Applications of measure rigidity: from number theory to statistical mechanics

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Lecture I: A basic survey

1. What is measure rigidity? What are homogeneous flows?

2. Margulis' proof of the Oppenheim conjecture

3. Quantum chaos

4. Randomness mod 1

5. The Lorentz gas ($\rightarrow$ Lecture II)

6. Frobenius numbers and circulant graphs ($\rightarrow$ Lecture III)

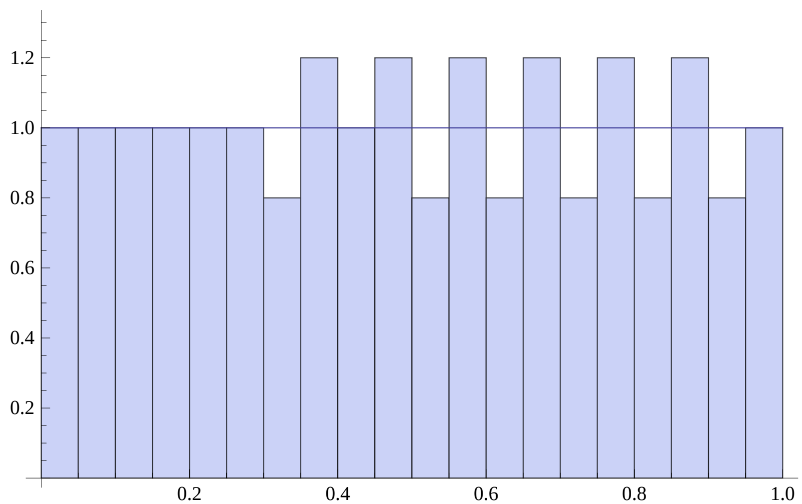
What is measure rigidity?

An illustration: Two proofs of a classic equidistribution theorem

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. (View this as a circle of length one.)

Theorem (Weyl, Sierpinsky, Bohl 1909-10). If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then, for any continuous $f : \mathbb{T} \to \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_{\mathbb{T}} f(x)\, dx.$$



$\alpha = \sqrt{2}$, $N = 100$

Hermann Weyl (1885-1955)

Proof #1—"harmonic analysis" (after Weyl 1914)

By Weierstrass' approximation theorem (*trigonometric polynomials are dense in* $C(\mathbb{T})$) it is enough to show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_{\mathbb{T}} f(x)\, dx \tag{1}$$

holds for finite Fourier series f of the form

$$f(x) = \sum_{k=-K}^{K} c_k\, \mathrm{e}^{2\pi \mathrm{i} kx}.$$

To establish eq. (1) we have to check that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_0 = c_0 = \int_{\mathbb{T}} f(x)\, dx \tag{2}$$

and secondly that for every $k \neq 0$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}^{2\pi \mathrm{i} k\, n\alpha} = \lim_{N\to\infty} \frac{\mathrm{e}^{2\pi \mathrm{i} k\alpha}}{N} \frac{1-\mathrm{e}^{2\pi \mathrm{i} k\, N\alpha}}{1-\mathrm{e}^{2\pi \mathrm{i} k\alpha}} = 0. \tag{3}$$

Eq. (2) is obvious and (3) follows from the formula for the geometric sum (which requires $\alpha \notin \mathbb{Q}$).

Proof #2—"measure rigidity"

The linear functional

$$f \mapsto \nu_N[f] := \frac{1}{N}\sum_{n=1}^{N} f(n\alpha)$$

defines a Borel probability measure on $\mathbb{T}$. Since $\mathbb{T}$ is compact, the sequence $(\nu_N)_N$ is relatively compact, i.e., every subsequence contains a convergent* subsequence $(\nu_{N_j})_j$. Suppose

$$\nu_{N_j} \to \nu.$$

What do we now about the probability measure ν?

in the weak-topology

Define the map

$$T_\alpha : \mathbb{T} \to \mathbb{T}, \qquad x \mapsto x + \alpha.$$

For each $f \in \mathsf{C}(\mathbb{T})$ we have

$$\begin{aligned}\nu_N[f \circ T_\alpha] &= \frac{1}{N}\sum_{n=1}^{N} f((n+1)\alpha) \\ &= \frac{1}{N}\sum_{n=1}^{N} f(n\alpha) + \frac{1}{N} f((N+1)\alpha) - \frac{1}{N} f(\alpha)\end{aligned}$$

and so $\nu_{N_j}[f] \to \nu[f]$ implies

$$\nu_{N_j}[f \circ T_\alpha] \to \nu[f].$$

Since $f \circ T_\alpha \in \mathsf{C}(\mathbb{T})$ we also have

$$\nu_{N_j}[f \circ T_\alpha] \to \nu[f \circ T_\alpha]$$

and so

$$\nu[f \circ T_\alpha] = \nu[f] \qquad \text{for all } f \in \mathsf{C}(\mathbb{T})$$

Conclusion: ν is T_α-invariant, and hence also invariant under the closure of the group

$$\langle T_\alpha \rangle = \{T_\alpha^n : n \in \mathbb{Z}\}.$$

If $\alpha \notin \mathbb{Q}$, it is well known that given any $y \in \mathbb{T}$ there is a subsequence of integers n_i such that

$$n_i \alpha \to y \text{ mod } 1.$$

This implies that the closure of $\langle T_\alpha \rangle$ is in fact the group of *all* translations of $\mathbb{T}$,

$$\overline{\langle T_\alpha \rangle} = \{T_y : y \in \mathbb{T}\}.$$

The only probability measure invariant under this group is Lebesgue measure, i.e. $\nu[f] = \int f(x)dx$. Thus the limit measure ν is unique, and therefore the full sequence must converge:

$$\nu_N[f] = \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) \to \int_{\mathbb{T}} f(x)\, dx.$$

Why measure rigidity?

In the above example there is only one measure ν that is invariant under the transformation T. We call this phenomenon *unique ergodicity*. This is a special case of *measure rigidity* = invariant ergodic measures are not abundant but appear as a countable family.

What is a homogeneous flow?

Consider a homogeneous space of the form

$$X = \Gamma\backslash G = \{\Gamma g : g \in G\}$$

where

- G a Lie group
- Γ a lattice in G

A lattice Γ in G is a discrete subgroup such that there is a fundamental domain $\mathcal{F}_\Gamma$ of the (left) Γ-action on G with finite left Haar measure. This in turn implies that G is unimodular, i.e. left Haar measure=right Haar measure.

A *homogeneous flow* is the action of a one-parameter subgroup $\{\Phi^t\}_{t\in\mathbb{R}}$ of G by right multiplication:

$$X \to X, \qquad x \mapsto x\Phi^t$$

The modular group

Our main example is

- $G = \mathrm{SL}(n, \mathbb{R})$
  (real $n \times n$ matrices with determinant one)
- $\Gamma = \mathrm{SL}(n, \mathbb{Z})$
  (the modular group)

The volume of $X = \Gamma \backslash G$ (with respect to Haar) was first computed for general n by Minkowski.

Hermann Minkowski

(1864-1909)

Examples of one-parameter subgroups:

$$\left\{ \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}) \right\}_{t \in \mathbb{R}} \qquad (\lambda_1 + \ldots + \lambda_n = 0)$$

or

$$\left\{ \begin{pmatrix} 1 & At \\ 0 & 1 \end{pmatrix} \right\}_{t \in \mathbb{R}} \qquad (A \text{ a fixed matrix})$$

The space of Euclidean lattices

Every Euclidean lattice $\mathcal{L} \subset \mathbb{R}^n$ of covolume one can be written as

$$\mathcal{L} = \mathbb{Z}^n M$$

for some M. The bijection

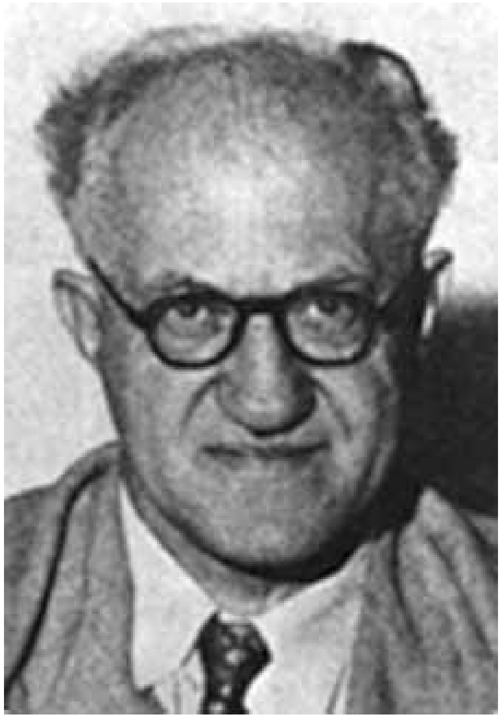
$$\begin{aligned} \Gamma\backslash G &\xrightarrow{\sim} \{\text{lattices of covolume one}\} \\ \Gamma M &\mapsto \mathbb{Z}^d M. \end{aligned}$$

allows us to identify the space of lattices with $\Gamma\backslash G$.

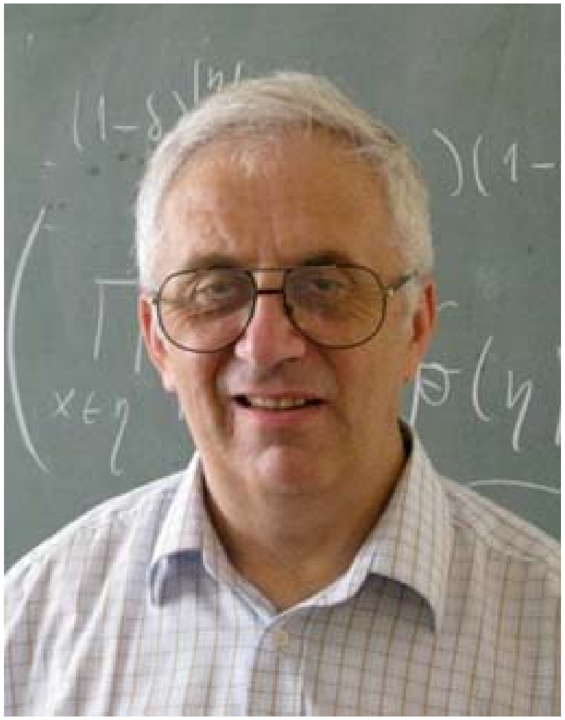
To find the inverse map, note that, for any basis $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_d$ of $\mathcal{L}$, the matrix $M = ({}^{\mathrm{t}}\boldsymbol{b}_1, \ldots, {}^{\mathrm{t}}\boldsymbol{b}_d)$ is in $\mathsf{SL}(n, \mathbb{R})$; the substitution $M \mapsto \gamma M$, $\gamma \in \Gamma$ corresponds to a base change of $\mathcal{L}$.

Margulis' proof of the Oppenheim conjecture

...was the first application of the theory of homogenous flows to a long-standing problem which had resisted attacks from analytic number theory. As we shall see, quantitative versions of the Oppenheim conjecture can be proved by means of measure rigidity.



Alexander Oppenheim (1903-1997)



Gregory Margulis (*1946)

Let

$$Q(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} q_{ij} x_i x_j$$

with $q_{ij} = q_{ji} \in \mathbb{R}$. We say Q is *indefinite* if the matrix (q_{ij}) has both positive and negative eigenvalues, and no zero eigenvalues. If there are p positive and $q = n - p$ negative eigenvalues, we say Q has *signature* (p, q). We say Q is irrational if (q_{ij}) is not proportional to a rational matrix.

Theorem (Margulis 1987). If Q is irrational* and indefinite with $n \geq 3$ then

$$\inf |Q(\mathbb{Z}^n \setminus \{0\})| = 0.$$

Oppenheim's original conjecture (PNAS 1929) was more cautious—it assumed $n \geq 5$. The assumption $n \geq 3$ in the above is however necessary; consider e.g. $Q(x_1, x_2) = x_1^2 - (1 + \sqrt{3})^2 x_2^2$.

*only required when $n = 3$ or 4

Values of quadratic forms vs. orbits of homogeneous flows

The key first step in Margulis' proof is to translate the problem to a question in homogeneous flows. This observation goes back to a paper by Cassels and Swinnerton-Dyer (1955) and was rediscovered by Raghunatan in the mid-1970s, leading him to formulate to influential conjectures on orbit closures of unipotent flows.

It is instructive to explain this first step in the case $n = 2$.

- **Observation 1:** Basic linear algebra shows that there is $M \in \mathrm{SL}(2,\mathbb{R})$ and $\lambda \neq 0$ so that

$$Q(\boldsymbol{x}) = \lambda Q_0(\boldsymbol{x}M), \qquad Q_0(x_1,x_2) := x_1 x_2.$$

  ($\boldsymbol{x} = (x_1,x_2)$ is viewed as a row vector)
- **Observation 2:**

$$Q_0(\boldsymbol{x}\Phi^t) = Q_0(\boldsymbol{x}), \qquad \text{for all} \quad \Phi^t = \begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix}, \quad t \in \mathbb{R}$$

- **Need to show:** Given any $\epsilon > 0$ find $t > 0$ so that the lattice $\mathbb{Z}^2 M \Phi^t$ contains a non-zero vector $\boldsymbol{y}$ such that

$$|Q_0(\boldsymbol{y})| < \epsilon.$$

- **Observation 3:** The above holds if the orbit

$$\{\mathbb{Z}^2 M \Phi^t\}_{t\in\mathbb{R}} \simeq \{\Gamma M \Phi^t\}_{t\in\mathbb{R}}$$

  is dense in $\Gamma\backslash G$ *. . . and this is where things go wrong for* $n = 2$ (but works out for $n \geq 3$ since then the action orthogonal group of Q_0 produces a dense orbit in $\Gamma\backslash G$).

Quantitative versions of the Oppenheim conjecture

Margulis in fact proved the stronger statement that for any irrational indefinite form Q in $n \geq 3$ variables $\overline{Q(\mathbb{Z}^n)} = \mathbb{R}$. Apart from being dense, can we say more about the distribution of the values of Q? The following theorems gives the answer for forms of signature (p,q) with $n \geq 4$ and $p \geq 3$.

Theorem (Eskin, Margulis & Mozes, Annals of Math 1998). If Q is irrational and indefinite with $p \geq 3$ then for any $a < b$

$$\lim_{T\to\infty} \frac{\#\{\boldsymbol{m} \in \mathbb{Z}^n : \|\boldsymbol{m}\| < T,\ a < Q(\boldsymbol{m}) < b\}}{\mathrm{vol}\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| < T,\ a < Q(\boldsymbol{x}) < b\}} = 1.$$

For signature (2,2) the statement only holds if Q is not too well approximable by rational forms (Eskin, Margulis, Mozes 2005) (which is true for almost all forms). The problem is open for signature (2,1).

Ratner's theorem (Annals of Math, 1991)

The key ingredient in the previous theorem is Ratner's celebrated classification of measures that are invariant and ergodic under unipotent flows. Ratner proves that any such measure ν is supported on the orbit

$$\Gamma\backslash\Gamma H \subset \Gamma\backslash G$$

for some (unique) closed connected subgroup H of G such that $\Gamma_H := \Gamma \cap H$ is a lattice in H and thus

$$\Gamma\backslash\Gamma H \simeq \Gamma_H\backslash H$$

is an embedded homogeneous space with finite H-invariant measure $\nu = \nu_H$ (the Haar measure of H).

Marina Ratner (*1938)

Examples:

$\left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right\}_{t\in\mathbb{R}}$ unipotent

$\left\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\right\}_{t\in\mathbb{R}}$ not

Quantum chaos

Quantum ergodicity vs. scars

$$-\Delta\varphi_j = \lambda_j \varphi_j, \quad \varphi_j\big|_{\partial\mathcal{D}} = 0, \quad 0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots \to \infty$$

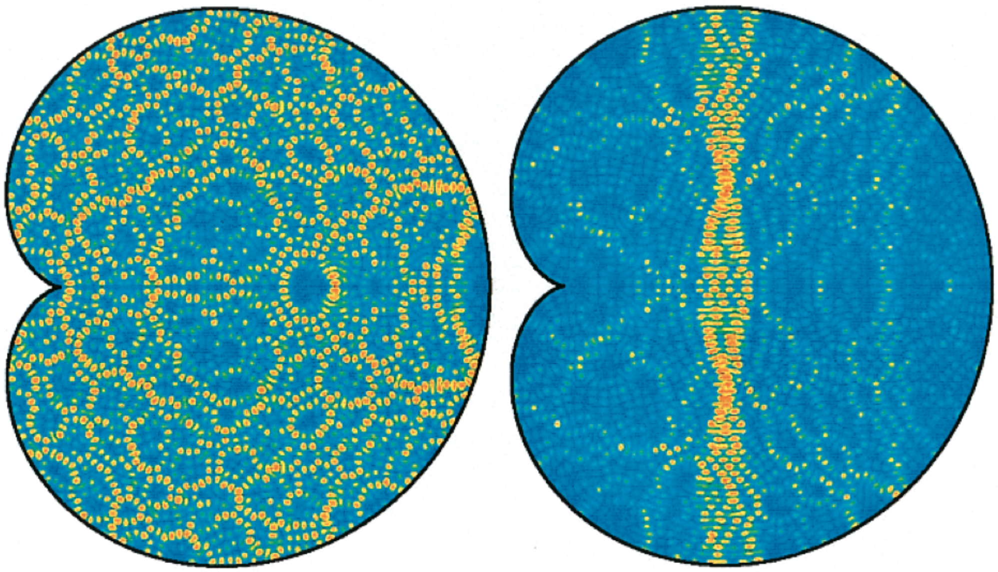


Fig. 1. Probability density of the 1,816th and 1,817th odd eigenstate of a quantum particle trapped in a chaotic heart-shaped region with Dirichlet boundary conditions. The probability of finding the particle at a given point is low in blue regions and high in red regions.

Numerics: A. Bäcker, TU Dresden

Random matrix conjecture (Bohigas, Giannoni & Schmit 1984)

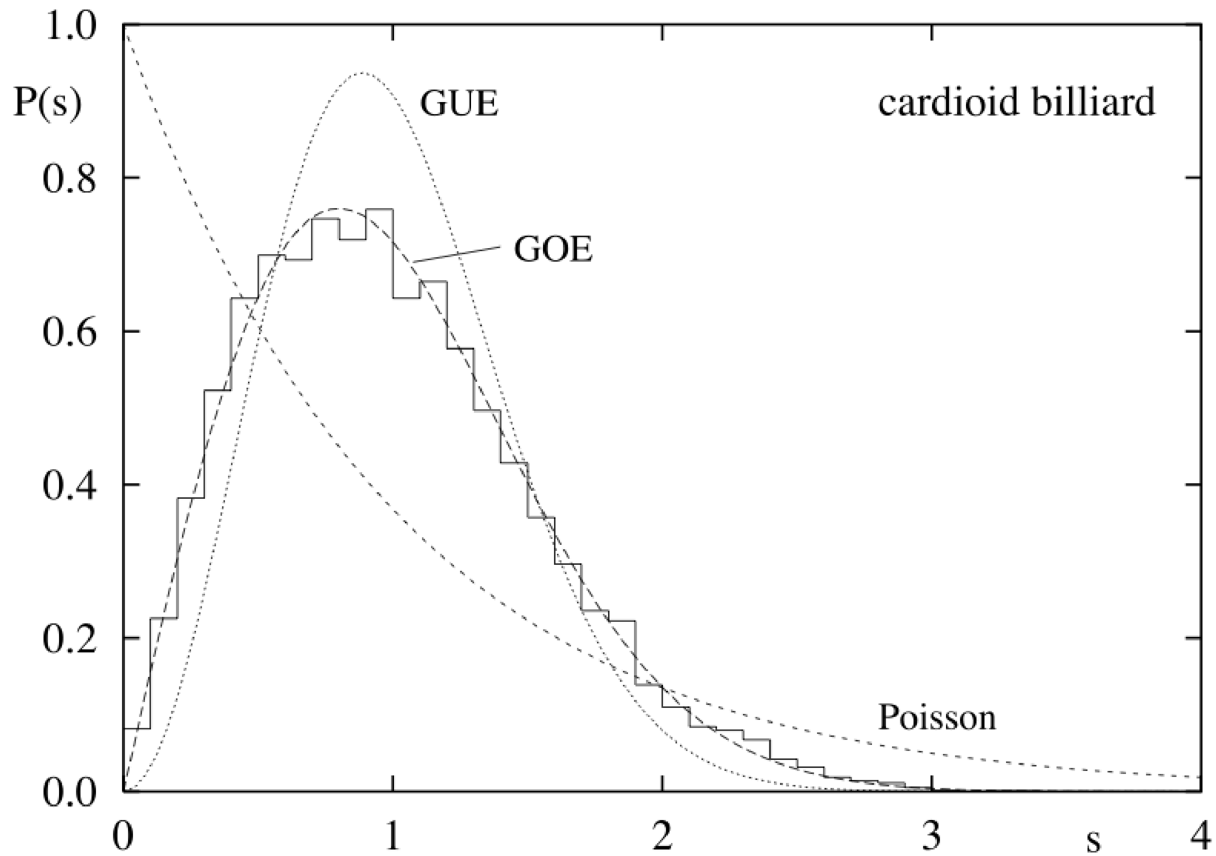


Fig. 2. Level spacing distribution for the energy spectrum of a quantum particle in the chaotic heart-shaped region of Fig. 1 vs. the level spacing distribution for Gaussian Unitary Ensemble, Gaussian Orthogonal Ensemble, and Poisson, respectively.

Berry-Tabor conjecture (Berry & Tabor 1977)

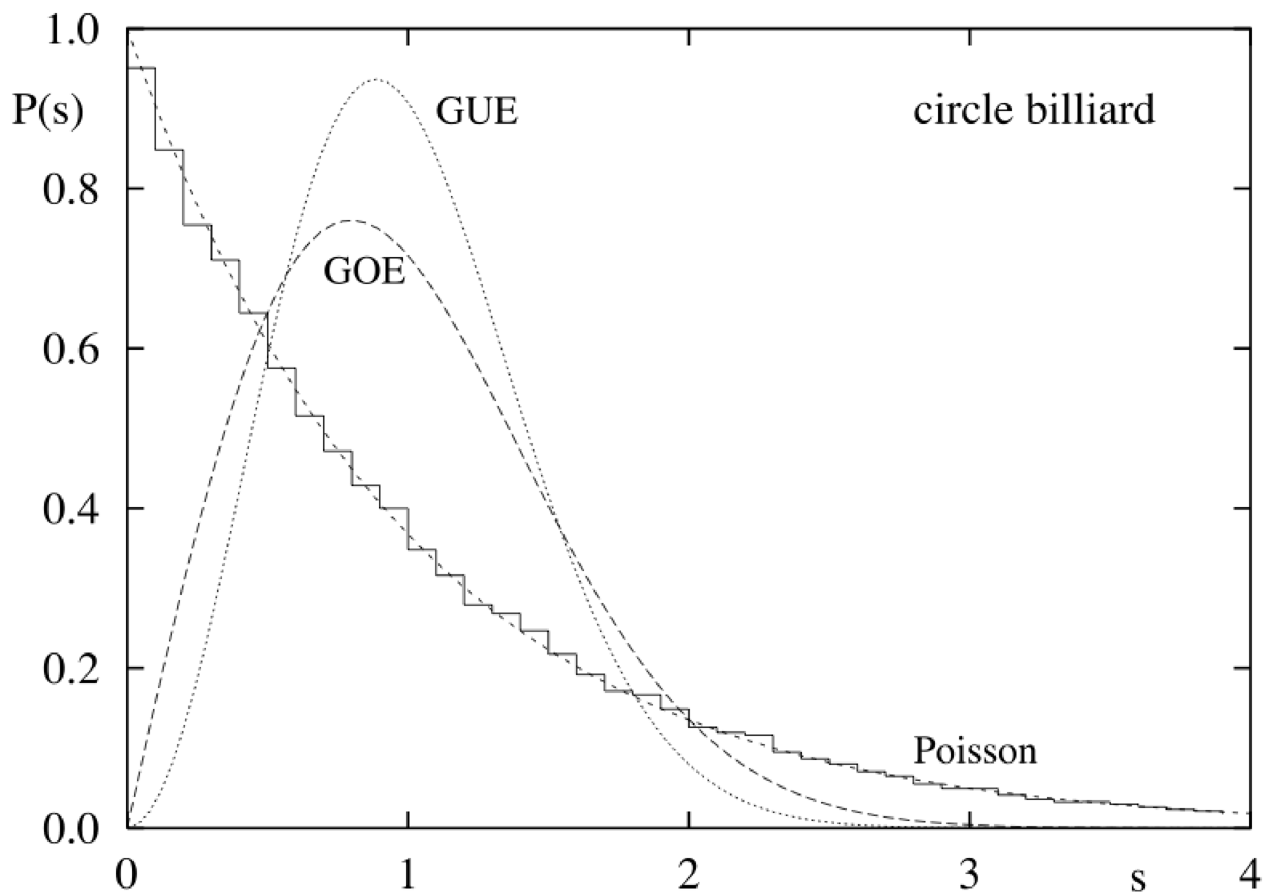


Fig. 3. Level spacing distribution for the energy spectrum of a quantum particle in a circular region vs. the level spacing distribution for Gaussian Unitary Ensemble, Gaussian Orthogonal Ensemble, and Poisson, respectively.

Quantum (unique) ergodicity

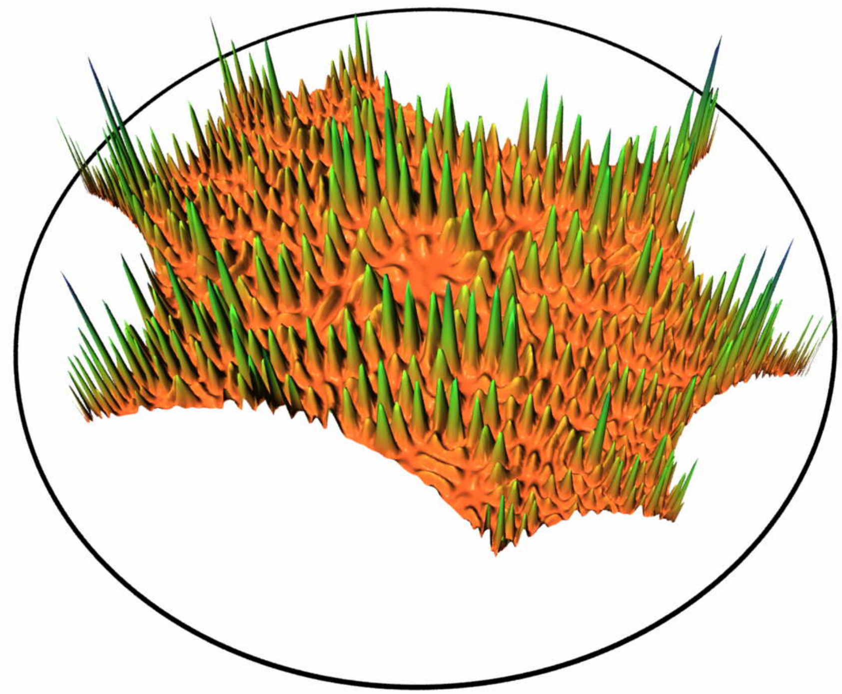
Let Δ be the Laplacian on a compact surface of constant negative curvature

$$-\Delta\varphi_j = \lambda_j\varphi_j, \quad \|\varphi_i\|_2 = 1$$

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \to \infty$$

What are possible limits of the sequence of probability measures $|\varphi_j(z)|^2 d\mu(z)$?

We know (by microlocal analysis): Any (microlocal lift) of a limit measure must be invariant under the geodesic flow. (Doesn't help much—many measures have this property!)



Numerics: R. Aurich, U Ulm

Quantum (unique) ergodicity

- Rudnick & Sarnak (Comm Math Phys 1994) : conjecture

$$|\varphi_j(z)|^2 d\mu(z) \to d\mu(z)$$

  along full sequence ("quantum unique ergodicity" or QUE).

- Shnirelman-Zelditch-Colin de Verdiere Theorem:

$$|\varphi_{j_i}(z)|^2 d\mu(z) \to d\mu(z)$$

  along a density-one subsequence j_i ("quantum ergodicity"); holds in much greater generality and only requires ergodicity of geodesic flow.

- E. Lindenstrauss (Annals Math 2006): If surface is arithmetic (of congruence type) then we have QUE. Proof uses measure rigidity of action of Hecke correspondences and geodesic flow.

- Anantharaman (Annals Math 2008) & with Nonnenmacher (Annals Fourier 2007): Kolmogorov-Sinai entropy of any limit is at least half the entropy of $d\mu(z)$. This means no strong scars.

Berry-Tabor conjecture: eigenvalue statistics for flat tori

Let $\triangle$ be the Laplacian on the flat torus $\mathbb{R}^d/\mathcal{L}$ ($\mathcal{L}$ =lattice of covolume 1)

$$-\Delta\varphi_j = \lambda_j\varphi_j, \quad \|\varphi_i\|_2 = 1$$

with eigenfunctions

$$\{\varphi_j(\boldsymbol{x})\} = \{\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{y}} : \boldsymbol{y} \in \mathcal{L}^*\}$$

and eigenvalues (counted with multiplicity)

$$\{\lambda_j\} = \{\|\boldsymbol{m}\|^2 : \boldsymbol{m} \in \mathcal{L}^*\}$$

Note

$$\#\{\lambda_j < \lambda\} \sim V_d\,\lambda^{d/2} \quad (\lambda \to \infty), \qquad V_d := \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$$

To compare the statistics with a Poisson process of intensity one, rescale the spectrum by setting

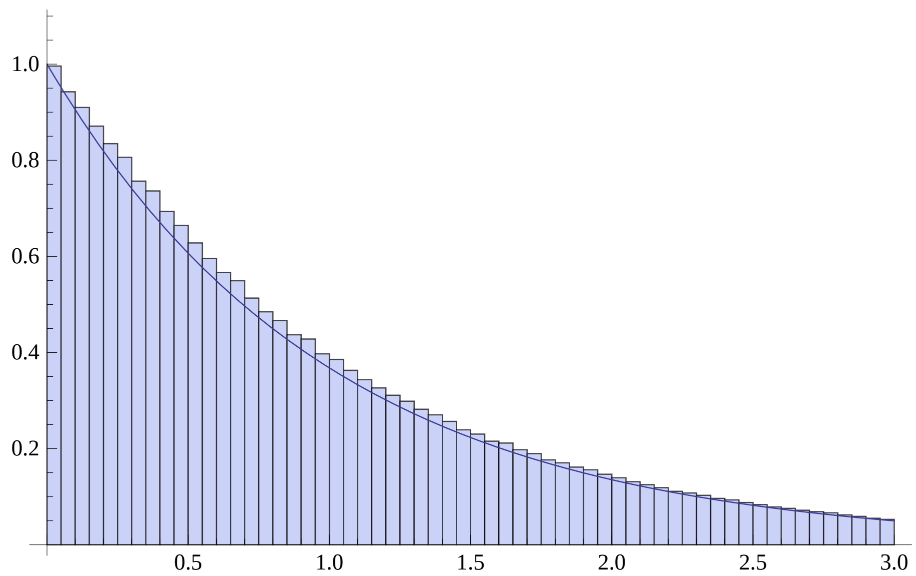
$$\xi_j = V_d\,\lambda_j^{d/2} \quad \text{so that} \quad \#\{\xi_j < \xi\} \sim \xi \quad (\xi \to \infty)$$

Gap distribution

Assume the lattice $\mathcal{L}$ is "generic". Is it true that for any interval $[a, b]$

$$\lim_{N\to\infty} \frac{\#\{j \le N : a < \xi_{j+1} - \xi_j < b\}}{N} = \int_a^b \mathrm{e}^{-s} ds \quad ?$$

(which is the answer for a Poisson process)



gap distribution for the recaled $\{\lambda_j\} = \{m^2 + \sqrt{2}\, n^2 : m, n \in \mathbb{N}\}$, $N = 2,643,599$

WE DON'T KNOW HOW TO PROVE THIS!

Simpler but still not easy: Two-point statistics

Is it true that for any interval $[a, b]$

$$\lim_{N\to\infty} \frac{\#\{(i,j) : i \neq j \leq N,\ a < \xi_i - \xi_j < b\}}{N} = b - a \quad ?$$

- Eskin, Margulis & Mozes (Annals Math 2005): YES for $d = 2$ and under diophantine conditions on $\mathcal{L}$—this reduces to quantitative Oppenheim for quadratic forms of signature (2,2)

- VanderKam (Duke Math J 1999, CMP 2000): YES for any d and almost all $\mathcal{L}$ (in measure); follows idea by Sarnak for $d = 2$

- JM (Duke Math J 2002, Annals Math 2003): YES for

$$\{\xi_j\} = \{V_d \|\boldsymbol{m} - \boldsymbol{\alpha}\|^d : \boldsymbol{m} \in \mathbb{Z}^d\}$$

  and any $d \geq 2$, provided $\boldsymbol{\alpha} \in \mathbb{R}^d$ is diophantine of type $(d-1)/(d-2)$. The proof is different from EMM's approach. It uses theta series and Ratner's measure classification theorem.

Randomness mod 1

Gap and two-point statistics

Let

$$\begin{array}{ccccc} \xi_{11} & & & & \\ \xi_{21} & \xi_{22} & & & \\ \vdots & \vdots & \ddots & & \\ \xi_{N1} & \xi_{N2} & \cdots & \xi_{NN} & \\ \vdots & \vdots & & & \ddots \end{array}$$

be a triangular array numbers $\xi_j = \xi_{Nj}$ in $[0,1]$ which is ordered ($\xi_j \le \xi_{j+1}$) and uniformly distributed, i.e. for any interval $[a,b] \subset [0,1]$

$$\lim_{N\to\infty} \frac{\#\{j \le N : a < \xi_j < b\}}{N} = b - a.$$

Do these have a Poisson limit? That is, for the gaps

$$\lim_{N\to\infty} \frac{\#\{j \le N : \frac{a}{N} < \xi_{j+1} - \xi_j < \frac{b}{N}\}}{N} = \int_a^b \mathrm{e}^{-s} ds \quad ?$$

For the two point statistics

$$\lim_{N\to\infty} \frac{\#\{(i,j) : i \neq j \le N,\ \frac{a}{N} < \xi_i - \xi_j < \frac{b}{N}\}}{N} = b - a \quad ?$$

Some results for fractional parts

- For almost all α, $\xi_j = \{2^j\alpha\}$ is Poisson for gaps and all n-point statistics (Rudnick & Zaharescu, Forum Math 2002); applies to other lacunary sequences in place of 2^n

- For almost all α, $\xi_j = \{j^2\alpha\}$ has Poisson pair correlation (Rudnick & Sarnak, CMP 1998); applies also to other polynomials such as $\xi_j = \{j^3\alpha\}$ etc

  for certain well approximable α, there are subsequences of N such that the gap statistics of $\xi_j = \{j^2\alpha\}$ both converges to Poisson and at the same time to a singular limit (Rudnick, Sarnak & Zaharescu, Inventiones 2001);

  algorithmic characterization of those α for which the two-point statistics is Poisson is given by Heath-Brown (Math Proc Camb Phil Soc 2010).

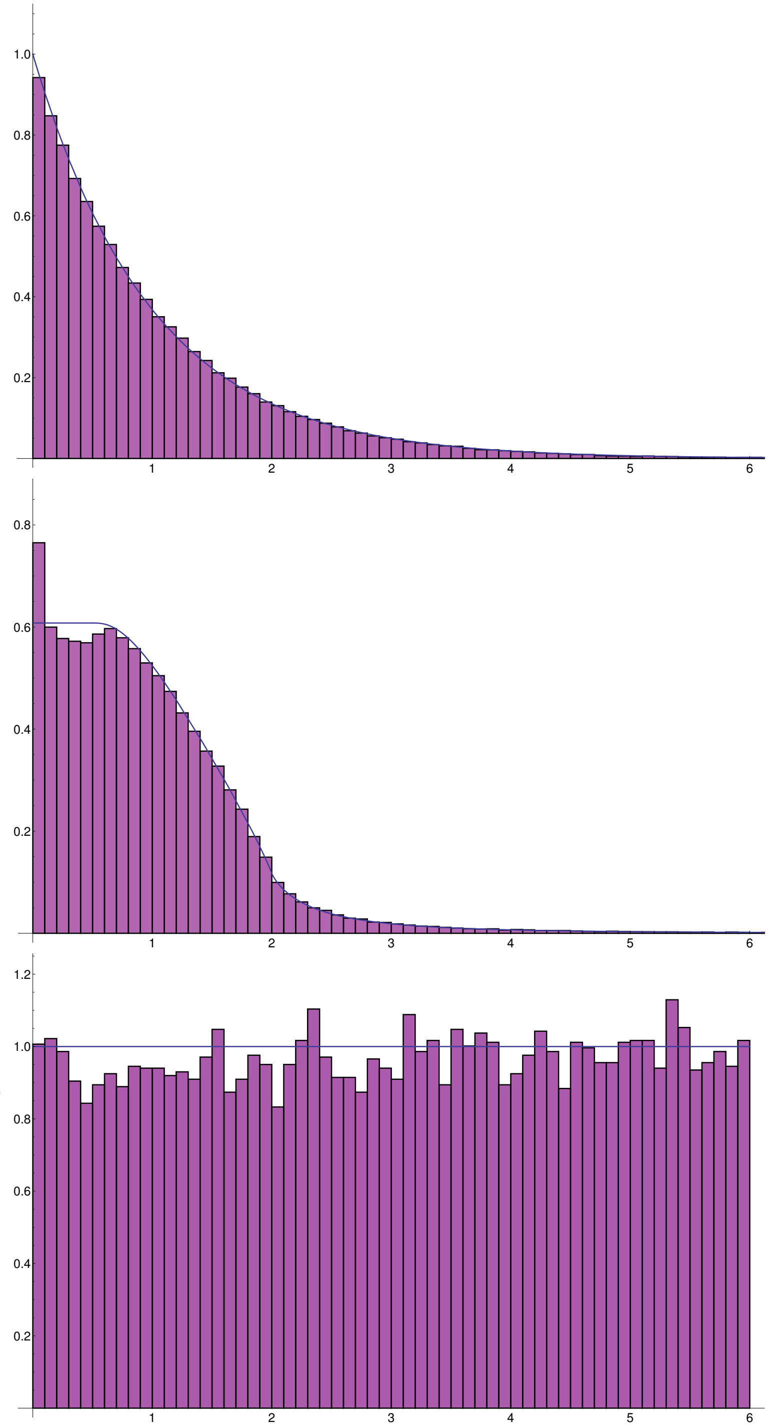
  WE DON'T KNOW WHETHER EVEN THE TWO-POINT STATISTICS ARE POISSON FOR $\alpha = \sqrt{2}$

Fractional parts of small powers

- For fixed $0 < \beta < 1$, $\beta \neq \frac{1}{2}$, the gap and two-point statistics of $\{n^{\beta}\}$ look Poisson numerically—-NO PROOFS! $\beta = \frac{1}{3} \rightarrow$

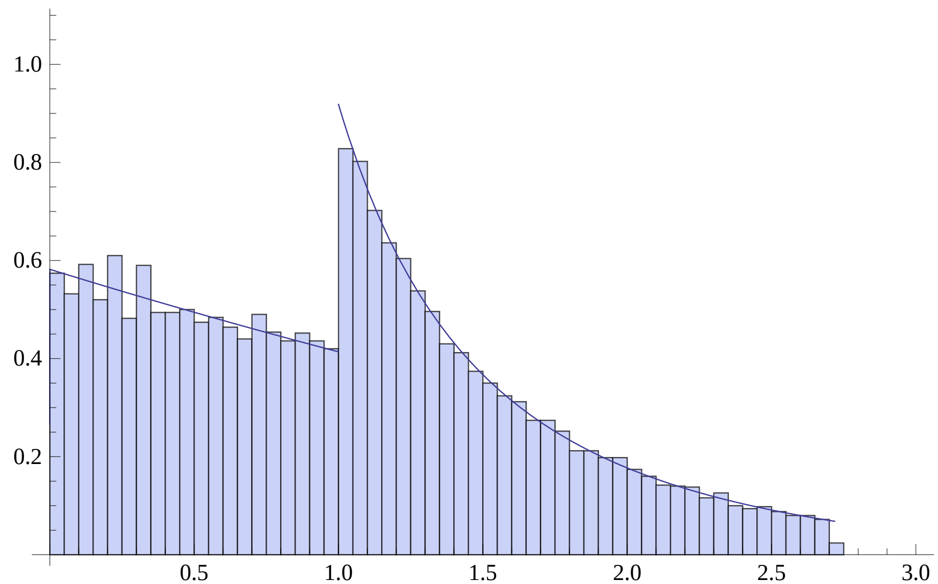
- For $\beta = \frac{1}{2}$, Elkies & McMullen (Duke Math J 2004) have shown that the gap distribution exists, and derived an explicit formula which is clearly different from the exponential. Their proof uses Ratner's measure classification theorem!

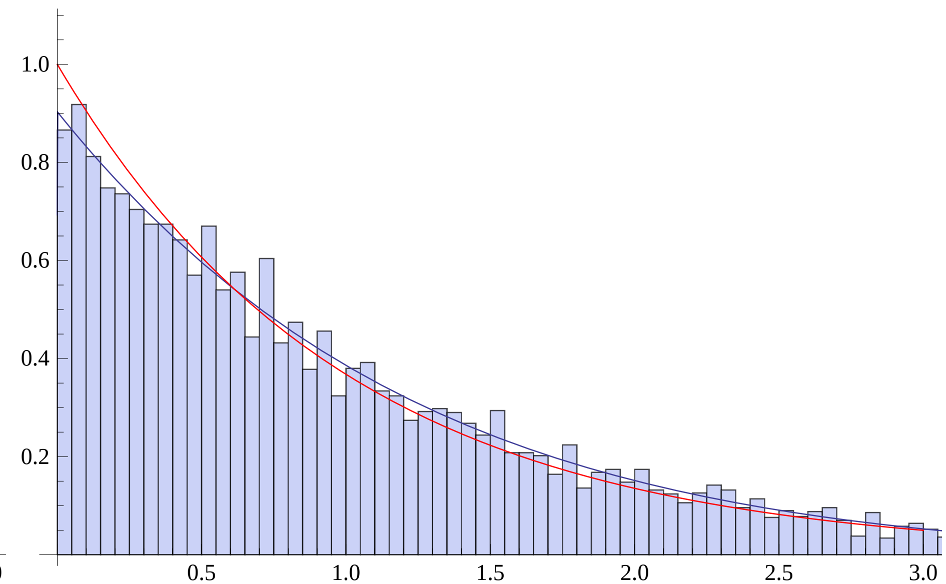
  At the same time, the two-point function converges to the Poisson answer (with El Baz & Vinogradov, preprint 2013). The proof requires upper bounds for the equidistribution of certain unipotent flows with respect to unbounded test functions.



A great student project—gaps between logs

- Study the distribution of gaps between the fractional parts of $\log n$:



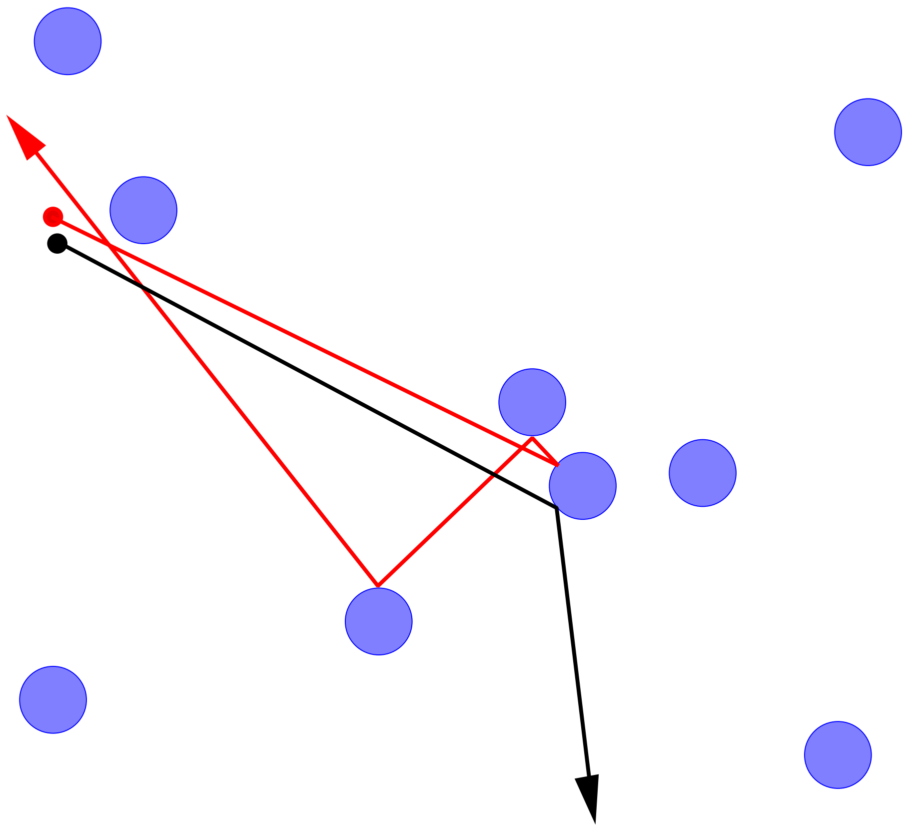


gaps for natural base e —— gaps for base $\mathrm{e}^{1/5}$ vs. exponential distribution

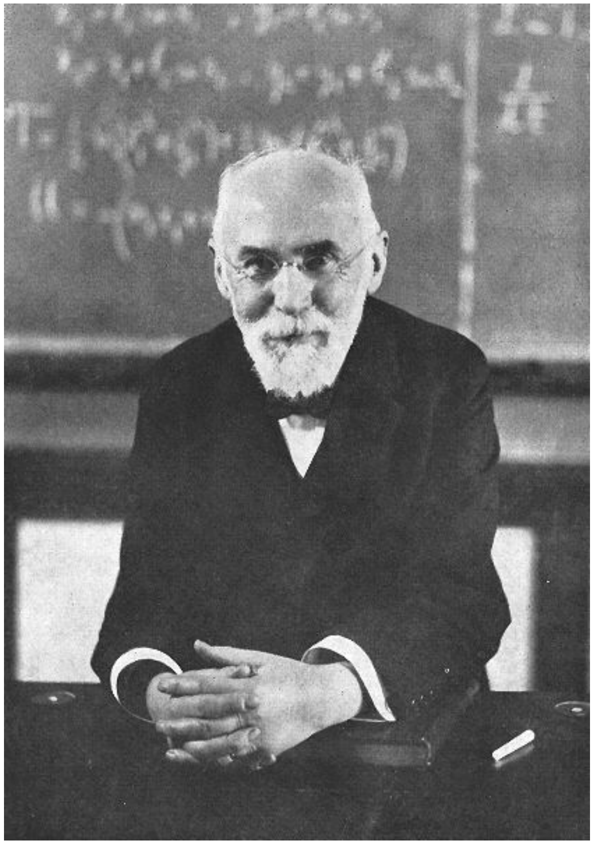
The proof is elementary and exploits Weyl equidistribution.

For details see JM & Strömbergsson, Bull LMS 2013

The Lorentz gas ($\longrightarrow$ Lecture II)



Arch. Neerl. (1905)



Hendrik Lorentz (1853-1928)

The Boltzmann-Grad limit

- Consider the dynamics in the limit of small scatterer radius ρ

- $\big(\boldsymbol{q}(t), \boldsymbol{v}(t)\big)$ = "microscopic" phase space coordinate at time t

- A dimensional argument shows that, in the limit $\rho \to 0$, the mean free path length (i.e., the average time between consecutive collisions) scales like $\rho^{-(d-1)}$ (= 1/total scattering cross section)

- We thus re-define position and time and use the "macroscopic" coordinates

$$\big(\boldsymbol{Q}(t), \boldsymbol{V}(t)\big) = \big(\rho^{d-1}\boldsymbol{q}(\rho^{-(d-1)}t), \boldsymbol{v}(\rho^{-(d-1)}t)\big)$$

The linear Boltzmann equation

- Time evolution of initial data $(\boldsymbol{Q}, \boldsymbol{V})$:

$$(\boldsymbol{Q}(t), \boldsymbol{V}(t)) = \Phi^t_\rho(\boldsymbol{Q}, \boldsymbol{V})$$

- Time evolution of a particle cloud with initial density $f \in \mathrm{L}^1$:

$$f^{(\rho)}_t(\boldsymbol{Q}, \boldsymbol{V}) := f\Big(\Phi^{-t}_\rho(\boldsymbol{Q}, \boldsymbol{V})\Big)$$

In his 1905 paper Lorentz suggested that $f^{(\rho)}_t$ is governed, as $\rho \to 0$, by the linear Boltzmann equation:

$$\boxed{\left[\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{Q}}\right] f_t(\boldsymbol{Q}, \boldsymbol{V}) = \int_{\mathrm{S}^{d-1}_1} \Big[f_t(\boldsymbol{Q}, \boldsymbol{V}_0) - f_t(\boldsymbol{Q}, \boldsymbol{V})\Big] \sigma(\boldsymbol{V}_0, \boldsymbol{V}) \, d\boldsymbol{V}_0}$$

where the collision kernel $\sigma(\boldsymbol{V}_0, \boldsymbol{V})$ is the cross section of the individual scatterer. E.g.: $\sigma(\boldsymbol{V}_0, \boldsymbol{V}) = \frac{1}{4}\|\boldsymbol{V}_0 - \boldsymbol{V}\|^{3-d}$ for specular reflection at a hard sphere

The linear Boltzmann equation—rigorous proofs

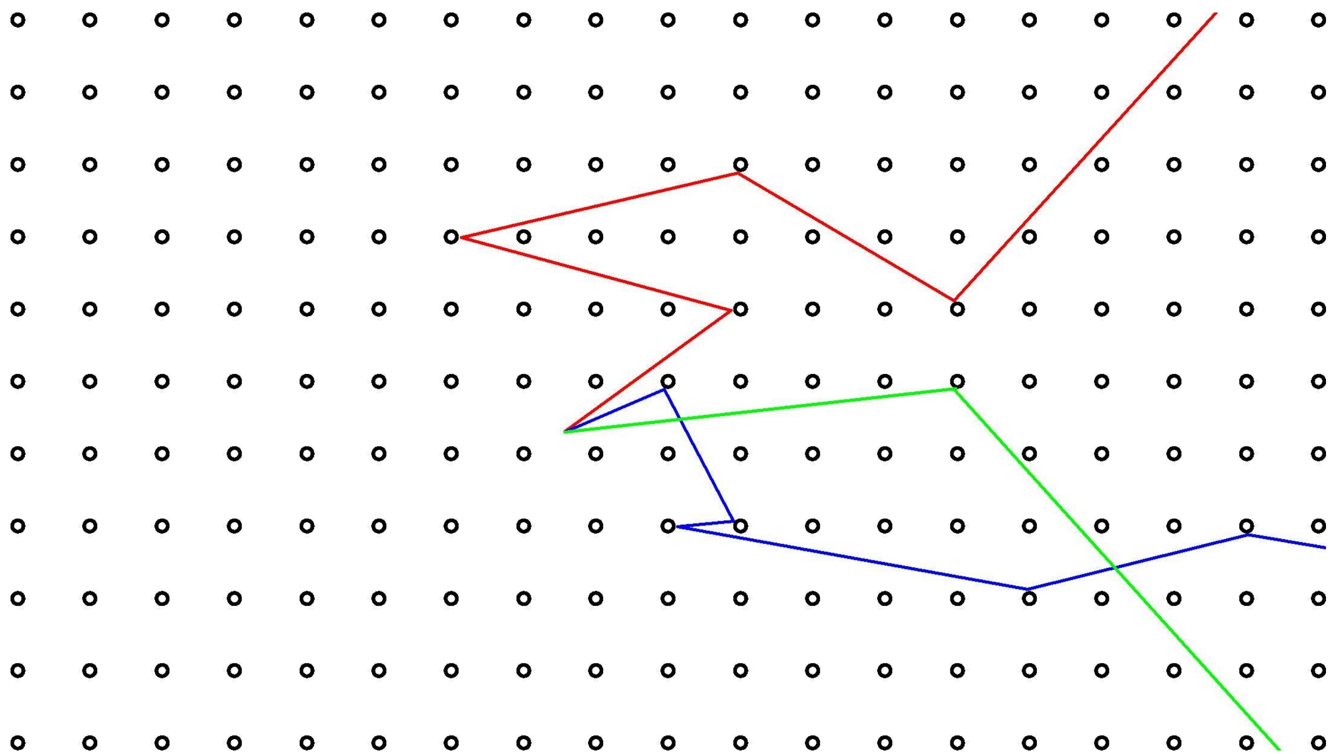
- Galavotti (Phys Rev 1969 & report 1972): Poisson distributed hard-sphere scatterers

- Spohn (Comm Math Phys 1978): extension to more general random scatterer configurations and potentials

- Boldrighini, Bunimovich and Sinai (J Stat Phys 1983): prove convergence for almost every scatterer configuration

The periodic Lorentz gas

Here the linear Boltzmann equation breaks down and a new transport equation will govern the Boltzmann-Grad limit (with Strömbergsson, Annals of Math 2010 & 2011, GAFA 2011).



The key ingredient in the proofs are equidistribution results for flows of homogeneous spaces. I will explain this tomorrow.

Frobenius numbers and circulant graphs (→ Lecture III)

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