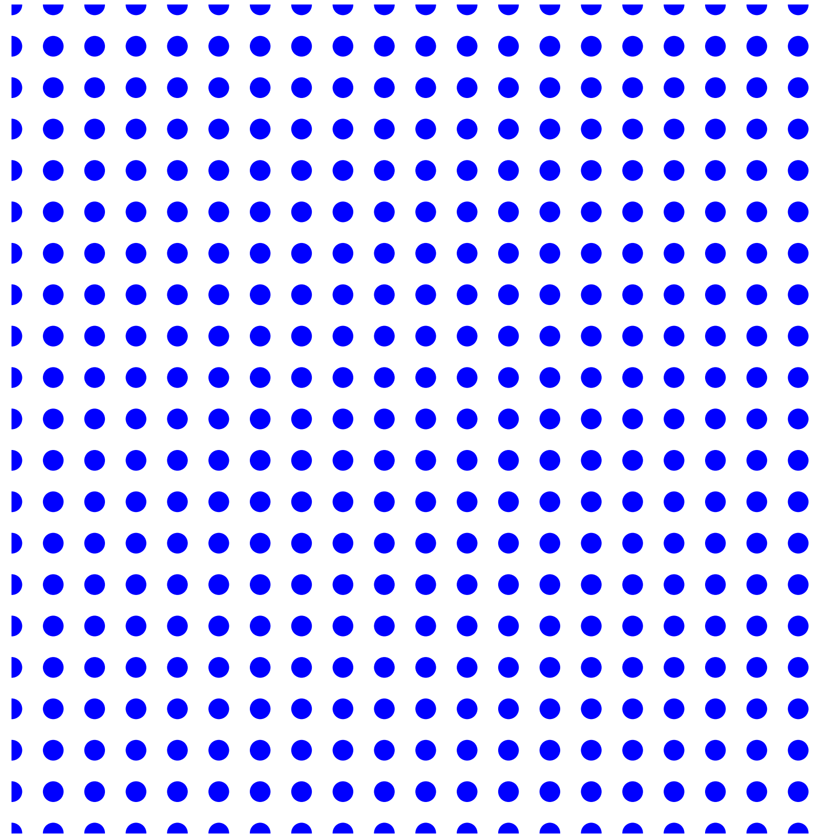
- Time evolution of initial data $(\boldsymbol{x}_0, \boldsymbol{y}_0)$:

$$\big(\boldsymbol{x}(t), \boldsymbol{y}(t)\big) = \Phi_r^t(\boldsymbol{x}_0, \boldsymbol{y}_0)$$

The Boltzmann-Grad limit



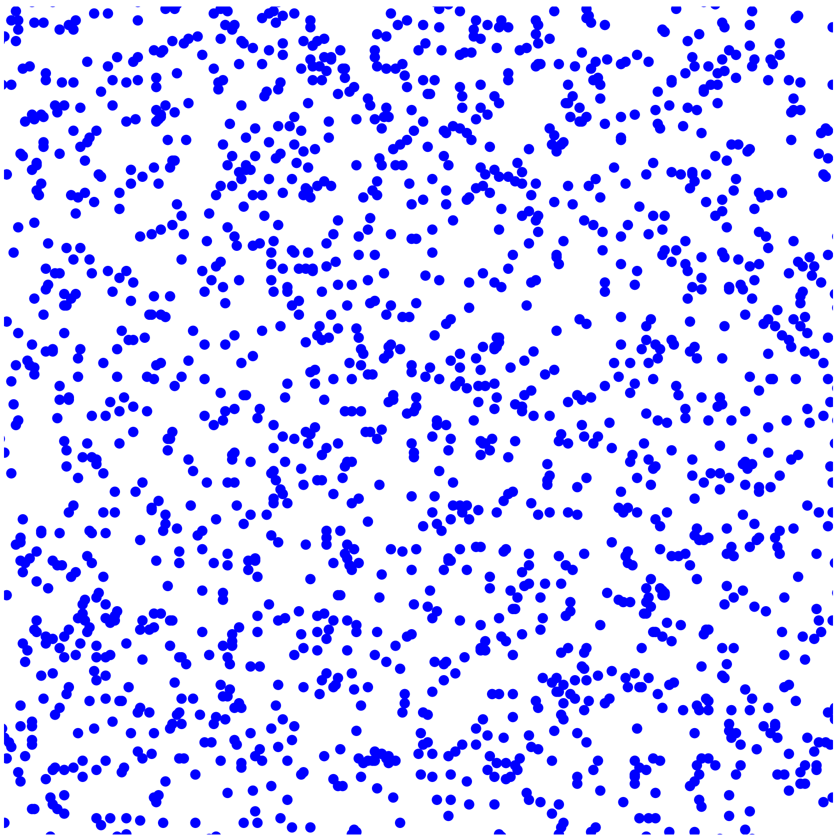
Fixed random scatterer configuration



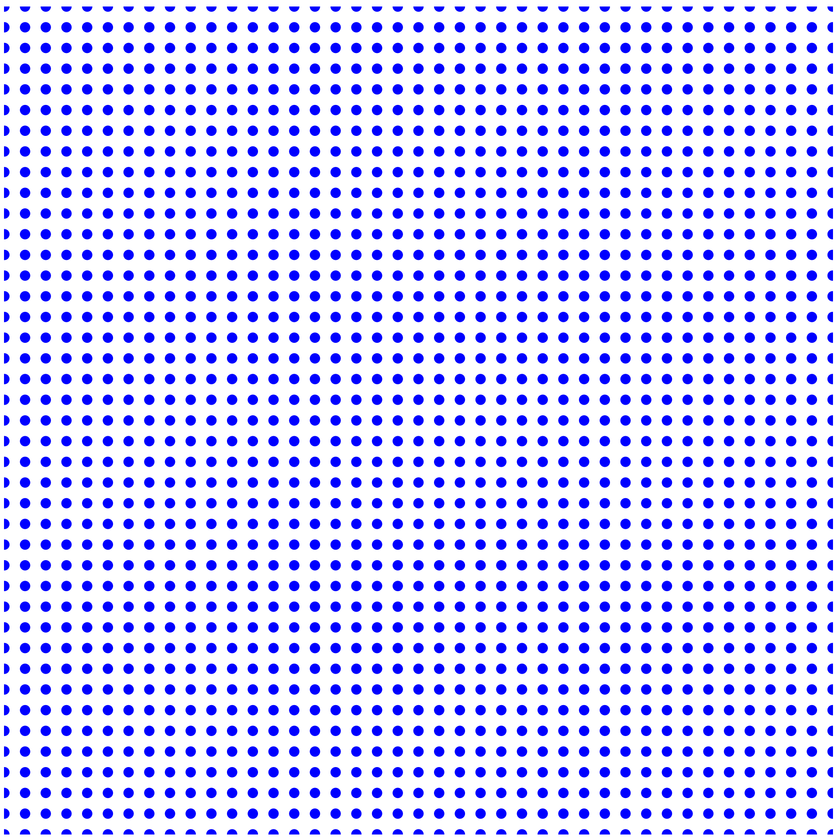
Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $r = 1/4$, mean free path$= \frac{1}{2r} = 2$

The Boltzmann-Grad limit



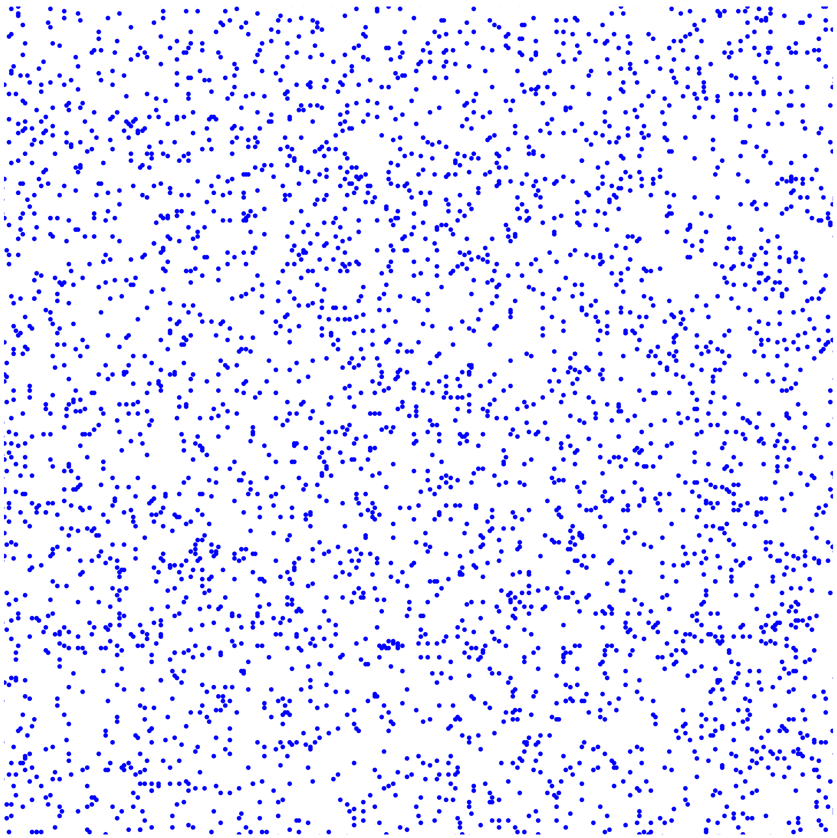
Fixed random scatterer configuration



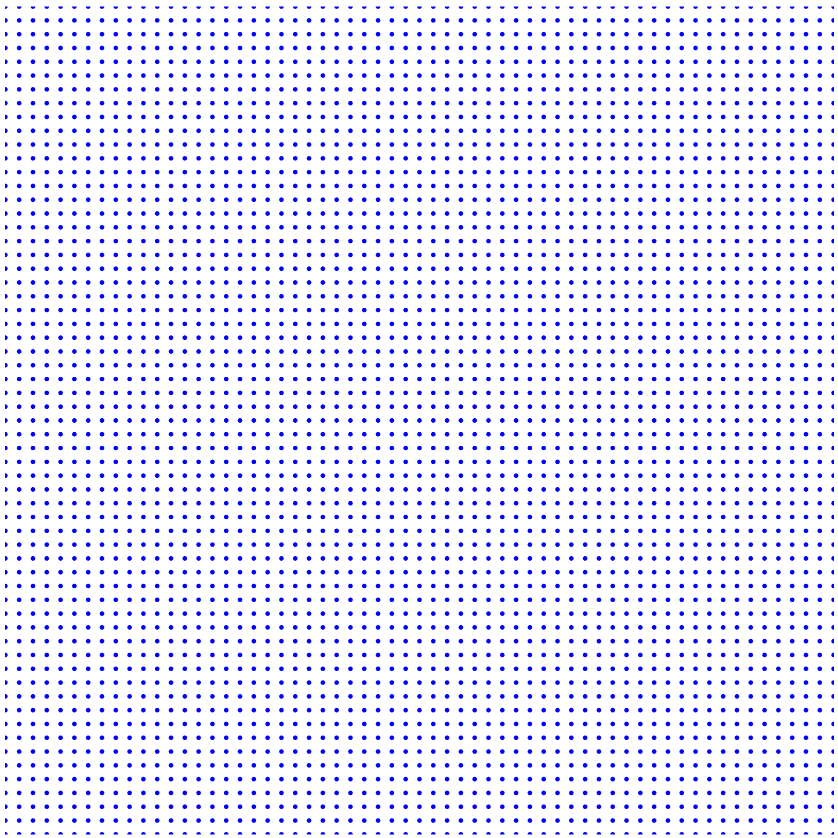
Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $r = 1/4$; $1/2$-zoom: macroscopic mean free path=1

The Boltzmann-Grad limit



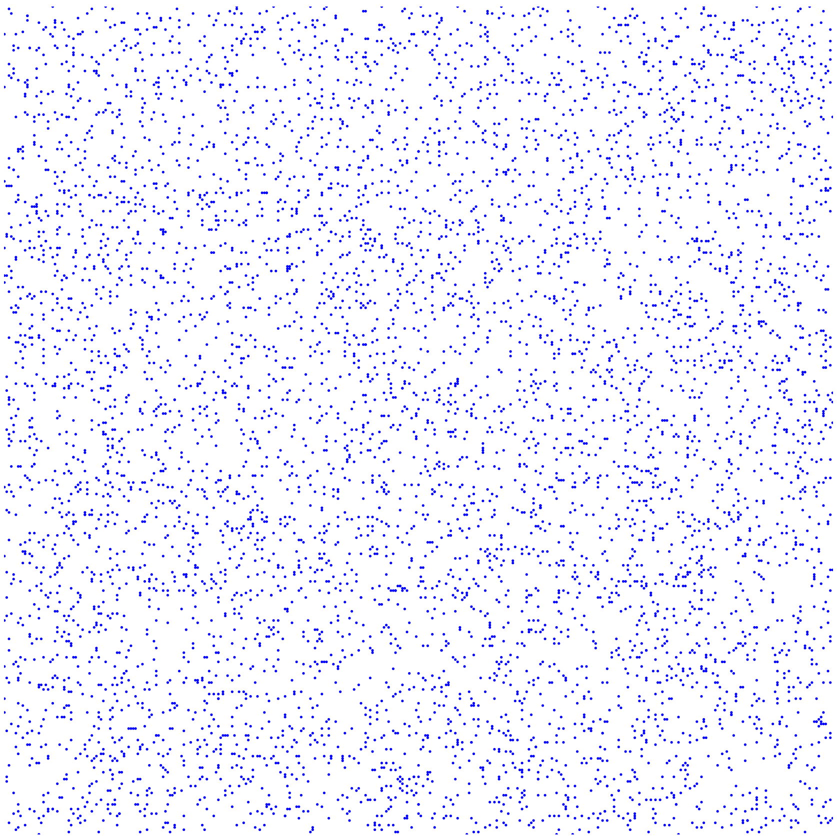
Fixed random scatterer configuration



Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $r = 1/6$; $1/3$-zoom: macroscopic mean free path=1

The Boltzmann-Grad limit



Fixed random scatterer configuration

Periodic scatterer configuration $\mathbb{Z}^2$

Scattering radius $r = 1/8$; $1/4$-zoom: macroscopic mean free path=1

The linear Boltzmann equation

Time evolution of a particle cloud with initial density $f \in \mathrm{L}^1$:

$$f_t^{(r)}(\boldsymbol{x},\boldsymbol{y}) := f\Big(\Phi_r^{-t}(\boldsymbol{x},\boldsymbol{y})\Big)$$

In his 1905 paper Lorentz suggested that $f_t^{(r)}$ is governed, as $r \to 0$, by the linear Boltzmann equation:

$$\Big(\partial_t + \boldsymbol{y}\cdot\nabla_x\Big) f(t,\boldsymbol{x},\boldsymbol{y}) = \int_{\mathbb{R}^d} \Big[\Sigma(\boldsymbol{y},\boldsymbol{y}') f(t,\boldsymbol{x},\boldsymbol{y}') - \Sigma(\boldsymbol{y}',\boldsymbol{y}) f(t,\boldsymbol{x},\boldsymbol{y})\Big]\, d\boldsymbol{y}'$$

where $\Sigma(\boldsymbol{y},\boldsymbol{y}')$ is the collision kernel (differential cross section) of the individual scatterer. E.g. $\Sigma(\boldsymbol{y},\boldsymbol{y}') = \frac{1}{4}\|\boldsymbol{y}-\boldsymbol{y}'\|^{3-d}$ for specular reflection at a hard sphere

Applications: Neutron transport, radiative transfer, . . .

The linear Boltzmann equation—rigorous proofs

Random

- Galavotti (Phys Rev 1969 & report 1972): Poisson distributed hard-sphere scatterer configuration
- Spohn (Comm Math Phys 1978): extension to more general random scatterer configurations and potentials
- Boldrighini, Bunimovich and Sinai (J Stat Phys 1983): prove convergence for almost every scatterer configuration (w.r.t. the Poisson process)

Periodic

- Golse (Ann Fac Toulouse 2008): failure of linear Boltzmann equation; Caglioti and Golse (Comptes Rendus 2008, J Stat Phys 2010): identified limit process in two dimensions
- JM and Strömbergsson (Nonlinearity 2008, Annals Math 2010, Annals Math 2011, GAFA 2011): proof of convergence of Lorentz gas to limit process (in arbitrary dimension); extension to quasicrystals and other aperiodic scatterer configurations (Memoirs AMS, to appear)

The quantum Lorentz gas

Random

- Spohn (J Stat Phys 1977): Gaussian random potentials, weak coupling limit & small times
- Erdös and Yau (Contemp Math 1998, Comm Pure Appl Math 2000): General random potentials, weak coupling limit
- Eng and Erdös (Rev Math Phys 2005): random scatterer configuration with smooth potential, Boltzmann-Grad limit

Periodic *“easy”...?* $>>>$**this talk**$<<<$

- Griffin and JM (Pure Applied Math 2019, J Stat Phys 2021): periodic scatterer configuration, new limit process in Boltzmann-Grad limit

The setting

- Schrödinger equation

$$\mathrm{i}\frac{h}{2\pi}\,\partial_t f(t,\boldsymbol{x}) = H_{h,\lambda} f(t,\boldsymbol{x}), \qquad f(0,\boldsymbol{x}) = f_0(\boldsymbol{x})$$

- quantum Hamiltonian

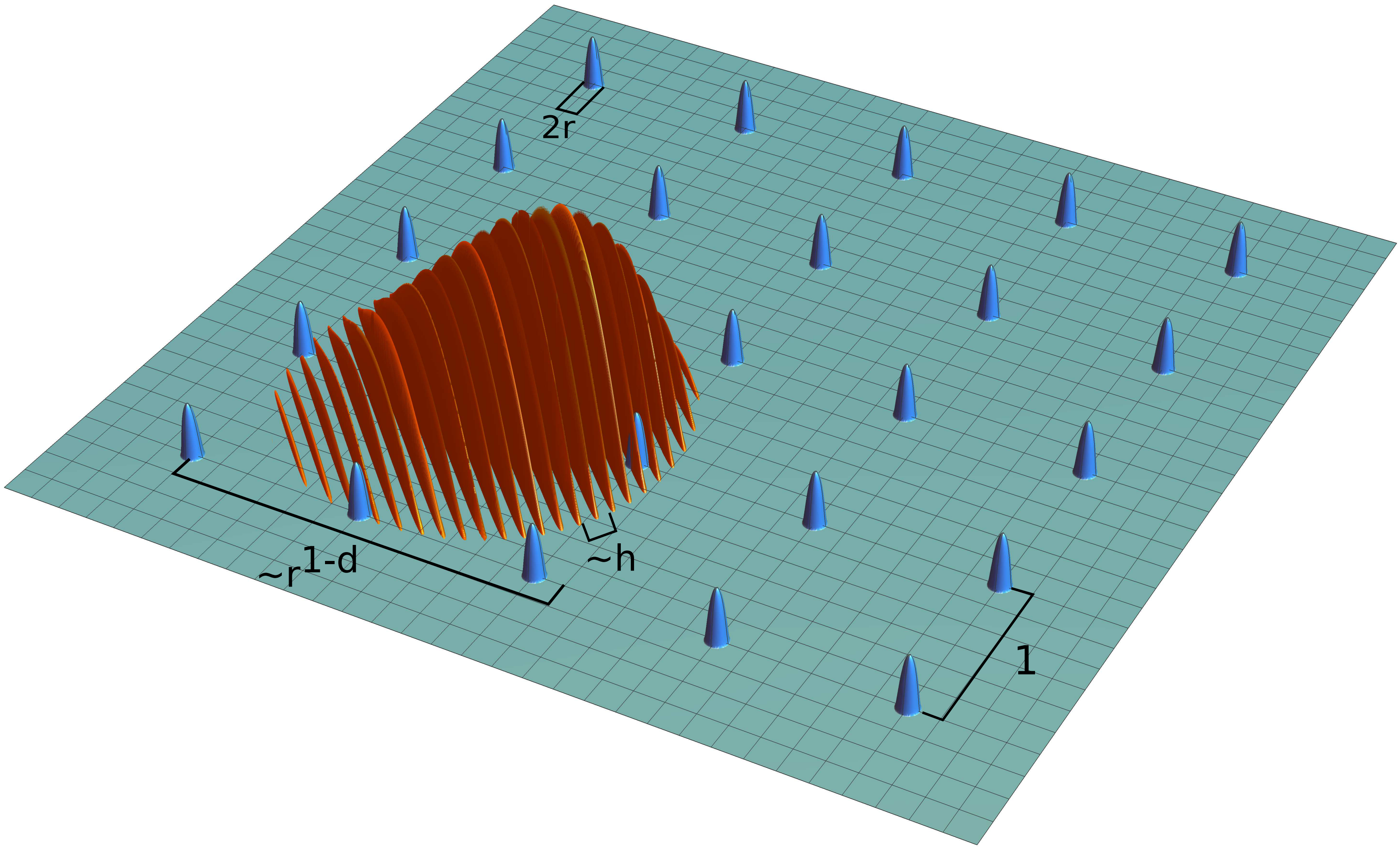
$$H_{h,\lambda} = -\frac{h^2}{8\pi^2}\,\Delta + \lambda V(\boldsymbol{x})$$

- potential

$$V(\boldsymbol{x}) = V_r(\boldsymbol{x}) = \sum_{\boldsymbol{m}\in\mathcal{P}} W(r^{-1}(\boldsymbol{x}+\boldsymbol{m})), \qquad W \in \mathcal{S}(\mathbb{R}^d)$$

  with $\mathcal{P}$ point set describing location of scatterers (e.g. $\mathcal{P} = \mathbb{Z}^d$)
- solution

$$f(t,\boldsymbol{x}) = U_{h,\lambda}(t) f_0(\boldsymbol{x}), \qquad U_{h,\lambda}(t) = \mathrm{e}^{-2\pi\mathrm{i}H_{h,\lambda}t/h}$$


2r
~h
$\sim r^{1-d}$
1

Observables

- time evolution of linear operators $A(t)$ ("quantum observables") given by Heisenberg evolution $A(t) = U_{h,\lambda}(t)\, A\, U_{h,\lambda}(t)^{-1}$.
- L^2 inner product on classical phase space

$$\langle a, b\rangle = \int_{\mathbb{R}^d\times\mathbb{R}^d} a(\boldsymbol{x},\boldsymbol{y})\,\overline{b(\boldsymbol{x},\boldsymbol{y})}\,d\boldsymbol{x}d\boldsymbol{y},$$

- Hilbert-Schmidt inner product $\langle A, B\rangle_{\mathrm{HS}} = \mathrm{Tr}\, AB^\dagger$.
- semiclassical Boltzmann-Grad scaling

$$D_{r,h} a(\boldsymbol{x},\boldsymbol{y}) = r^{d(d-1)/2} h^{d/2}\, a(r^{d-1}\boldsymbol{x}, h\boldsymbol{y}),$$

- standard Weyl quantisation of $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\mathrm{Op}(a) f(\boldsymbol{x}) = \int_{\mathbb{R}^d\times\mathbb{R}^d} a(\tfrac{1}{2}(\boldsymbol{x}+\boldsymbol{x}'),\boldsymbol{y})\, \mathrm{e}((\boldsymbol{x}-\boldsymbol{x}')\cdot\boldsymbol{y})\, f(\boldsymbol{x}')\, d\boldsymbol{x}'d\boldsymbol{y}$$

- Set $\mathrm{Op}_{r,h} = \mathrm{Op} \circ D_{r,h}$ and $\mathrm{Op}_h = \mathrm{Op}_{1,h}$.

A limiting transport process?

Pick your favourite scatterer configuration $\mathcal{P}$ (random or deterministic).

Questions.

(i) Does there exist a family of operators $L(t) : \mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d) \to \mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)$ such that for all $a, b \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, $A = \mathrm{Op}_{r,h}(a)$, $B = \mathrm{Op}_{r,h}(b)$,

$$\lim_{r \to 0} \langle A(tr^{-(d-1)}), B \rangle_{\mathrm{HS}} = \langle L(t)a, b \rangle \quad ?$$

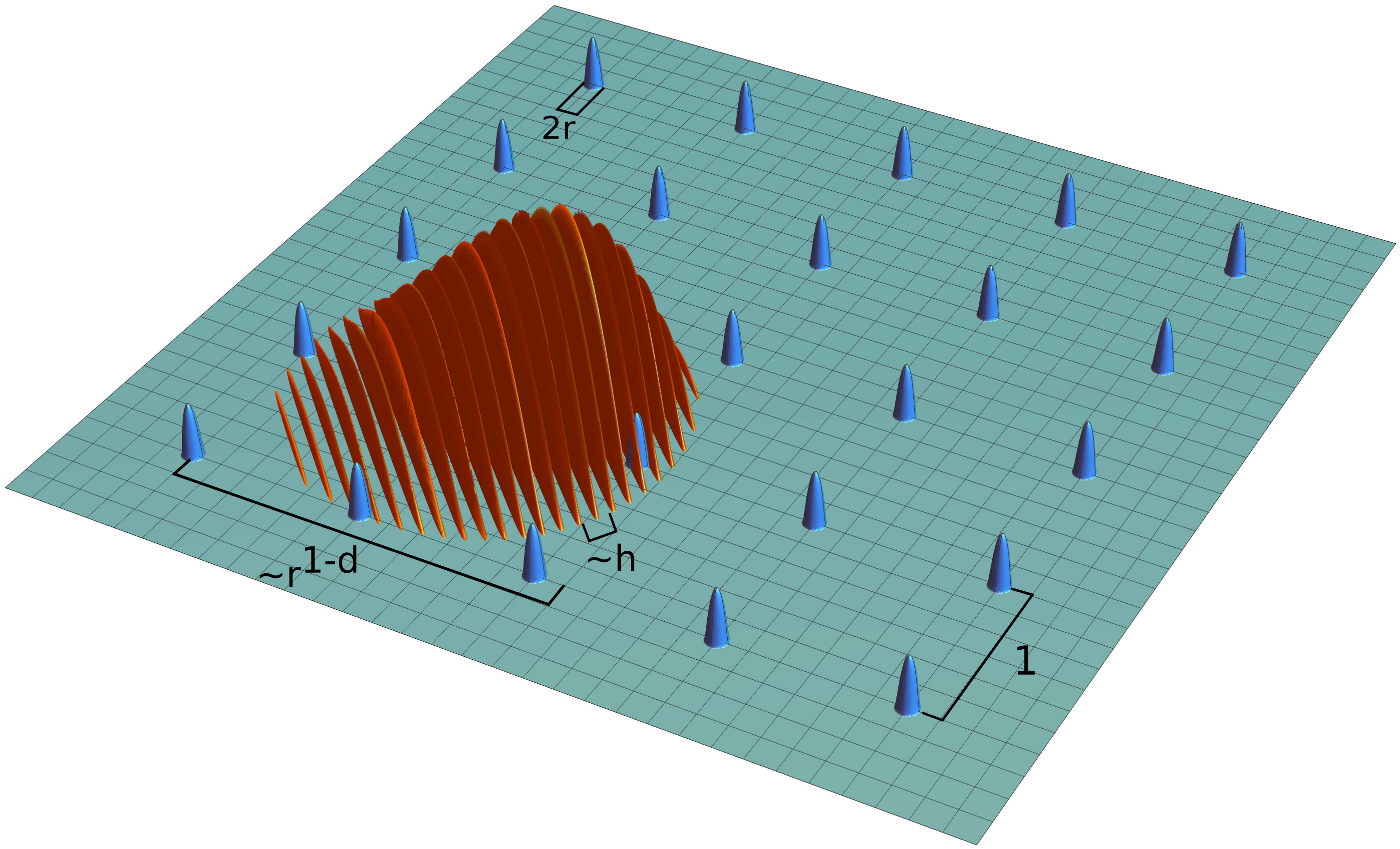
(ii) Is $f(t, \boldsymbol{x}, \boldsymbol{y}) = L(t)a(\boldsymbol{x}, \boldsymbol{y})$ a solution of the linear Boltzmann equation?

For random scatterer configurations Eng and Erdös (Rev Math Phys 2005) have proved convergence (in the annealed case) to a limit $L(t)$, which in fact is a solution to the linear Boltzmann equation with the standard quantum mechanical collision kernel

$$\Sigma(\boldsymbol{y}, \boldsymbol{y}') = 8\pi^2 \, \delta(\|\boldsymbol{y}\|^2 - \|\boldsymbol{y}'\|^2) \, |T(\boldsymbol{y}, \boldsymbol{y}')|^2.$$

Here $T(\boldsymbol{y}, \boldsymbol{y}')$ is the (single scatterer) T-matrix.

$>>>$**Semiclassical propagation with quantum scattering**$<<<$



>>>**Semiclassical propagation with quantum scattering**<<<

A limiting transport process!

Consider the periodic scatterer configuration $\mathcal{P} = \mathbb{Z}^d$ (or any other lattice in $\mathbb{R}^d$ of full rank).

Theorem (Griffin & JM, J Stat Phys 2021).
Conditional on a generalised Berry-Tabor conjecture:
(i) There exists a family of operators $L(t) : \mathsf{L}^1(\mathbb{R}^d \times \mathbb{R}^d) \to \mathsf{L}^1(\mathbb{R}^d \times \mathbb{R}^d)$ such that for all $a, b \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, $A = \mathrm{Op}_{r,h}(a)$, $B = \mathrm{Op}_{r,h}(b)$, $t > 0$ and $0 < \lambda \leq \lambda_0$ (λ_0 sufficently small)

$$\lim_{r \to 0} \langle A(tr^{-(d-1)}), B \rangle_{\mathrm{HS}} = \langle L(t)a, b \rangle$$

(ii) $f(t, \boldsymbol{x}, \boldsymbol{y}) = L(t)a(\boldsymbol{x}, \boldsymbol{y})$ is **NOT** a solution of the linear Boltzmann equation.

The statement can be proved unconditionally up to second order in perturbation theory (in λ): Griffin and JM, Pure & Applied Analysis 2019.

Collision series for linear Boltzmann

Total scattering cross section $\Sigma_{\mathrm{tot}}(\boldsymbol{y}) = \int_{\mathbb{R}^d} \Sigma(\boldsymbol{y}', \boldsymbol{y})\, d\boldsymbol{y}'$

Collision series for solution of the linear Boltzmann equation

$$f_{\mathrm{LB}}(t, \boldsymbol{x}, \boldsymbol{y}) = \sum_{k=1}^{\infty} f_{\mathrm{LB}}^{(k)}(t, \boldsymbol{x}, \boldsymbol{y})$$

with the zero-collision term

$$f_{\mathrm{LB}}^{(1)}(t, \boldsymbol{x}, \boldsymbol{y}) = a(\boldsymbol{x} - t\boldsymbol{y}, \boldsymbol{y})\, \mathrm{e}^{-t\Sigma_{\mathrm{tot}}(\boldsymbol{y})},$$

and the $(k-1)$-collision term ...

Collision series for linear Boltzmann

...and the $(k-1)$-collision term

$$f_{\mathrm{LB}}^{(k)}(t,\boldsymbol{x},\boldsymbol{y}) = \int_{(\mathbb{R}^d)^k} \int_{\mathbb{R}^k_{\geq 0}} \delta(\boldsymbol{y}-\boldsymbol{y}_1)\, a\Big(\boldsymbol{x} - \sum_{j=1}^{k} u_j \boldsymbol{y}_j, \boldsymbol{y}_k\Big) \times \rho_{\mathrm{LB}}^{(k)}(\boldsymbol{u},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_k)\, \delta\Big(t - \sum_{j=1}^{k} u_j\Big)\, d\boldsymbol{u}\, d\boldsymbol{y}_1 \cdots d\boldsymbol{y}_k$$

with

$$\rho_{\mathrm{LB}}^{(k)}(\boldsymbol{u},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_k) = \prod_{i=1}^{k} \mathrm{e}^{-u_i \Sigma_{\mathrm{tot}}(\boldsymbol{y}_i)} \prod_{j=1}^{k-1} \Sigma(\boldsymbol{y}_j, \boldsymbol{y}_{j+1}).$$

The product form of the density $\rho_{\mathrm{LB}}^{(k)}$ shows that the corresponding random flight process is Markovian, and describes a particle moving along a random piecewise linear curve with momenta $\boldsymbol{y}_i$ and exponentially distributed flight times u_i.

Collision series for our limit process

Collision series

$$f(t,\boldsymbol{x},\boldsymbol{y}) = \sum_{k=1}^{\infty} f^{(k)}(t,\boldsymbol{x},\boldsymbol{y})$$

with the zero-collision term (as for LB)

$$f^{(1)}(t,\boldsymbol{x},\boldsymbol{y}) = f^{(1)}_{\mathrm{LB}}(t,\boldsymbol{x},\boldsymbol{y}) = a(\boldsymbol{x}-t\boldsymbol{y},\boldsymbol{y})\,\mathrm{e}^{-t\Sigma_{\mathrm{tot}}(\boldsymbol{y})},$$

and the $(k-1)$-collision term ...

$$f^{(k)}(t,\boldsymbol{x},\boldsymbol{y}) = \frac{1}{k!}\sum_{\ell,m=1}^{k}\int_{(\mathbb{R}^d)^k}\int_{\mathbb{R}^k_{\geq 0}} \delta(\boldsymbol{y}-\boldsymbol{y}_\ell)\, a\Big(\boldsymbol{x}-\sum_{j=1}^{k} u_j\boldsymbol{y}_j,\boldsymbol{y}_m\Big) \times \rho^{(k)}_{\ell m}(\boldsymbol{u},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_k)\,\delta\Big(t-\sum_{j=1}^{k}u_j\Big)\,d\boldsymbol{u}\,d\boldsymbol{y}_1\cdots d\boldsymbol{y}_k,$$

with the collision densities ...

Collision series for our limit process

...with the **positive** collision densities

$$\rho_{\ell m}^{(k)}(\boldsymbol{u}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_k) = \left| g_{\ell m}^{(k)}(\boldsymbol{u}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_k) \right|^2 \omega_k(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k) \prod_{i=1}^{k} \mathrm{e}^{-u_i \Sigma_{\mathrm{tot}}(\boldsymbol{y}_i)}.$$

Here

$$\omega_k(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k) = \prod_{j=1}^{k-1} \delta\Big(\tfrac{1}{2}\|\boldsymbol{y}_j\|^2 - \tfrac{1}{2}\|\boldsymbol{y}_{j+1}\|^2\Big)$$

and $g_{\ell m}^{(k)}$ are the coefficients of the matrix valued function

$$\mathbb{G}^{(k)}(\boldsymbol{u}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_k) = \frac{1}{(2\pi \mathrm{i})^k} \oint \cdots \oint \Big(\mathbb{D}(\boldsymbol{z}) - \mathbb{W}\Big)^{-1} \exp(\boldsymbol{u} \cdot \boldsymbol{z})\, dz_1 \cdots dz_k,$$

where $\mathbb{D}(\boldsymbol{z}) = \mathrm{diag}(z_1, \ldots, z_k)$ and $\mathbb{W} = \mathbb{W}(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_k)$ with entries

$$w_{ij} = \begin{cases} 0 & (i = j) \\ -2\pi \mathrm{i} T(\boldsymbol{y}_i, \boldsymbol{y}_j) & (i \neq j). \end{cases}$$

$>>>$**Strong correlation with past momenta**$<<<$

Collision series for our limit process

Explicitly, for the one collision terms

$$\rho_{11}^{(2)}(\boldsymbol{u}, \boldsymbol{y}_1, \boldsymbol{y}_2) = \rho_{\mathrm{LB}}^{(2)}(\boldsymbol{u}, \boldsymbol{y}_1, \boldsymbol{y}_2) \left| \frac{u_1 T(\boldsymbol{y}_2, \boldsymbol{y}_1)}{u_2 T(\boldsymbol{y}_1, \boldsymbol{y}_2)} \right| \times \left| J_1\Big(4\pi [u_1 u_2 T(\boldsymbol{y}_1, \boldsymbol{y}_2) T(\boldsymbol{y}_2, \boldsymbol{y}_1)]^{1/2}\Big) \right|^2.$$

and

$$\rho_{12}^{(2)}(\boldsymbol{u}, \boldsymbol{y}_1, \boldsymbol{y}_2) = \rho_{\mathrm{LB}}^{(2)}(\boldsymbol{u}, \boldsymbol{y}_1, \boldsymbol{y}_2) \left| J_0\Big(4\pi [u_1 u_2 T(\boldsymbol{y}_1, \boldsymbol{y}_2) T(\boldsymbol{y}_2, \boldsymbol{y}_1)]^{1/2}\Big) \right|^2$$

with J_k the standard Bessel functions.

The remaining matrix elements can be computed via the identities

$$\rho_{22}^{(2)}(u_1, u_2, \boldsymbol{y}_1, \boldsymbol{y}_2) = \rho_{11}^{(2)}(u_2, u_1, \boldsymbol{y}_2, \boldsymbol{y}_1),$$

$$\rho_{21}^{(2)}(u_1, u_2, \boldsymbol{y}_1, \boldsymbol{y}_2) = \rho_{12}^{(2)}(u_2, u_1, \boldsymbol{y}_2, \boldsymbol{y}_1).$$

- Above formulas strikingly similar to those for two-point spectral statistics in diffractive systems (Bogomolny and Giraud, Nonlinearity 2002)

Key steps in proof

- Use Floquet-Bloch decomposition to reduce problem to L^2 subspaces of functions

$$\psi(\boldsymbol{x}+\boldsymbol{k}) = \mathrm{e}(\boldsymbol{k}\cdot\boldsymbol{\alpha})\psi(\boldsymbol{x}), \quad \forall \boldsymbol{k} \in \mathbb{Z}^d$$

with $\boldsymbol{\alpha} \in [0,1)^d$

- Consider each $\boldsymbol{\alpha}$-subspace separately (random or fixed)
- Use iterated application of Duhamel formula for quantum propagator,

$$U_{\lambda,h}(t) = U_{0,h}(t) - 2\pi\mathrm{i}\lambda \int_0^t U_{\lambda,h}(t-s)\,\mathrm{Op}(V)U_{0,h}(s)ds,$$

to produce perturbation expansion

- The eigenphases of $U_{0,h}(t)$ restricted to $\boldsymbol{\alpha}$-subspace are of the form

$$\pi\, t\, \|\boldsymbol{m}+\boldsymbol{\alpha}\|^2, \quad \boldsymbol{m} \in \mathbb{Z}^d$$

Key steps in proof

- Set $\mathcal{P}_{\boldsymbol{\alpha}} = \mathbb{Z}^d + \boldsymbol{\alpha}$
- The $(n-1)$ collision term can be expressed in the form

$$r^d \sum_{\substack{\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n=\boldsymbol{p}_0\in\mathcal{P}_{\boldsymbol{\alpha}} \\ \text{non-consec}}} H_{t,\ell,n}\Big(r^{2-d}(\tfrac{1}{2}\|\boldsymbol{p}_0\|^2,\ldots,\tfrac{1}{2}\|\boldsymbol{p}_n\|^2), r\boldsymbol{p}_0,\ldots,r\boldsymbol{p}_n\Big)$$

  form some (not so well behaved) function $H_{t,\ell,n}$, which has translation invariance in the first coordinates so that it only depends on the differences between the $\|\boldsymbol{p}_j\|^2$
- The above expression is thus the n-point correlation density of $\mathcal{P}$ tested against $H_{t,\ell,n}$ — measured on the scale of their mean separation
- **Our key assumption in this work is that we can replace $\mathcal{P}_{\boldsymbol{\alpha}}$, for typical (or random) $\boldsymbol{\alpha}$ by a Poisson point process in $\mathbb{R}^d$ of intensity one**

How random is $\mathcal{P}_{\boldsymbol{\alpha}} = \mathbb{Z}^d + \boldsymbol{\alpha}$?

Illustrative example for $d = 2$:

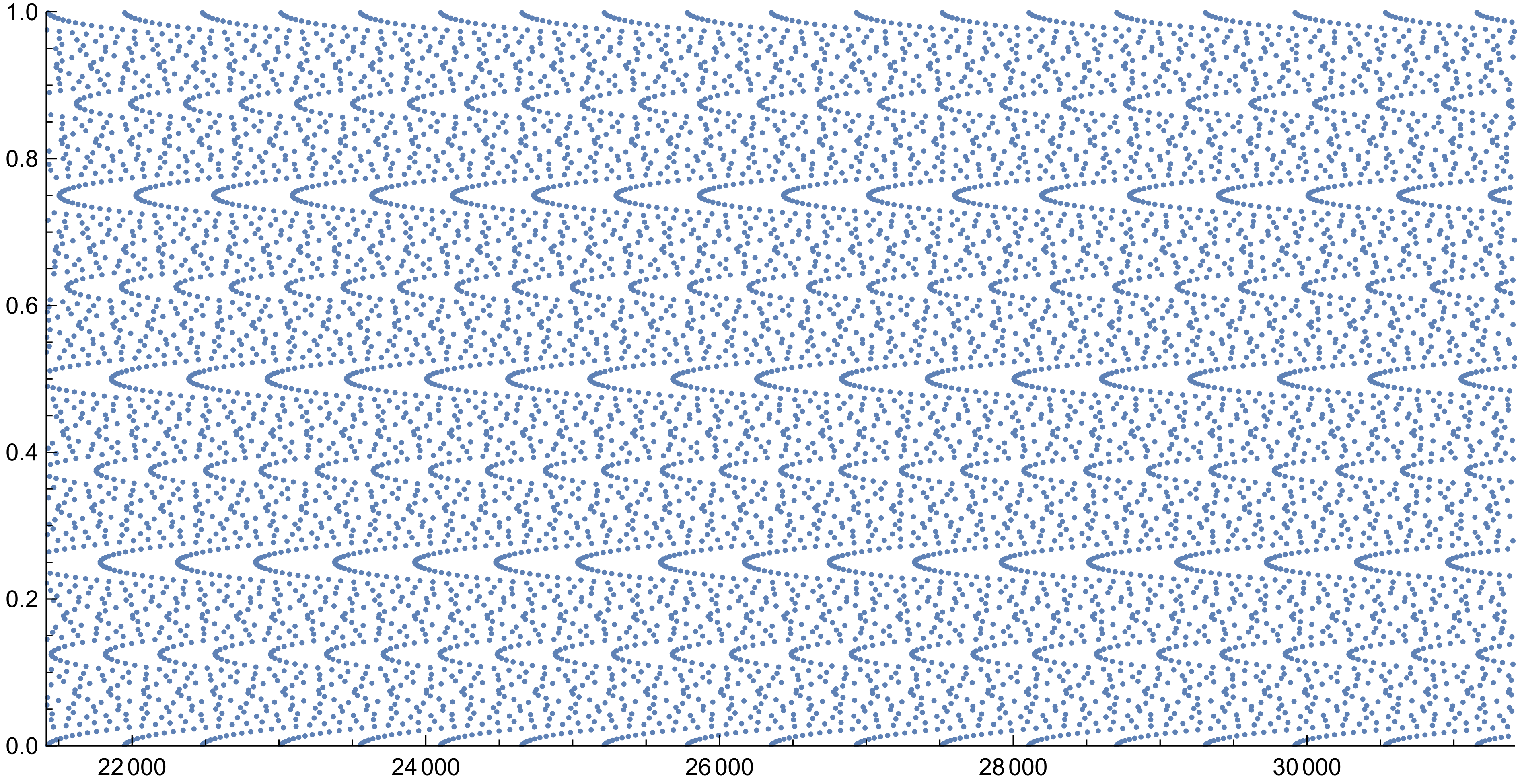
- Fix $\boldsymbol{\alpha} = (\sqrt{2}, \sqrt{3}) \longleftarrow$ not even generic/random

- Consider the sequence $(\lambda_i, \theta_i)_{i \in \mathbb{N}}$ of elements of the set

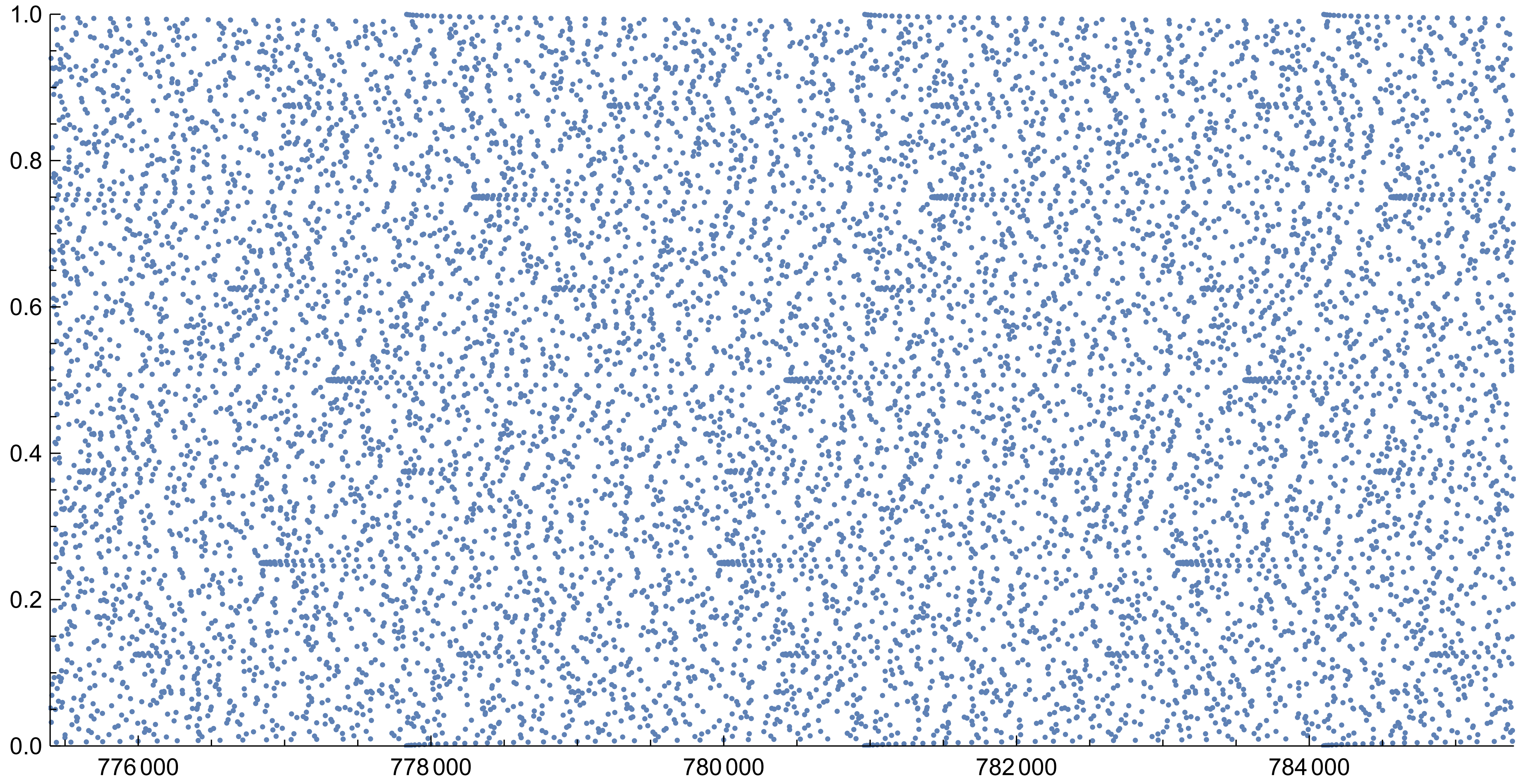
$$\left\{ \left(\pi \|\boldsymbol{n} + \boldsymbol{\alpha}\|^2, \frac{1}{2\pi} \arg(\boldsymbol{n} + \boldsymbol{\alpha}) \right) \in \mathbb{R}_{\geq 0} \times [0,1) \,\middle|\, \boldsymbol{n} \in \mathbb{Z}^2 \right\}$$

  arranged in increasing order according to the first component

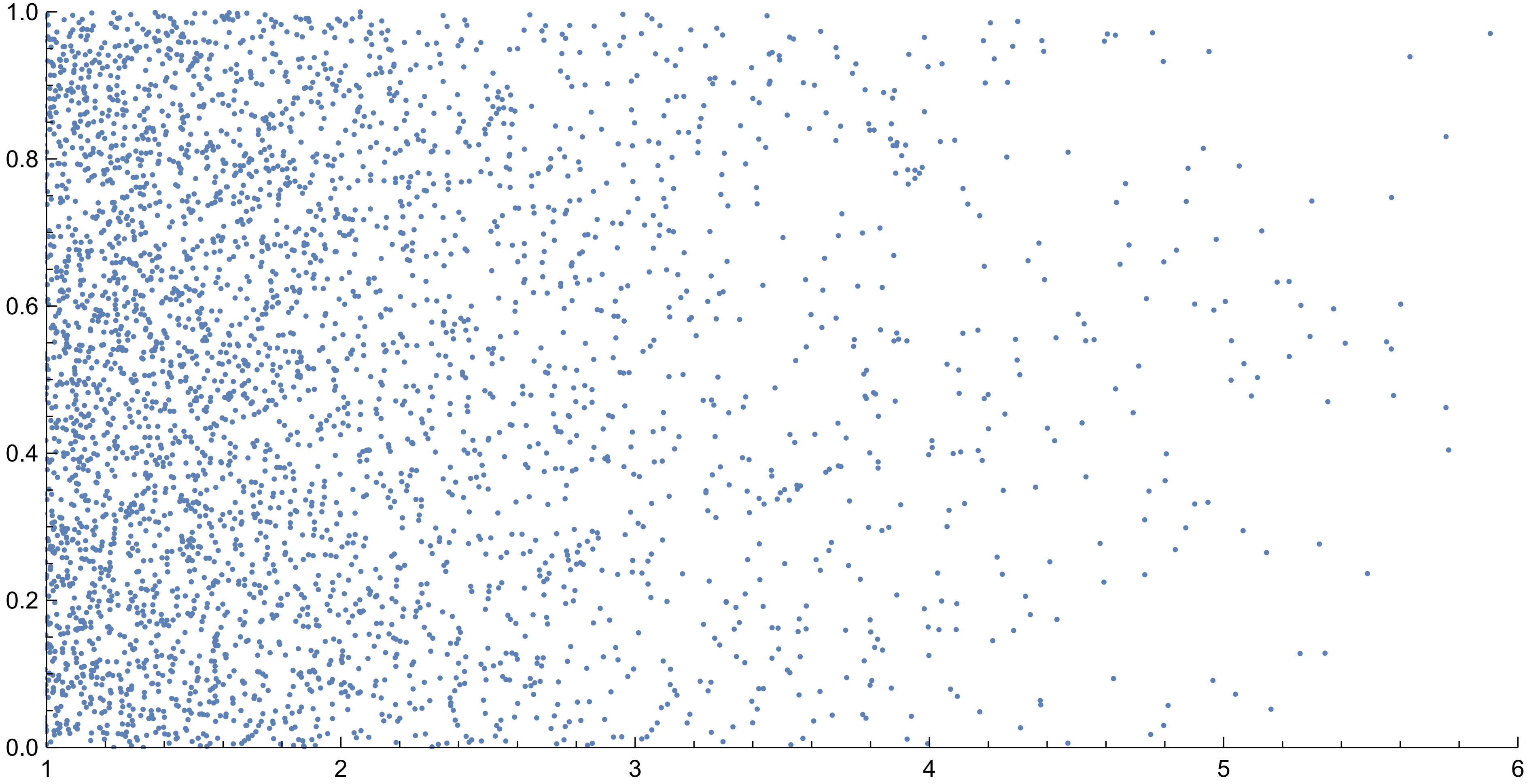
- Our assumption is concerned with the distribution of points (λ_i, θ_i) restricted to a strip $[R - \Delta R, R) \times [0,1)$ for $\Delta R > 0$ fixed and $R \to \infty$



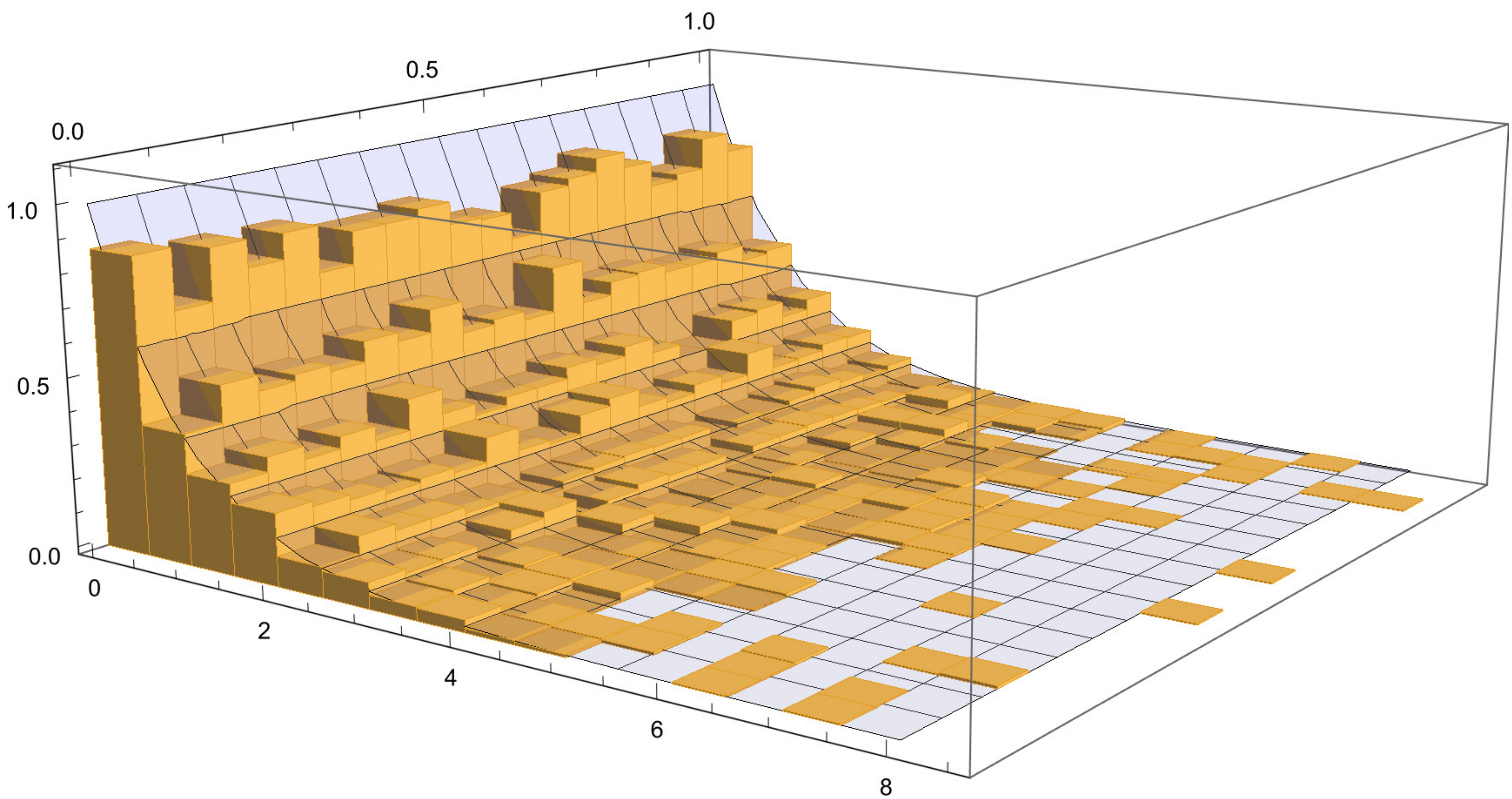
Scatter plots of (λ_i, θ_i) in the strip $[R - \Delta R, R) \times [0, 1)$ for $R = \pi \times 100^2$, with $\Delta R = 10^4$. For large R we expect the point set to be modelled by a Poisson point process.



Scatter plots of (λ_i, θ_i) in the strip $[R - \Delta R, R) \times [0, 1)$ for $R = \pi \times 500^2$, with $\Delta R = 10^4$. For large R we expect the point set to be modelled by a Poisson point process.



Scatter plot for the sequence $(\lambda_{i+1} - \lambda_i, \theta_i)$ for $R = \pi \times 500^2$ and $\Delta R = 10^4$



Histogram for the sequence $(\lambda_{i+1} - \lambda_i, \theta_i)$ for $R = \pi \times 500^2$ and $\Delta R = 10^4$

Theoretical evidence

- Our key assumption can be established for two-point correlations, in the case of random, generic and Diophantine $\boldsymbol{\alpha} \in \mathbb{R}^d$ (JM, Annals Math 2003, Duke Math J 2002)

- Can be used to prove macroscopic limit up to order λ^2 (Griffin & JM, Pure & Applied Analysis 2019)

Outlook

- Can our hypothesis on the Poisson nature of

$$\|\boldsymbol{m}+\boldsymbol{\alpha}\|^2, \quad \boldsymbol{m}\in\mathbb{Z}^d$$

 can be made rigorous?

- Is the long-time limit of the macroscopic process (super-) diffusive?

  (Cf. superdiffusive CLT for kinetic limit of classical periodic Lorentz gas, JM & Toth 2016)

- Other scalling limits: $h \ll r$ or $r \ll h$

- Extension to quasicrystals or other scatterer configurations with long-range correlations

Further reading

- J. Griffin and J. Marklof, Quantum transport in a low-density periodic potential: homogenisation via homogeneous flows, Pure and Applied Analysis 1 (2019) 571-614

- J. Griffin and J. Marklof, Quantum transport in a crystal with short-range interactions: The Boltzmann-Grad limit, Journal of Statistical Physics 184 (2021) no. 16; 46pp