Lecture II:
The Lorentz gas

Applications of measure rigidity: from number theory to statistical mechanics
Simons Lectures, Stony Brook
October 2013

Jens Marklof

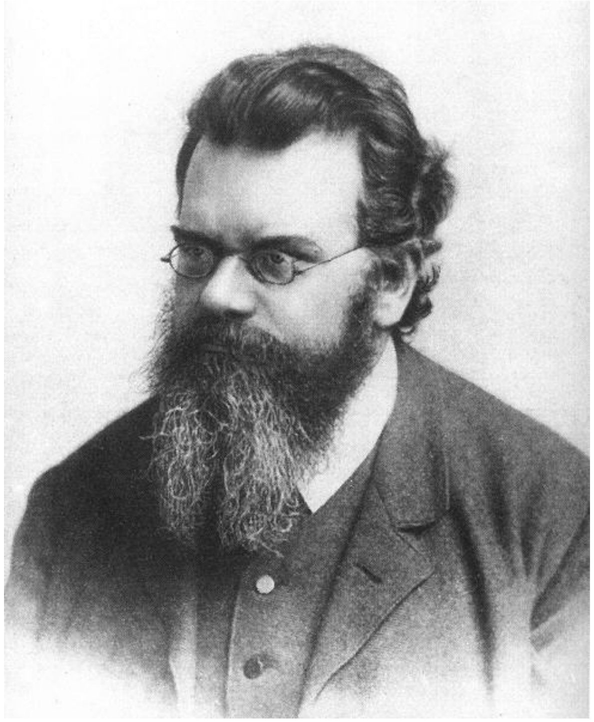
University of Bristol
http://www.maths.bristol.ac.uk

supported by Royal Society and ERC

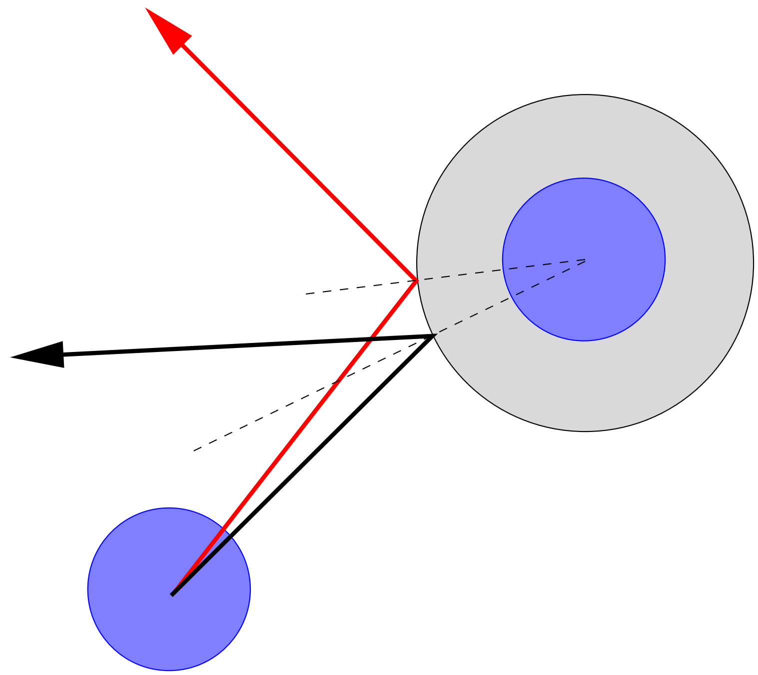
Boltzmann's statistical mechanics

Boltzmann proposed to explain the motion of a gas cloud by using the dynamics of microscopic particles—atoms and molecules, whose existence was highly disputed during Boltzmann's lifetime.

In his 1872 paper, Boltzmann derived the famous nonlinear *Boltzmann equation* in the limit of low particle densities, assuming that the dynamics of the colliding gas molecules is chaotic.



Ludwig Boltzmann (1844-1906)



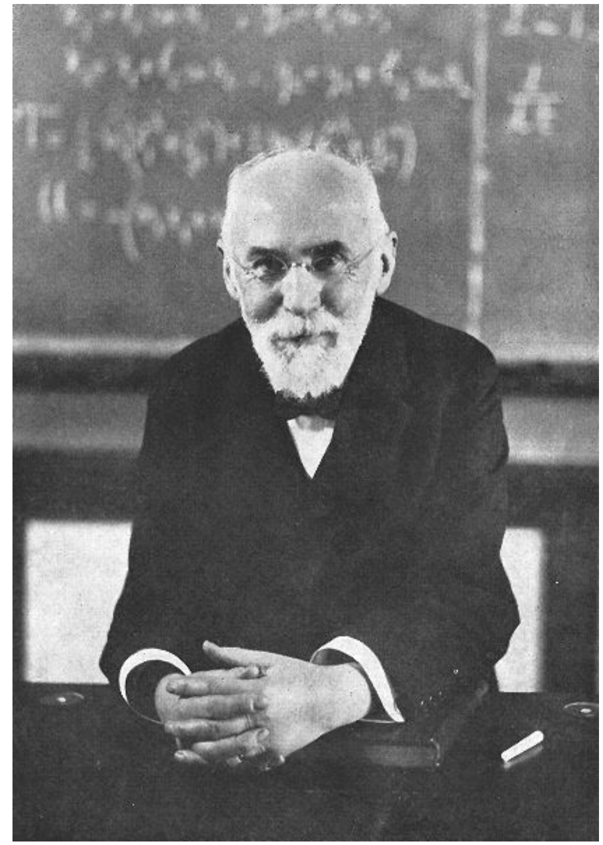
The Boltzmann gas: Sensitive dependence in two-molecule collision.

The first rigorous justification of the Boltzmann equation was given by Oscar Lanford in 1975 for the dynamics over very short time intervals (a fraction of the mean collision time). The problem for longer time scales is still wide open.

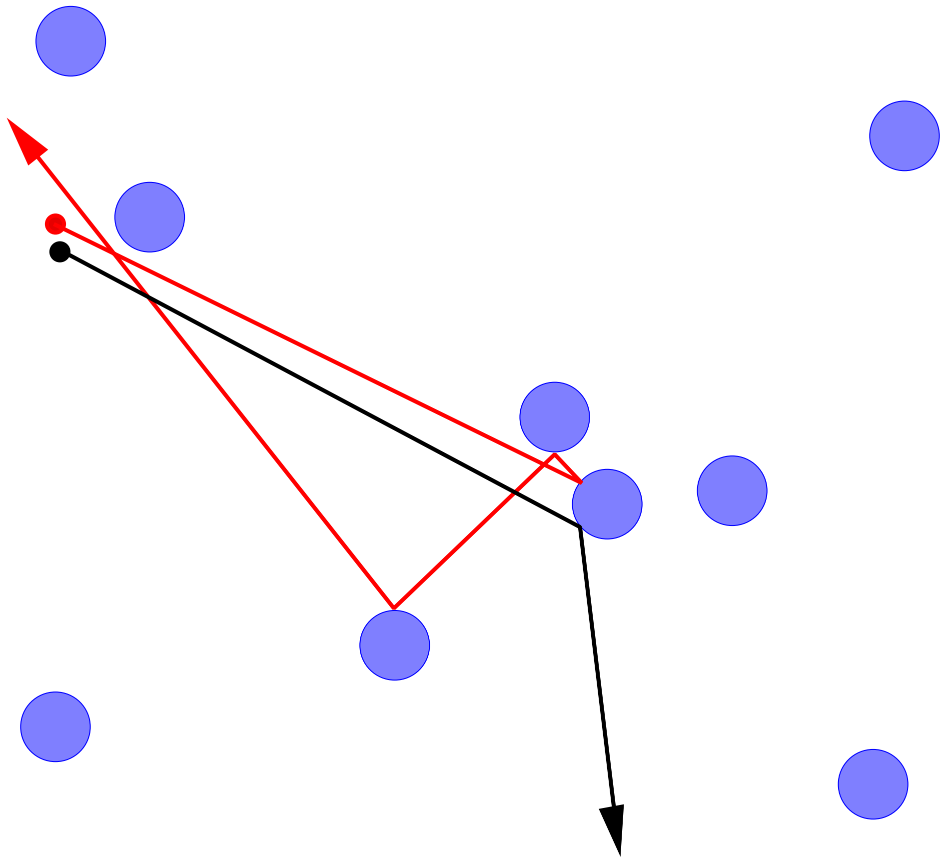
The Lorentz gas

In an attempt to describe the evolution of a dilute electron gas in a metal, Lorentz proposed in 1905 a model, where the heavier atoms are assumed to be fixed, whereas the electrons are interacting with the atoms but not with each other. For simplicity, Lorentz assumed like Boltzmann that the atoms can be modeled by elastic spheres.

The Lorentz gas is still one of the iconic models for chaotic diffusion, both in a random and periodic configuration of scatterers.

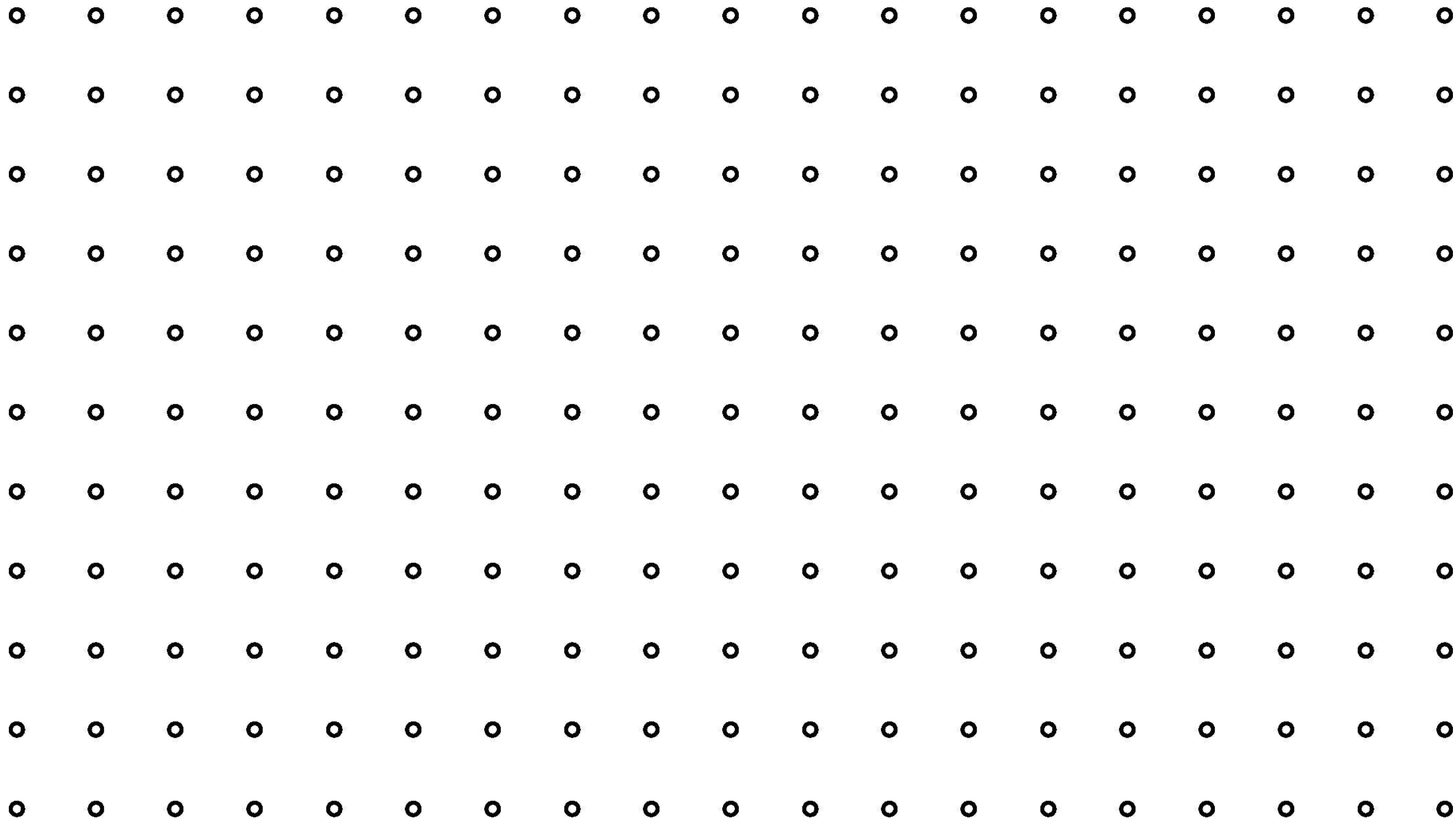


Hendrik Lorentz (1853-1928)

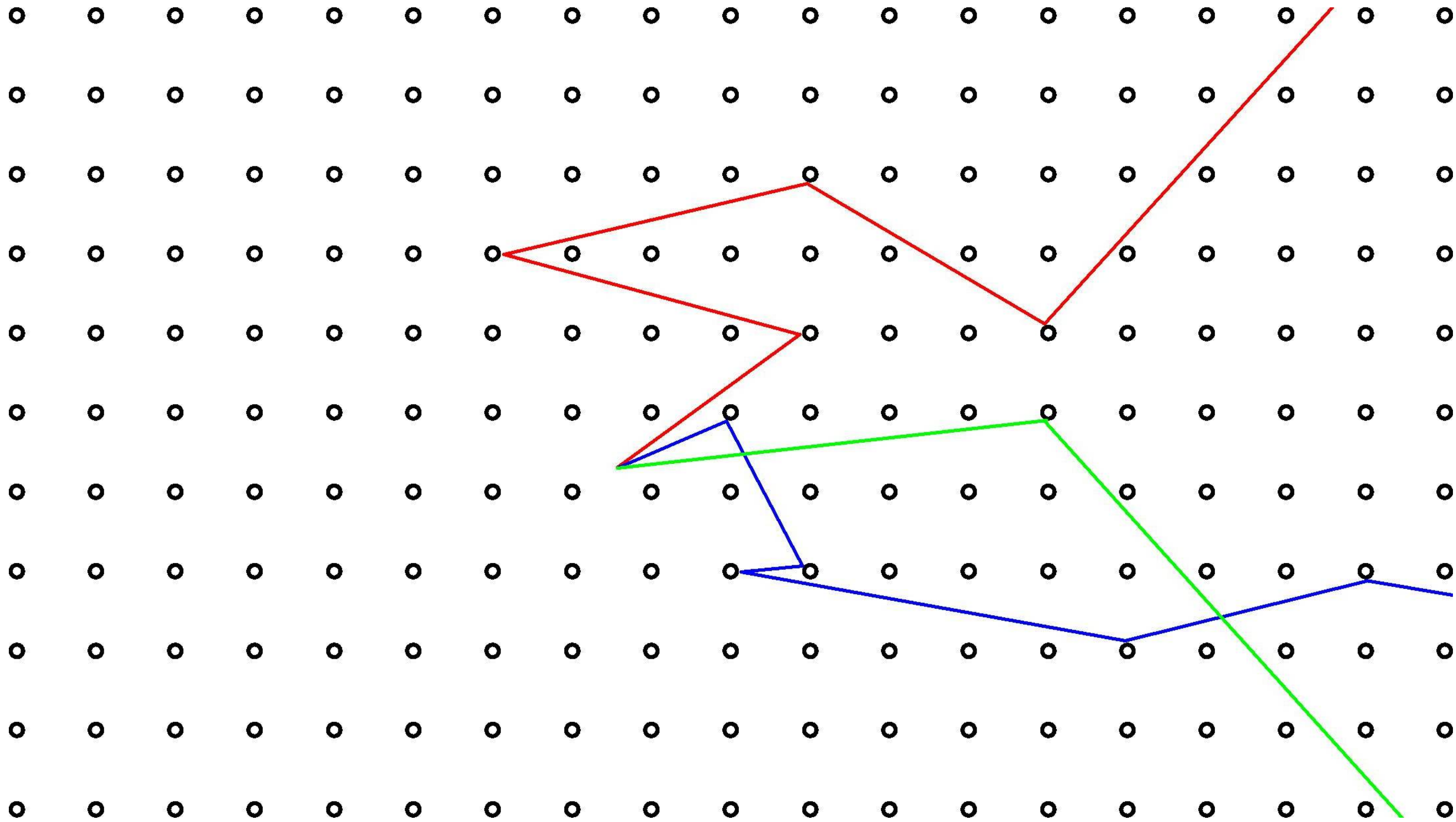


The Lorentz gas with randomly positioned scatterers.

The periodic Lorentz gas



The periodic Lorentz gas



The Boltzmann-Grad limit

- consider dynamics of macroscopic particle cloud in the limit of small scatterer radius ρ

- $(\boldsymbol{q}(t), \boldsymbol{v}(t))$ = "microscopic" phase space coordinate at time t

- the mean free path length and the time between collisions scales like $\rho^{-(d-1)}$

- rescale position and time to the "macroscopic" coordinates

$$(\boldsymbol{Q}(t), \boldsymbol{V}(t)) = (\rho^{d-1}\boldsymbol{q}(\rho^{-(d-1)}t), \boldsymbol{v}(\rho^{-(d-1)}t))$$

- define macroscopic Billiard flow Φ_ρ^t by

$$(\boldsymbol{Q}(t), \boldsymbol{V}(t)) = \Phi_\rho^t(\boldsymbol{Q}_0, \boldsymbol{V}_0)$$

The linear Boltzmann equation

- Time evolution of initial data $(\boldsymbol{Q}, \boldsymbol{V})$:

$$(\boldsymbol{Q}(t), \boldsymbol{V}(t)) = \Phi_\rho^t(\boldsymbol{Q}, \boldsymbol{V})$$

- Time evolution of a particle cloud with initial density $f \in \mathrm{L}^1$:

$$f_t = \mathrm{L}_\rho^t f, \qquad [L_\rho^t f](\boldsymbol{Q}, \boldsymbol{V}) := f\Big(\Phi_\rho^{-t}(\boldsymbol{Q}, \boldsymbol{V})\Big)$$

In his 1905 paper Lorentz suggested that f_t is governed, as $\rho \to 0$, by the linear Boltzmann equation:

$$\boxed{\left[\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{Q}}\right] f_t(\boldsymbol{Q}, \boldsymbol{V}) = \int_{\mathrm{S}_1^{d-1}} \Big[f_t(\boldsymbol{Q}, \boldsymbol{V}_0) - f_t(\boldsymbol{Q}, \boldsymbol{V})\Big] \sigma(\boldsymbol{V}_0, \boldsymbol{V}) d\boldsymbol{V}_0}$$

where the collision kernel $\sigma(\boldsymbol{V}_0, \boldsymbol{V})$ is the cross section of the individual scatterer. E.g.: $\sigma(\boldsymbol{V}_0, \boldsymbol{V}) = \frac{1}{4}\|\boldsymbol{V}_0 - \boldsymbol{V}\|^{3-d}$ for specular reflection at a hard sphere

The linear Boltzmann equation—rigorous proofs

- Galavotti (Phys Rev 1969 & report 1972): Poisson distributed hard-sphere scatterers

- Spohn (Comm Math Phys 1978): extension to more general random scatterer configurations and potentials

- Boldrighini, Bunimovich and Sinai (J Stat Phys 1983): prove convergence for almost every scatterer configuration

Failure of the Boltzmann equation: The periodic Lorentz gas

A cloud of particles with initial density $f(\boldsymbol{Q},\boldsymbol{V})$ evolves in time t to

$$f_t(\boldsymbol{Q},\boldsymbol{V}) = [L_\rho^t f](\boldsymbol{Q},\boldsymbol{V}) = f\Big(\Phi_\rho^{-t}(\boldsymbol{Q},\boldsymbol{V})\Big).$$

Theorem A (JM & Strömbergsson, Annals Math 2011)
For every $t > 0$ there exists a linear operator

$$L^t : \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d)) \to \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d)),$$

such that for every $f \in \mathrm{L}^1(\mathrm{T}^1(\mathbb{R}^d))$ and any set $\mathcal{A} \subset \mathrm{T}^1(\mathbb{R}^d)$ with boundary of Lebesgue measure zero,

$$\lim_{\rho\to 0} \int_{\mathcal{A}} [L_\rho^t f](\boldsymbol{Q},\boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V} = \int_{\mathcal{A}} [L^t f](\boldsymbol{Q},\boldsymbol{V})\, d\boldsymbol{Q}\, d\boldsymbol{V}.$$

The operator L^t thus describes the macroscopic diffusion of the Lorentz gas in the Boltzmann-Grad limit $\rho \to 0$.

Note: The family $\{L^t\}_{t\geq 0}$ does *not* form a semigroup.

A generalization of the linear Boltzmann equation

In the case of the periodic Lorentz gas L^t does not form a semigroup, and hence in particular the linear Boltzmann equation does not hold. This problem is resolved by considering extended phase space coordinates $(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+)$ where

$(\boldsymbol{Q}, \boldsymbol{V}) \in \mathrm{T}^1(\mathbb{R}^d)$ — usual position and momentum
$\xi \in \mathbb{R}_+$ — flight time until the next scatterer
$\boldsymbol{V}_+ \in \mathrm{S}_1^{d-1}$ — velocity after the next hit

We prove the following generalization of the linear Boltzmann equation in the extended phase space:

$$\boxed{\begin{aligned}&\left[\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{Q}} - \frac{\partial}{\partial \xi}\right] f_t(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+) \\ &\qquad = \int_{\mathrm{S}_1^{d-1}} f_t(\boldsymbol{Q}, \boldsymbol{V}_0, 0, \boldsymbol{V}) p_{\boldsymbol{0}}(\boldsymbol{V}_0, \boldsymbol{V}, \xi, \boldsymbol{V}_+) d\boldsymbol{V}_0\end{aligned}}$$

with a new collision kernel $p_{\boldsymbol{0}}(\boldsymbol{V}_0, \boldsymbol{V}, \xi, \boldsymbol{V}_+)$, which can be expressed as a product of the scattering cross section of an individual scatterer and a certain transition probability for hitting a given point the next scatterer after time ξ.

Why "a generalization" of the linear Boltzmann equation?

$$\left[\frac{\partial}{\partial t} + \boldsymbol{V} \cdot \nabla_{\boldsymbol{Q}} - \frac{\partial}{\partial \xi}\right] f_t(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+)$$
$$= \int_{\mathsf{S}_1^{d-1}} f_t(\boldsymbol{Q}, \boldsymbol{V}_0, 0, \boldsymbol{V}) p_{\boldsymbol{0}}(\boldsymbol{V}_0, \boldsymbol{V}, \xi, \boldsymbol{V}_+) d\boldsymbol{V}_0$$

Substituting in the above the transition density for the random (rather than periodic) scatterer configuration

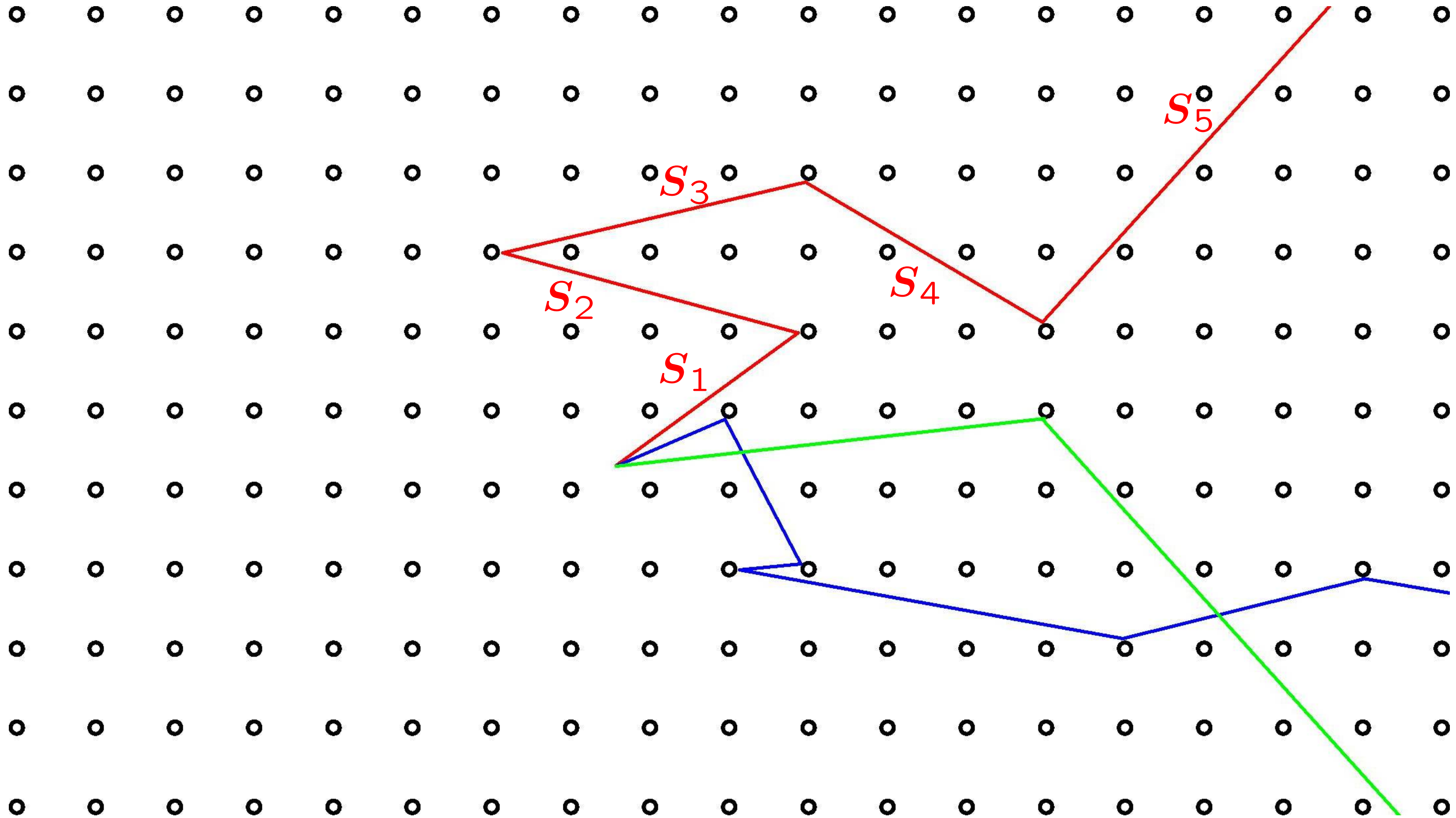
$$f_t(\boldsymbol{Q}, \boldsymbol{V}, \xi, \boldsymbol{V}_+) = g_t(\boldsymbol{Q}, \boldsymbol{V}) \sigma(\boldsymbol{V}, \boldsymbol{V}_+) \mathrm{e}^{-\overline{\sigma}\xi}, \qquad \overline{\sigma} = \mathrm{vol}(\mathcal{B}_1^{d-1}),$$

$$p_{\boldsymbol{0}}(\boldsymbol{V}_0, \boldsymbol{V}, \xi, \boldsymbol{V}_+) = \sigma(\boldsymbol{V}, \boldsymbol{V}_+) \mathrm{e}^{-\overline{\sigma}\xi}$$

yields the classical linear Boltzmann equation for $g_t(\boldsymbol{Q}, \boldsymbol{V})$.

The key theorem:

Joint distribution of path segments



Joint distribution for path segments

The following theorem proves the existence of a Markov process that describes the dynamics of the Lorentz gas in the Boltzmann-Grad limit.

Theorem B (JM & Strömbergsson, Annals of Math 2011). Fix an a.c. Borel probability measure Λ on $\mathrm{T}^1(\mathbb{R}^d)$. Then, for each $n \in \mathbb{N}$ there exists a probability density $\Psi_{n,\Lambda}$ on $\mathbb{R}^{nd}$ such that, for any set $\mathcal{A} \subset \mathbb{R}^{nd}$ with boundary of Lebesgue measure zero,

$$\lim_{\rho\to 0} \Lambda\Big(\Big\{(\boldsymbol{Q}_0,\boldsymbol{V}_0) \in \mathrm{T}^1(\mathbb{R}^d) : (\boldsymbol{S}_1,\ldots,\boldsymbol{S}_n) \in \mathcal{A}\Big\}\Big) = \int_{\mathcal{A}} \Psi_{n,\Lambda}(\boldsymbol{S}_1',\ldots,\boldsymbol{S}_n')\, d\boldsymbol{S}_1' \cdots d\boldsymbol{S}_n',$$

and, for $n \geq 3$,

$$\Psi_{n,\Lambda}(\boldsymbol{S}_1,\ldots,\boldsymbol{S}_n) = \Psi_{2,\Lambda}(\boldsymbol{S}_1,\boldsymbol{S}_2) \prod_{j=3}^{n} \Psi(\boldsymbol{S}_{j-2},\boldsymbol{S}_{j-1},\boldsymbol{S}_j),$$

where Ψ is a continuous probability density independent of Λ (and the lattice).

Theorem A follows from Theorem B by technical probabilistic arguments.

First step: The distribution of free path lengths

References

- Polya (Arch Math Phys 1918): “Visibility in a forest” $(d = 2)$
- Dahlquist (Nonlinearity 1997); Boca, Cobeli, Zaharescu (CMP 2000); Caglioti, Golse (CMP 2003); Boca, Gologan, Zaharescu (CMP 2003); Boca, Zaharescu (CMP 2007): Limit distributions for the free path lengths for various sets of initial data $(d = 2)$
- Dumas, Dumas, Golse (J Stat Phys 1997): Asymptotics of mean free path lengths $(d \geq 2)$
- Bourgain, Golse, Wennberg (CMP 1998); Golse, Wennberg (CMP 2000): bounds on possible weak limits $(d \geq 2)$
- Boca & Gologan (Annales I Fourier 2009), Boca (NY J Math 2010): honeycomb lattice
- JM & Strömbergsson (Annals of Math 2010, 2011, GAFA 2011): proof of limit distribution and tail estimates in arbitrary dimension

Polya's forest

The distribution of free path lengths in any dimension

- $\tau_1 = \tau_1(\boldsymbol{q}, \boldsymbol{v})$ — free path length for initial condition $(\boldsymbol{q}, \boldsymbol{v})$

Theorem C (JM & Strömbergsson, Annals Math 2010)
Fix a lattice L and the initial position $\boldsymbol{q}$. Let λ be any a.c. Borel probability measure on S_1^{d-1}. Then, for every $\xi > 0$, the limit

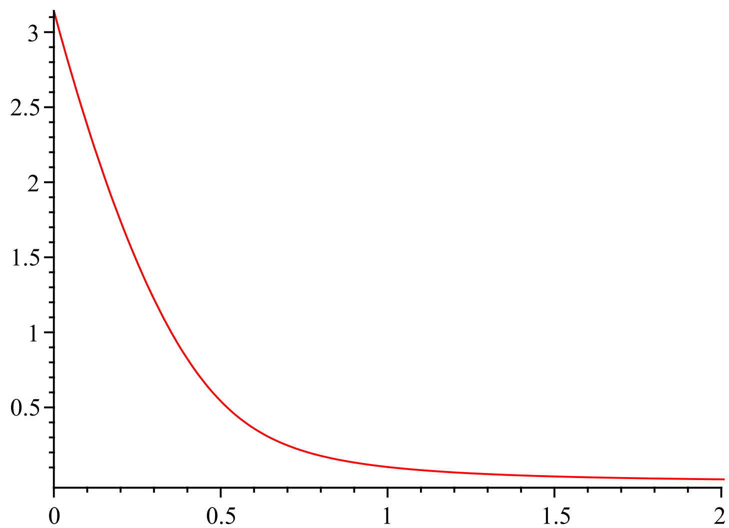
$$\lim_{\rho\to 0} \lambda(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} : \rho^{d-1}\tau_1 \geq \xi\})$$

exists, is continuous in ξ, independent of λ.

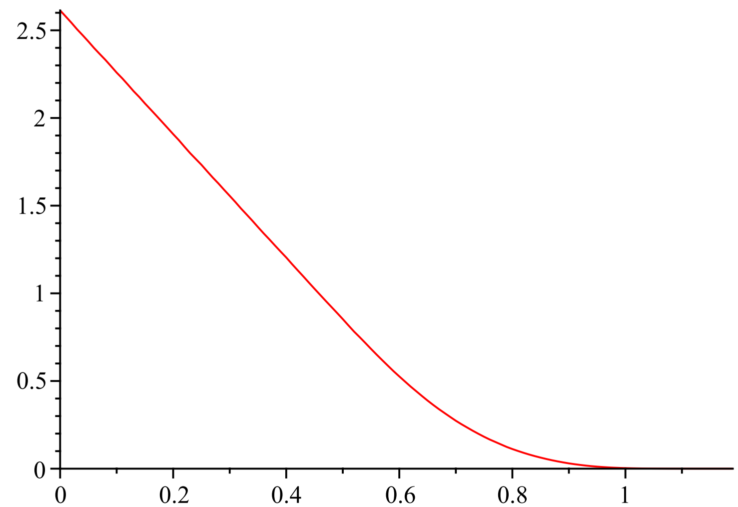
If $\boldsymbol{q} \in L$, it is given by the probability $F_0(\xi)$ that a random lattice $\widetilde{L}$ avoids the cylinder $Z(\xi) = \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 < x_1 < \xi, \|(x_2, \ldots, x_d)\| < 1\right\}$.

If $\boldsymbol{q} \notin \mathbb{Q}L$, it is given by the probability $F(\xi)$ that a random affine lattice $\widetilde{L}_\alpha$ avoids the cylinder $Z(\xi)$.

Limiting densities for $d = 3$



$$\Phi(\xi) = -\frac{d}{d\xi}F(\xi)$$



$$\Phi_{\mathbf{0}}(\xi) = -\frac{d}{d\xi}F_{\mathbf{0}}(\xi)$$

For random scatterer configuarions:

$\Phi(\xi) = \overline{\sigma}\mathrm{e}^{-\overline{\sigma}\xi}$, $\Phi_{\mathbf{0}}(\xi) = \overline{\sigma}\mathrm{e}^{-\overline{\sigma}\xi}$ with $\overline{\sigma} = \mathrm{vol}(\mathcal{B}_1^{d-1})$

Tail asymptotics (JM & Strömbergsson, GAFA 2011).

$$\Phi(\xi) = \frac{\pi^{\frac{d-1}{2}}}{2^d d\, \Gamma(\frac{d+3}{2})\, \zeta(d)}\, \xi^{-2} + O\Big(\xi^{-2-\frac{2}{d}}\Big) \qquad \text{as } \xi \to \infty$$

$$\Phi_{\mathbf{0}}(\xi) = 0 \qquad \text{for } \xi \text{ sufficiently large}$$

$$\Phi(\xi) = \overline{\sigma} - \frac{\overline{\sigma}^2}{\zeta(d)}\, \xi + O\Big(\xi^2\Big) \qquad \text{as } \xi \to 0$$

$$\Phi_{\mathbf{0}}(\xi) = \frac{\overline{\sigma}}{\zeta(d)} + O(\xi) \qquad \text{as } \xi \to 0.$$

with $\overline{\sigma} = \mathrm{vol}(\mathcal{B}_1^{d-1}) = \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)}$.

$1/\zeta(d)$ is the relative density of primitive lattice points (i.e., the lattice points visible from the origin).

The distribution of free path lengths

Theorem C (JM & Strömbergsson, Annals Math 2010)
Fix a lattice L and the initial position $\boldsymbol{q}$. Let λ be any a.c. Borel probability measure on S_1^{d-1}. Then, for every $\xi > 0$, the limit

$$\lim_{\rho\to 0} \lambda(\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \rho^{d-1}\tau_1 \geq \xi\})$$

exists, is continuous in ξ, independent of λ.

If $\boldsymbol{q} \in L$, it is given by the probability $F_{\mathbf{0}}(\xi)$ that a random lattice $\tilde{L}$ avoids the cylinder $Z(\xi) = \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d : 0 < x_1 < \xi, \|(x_2,\ldots,x_d)\| < 1\right\}$.

If $\boldsymbol{q} \notin \mathbb{Q}L$, it is given by the probability $F(\xi)$ that a random affine lattice $\tilde{L}_\alpha$ avoids the cylinder $Z(\xi)$.

The distribution of free path lengths

Theorem C (JM & Strömbergsson, Annals Math 2010)
Fix a lattice L and the initial position $\boldsymbol{q}$. Let λ be any a.c. Borel probability measure on S_1^{d-1}. Then, for every $\xi > 0$, the limit

$$\lim_{\rho\to 0} \lambda(\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \rho^{d-1}\tau_1 \geq \xi\})$$

exists, is continuous in ξ, independent of λ.

If $\boldsymbol{q} \in L$, it is given by the probability $F_0(\xi)$ that a random lattice $\tilde{L}$ avoids the cylinder $Z(\xi) = \Big\{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 < x_1 < \xi, \|(x_2, \ldots, x_d)\| < 1\Big\}$.

If $\boldsymbol{q} \notin \mathbb{Q}L$, it is given by the probability $F(\xi)$ that a random affine lattice $\tilde{L}_{\boldsymbol{\alpha}}$ avoids the cylinder $Z(\xi)$.

What is a random lattice?

- $L \subset \mathbb{R}^d$—euclidean lattice of covolume one

- recall $L = \mathbb{Z}^d M$ for some $M \in \mathsf{SL}(d, \mathbb{R})$, therefore the homogeneous space $X_1 = \mathsf{SL}(d, \mathbb{Z}) \backslash \mathsf{SL}(d, \mathbb{R})$ parametrizes the space of lattices of covolume one

- μ_1—right-$\mathsf{SL}(d, \mathbb{R})$ invariant prob measure on X_1 (Haar)

- $F_{\mathbf{0}}(\xi) := \mu_1(\{M \in X_1 : \mathbb{Z}^d M \cap Z(\xi) = \emptyset\})$

What is a random affine lattice?

- $\mathrm{ASL}(d,\mathbb{R}) = \mathrm{SL}(d,\mathbb{R}) \ltimes \mathbb{R}^d$—the semidirect product group with multiplication law

$$(M,\boldsymbol{x})(M',\boldsymbol{x}') = (MM', \boldsymbol{x}M' + \boldsymbol{x}').$$

An action of $\mathrm{ASL}(d,\mathbb{R})$ on $\mathbb{R}^d$ can be defined as

$$\boldsymbol{y} \mapsto \boldsymbol{y}(M,\boldsymbol{x}) := \boldsymbol{y}M + \boldsymbol{x}.$$

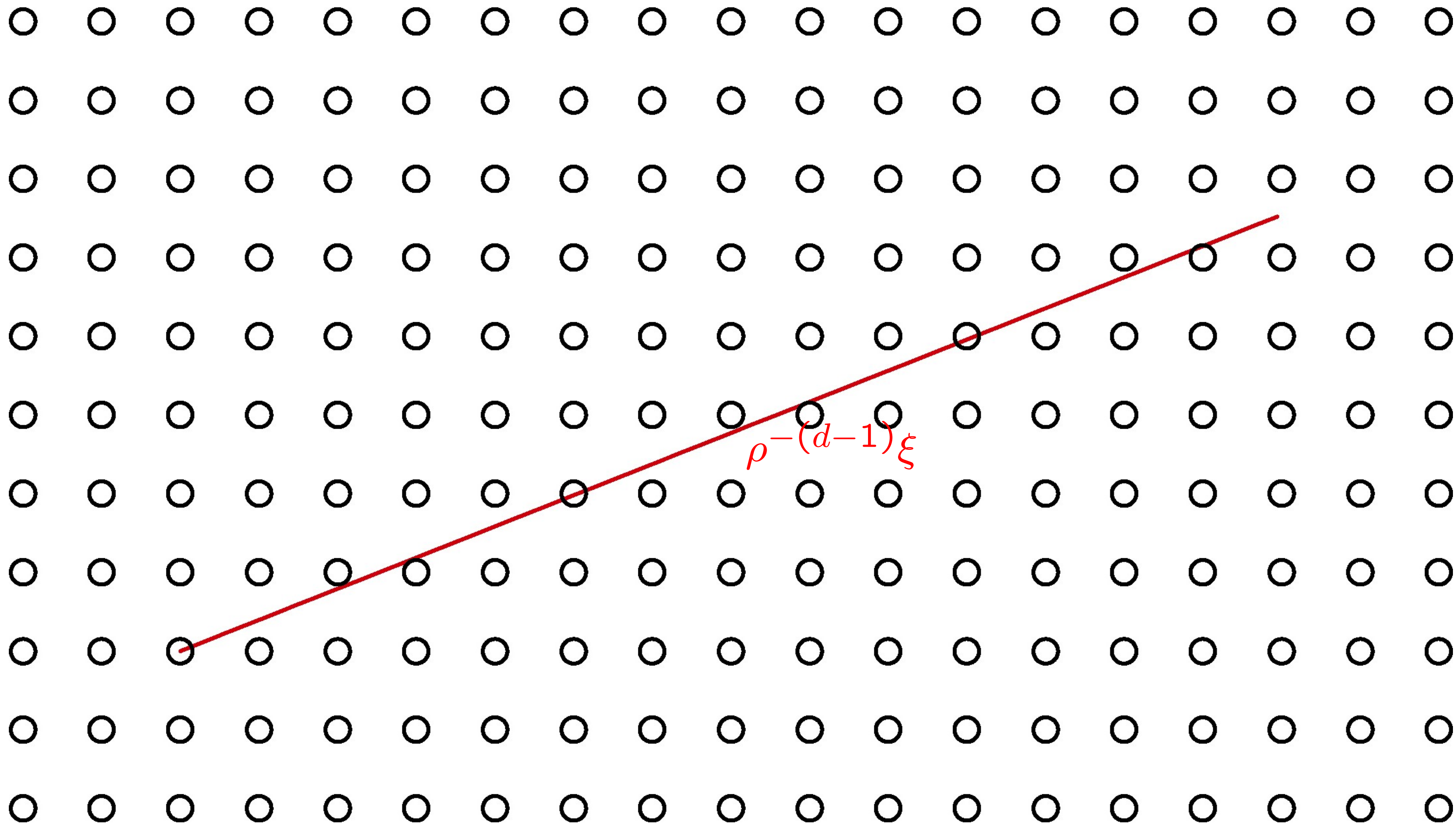
- the space of affine lattices is then represented by $X = \mathrm{ASL}(d,\mathbb{Z})\backslash \mathrm{ASL}(d,\mathbb{R})$ where $\mathrm{ASL}(d,\mathbb{Z}) = \mathrm{SL}(d,\mathbb{Z}) \ltimes \mathbb{Z}^d$, i.e., $L_{\boldsymbol{\alpha}} := (\mathbb{Z}^d + \boldsymbol{\alpha})M$

- μ—right-$\mathrm{ASL}(d,\mathbb{R})$ invariant prob measure on X

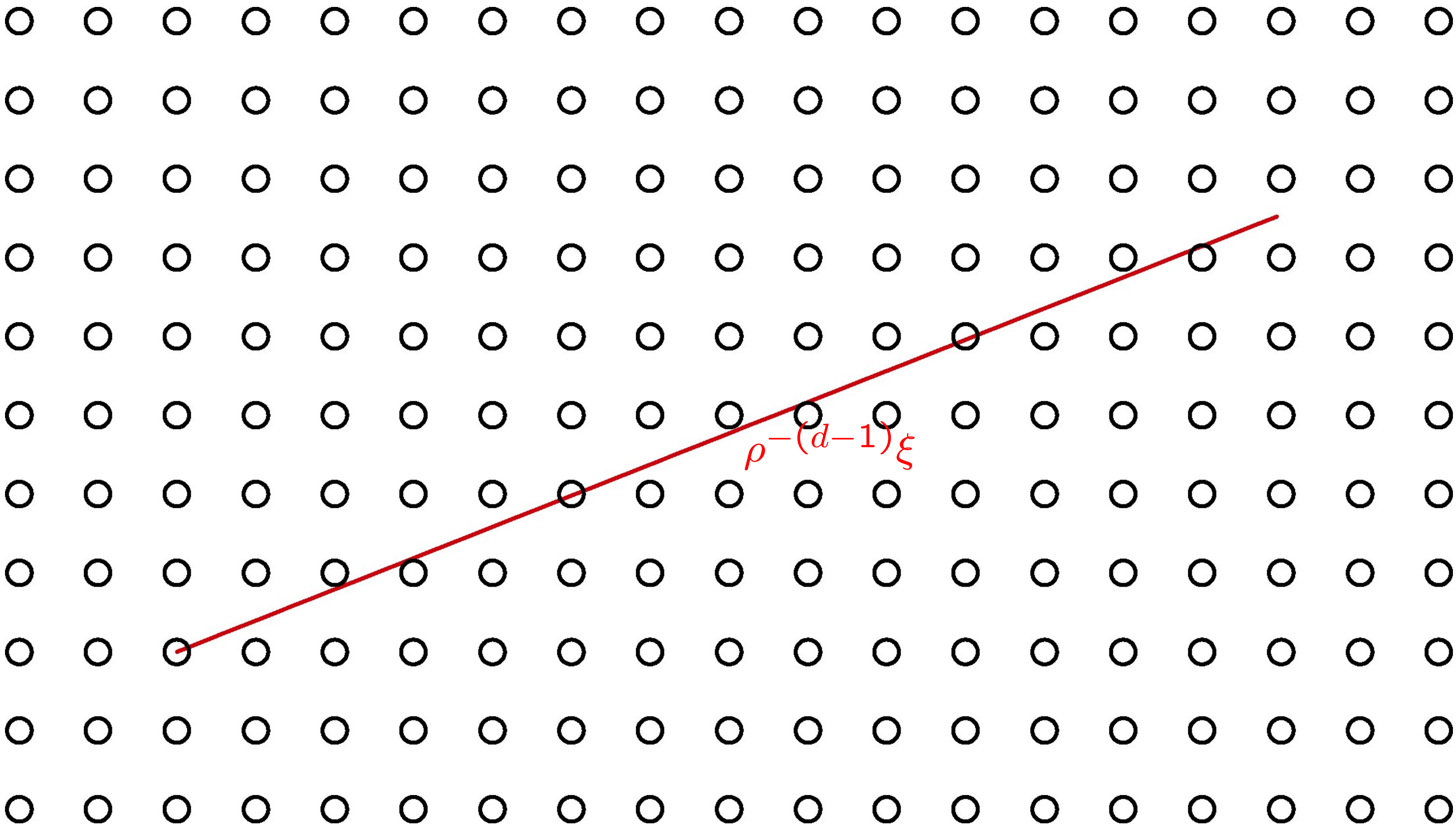
- $F(\xi) := \mu(\{(M,\boldsymbol{x}) \in X : (\mathbb{Z}^d M + \boldsymbol{x}) \cap Z(\xi) = \emptyset\})$

Outline of proof of Theorem C

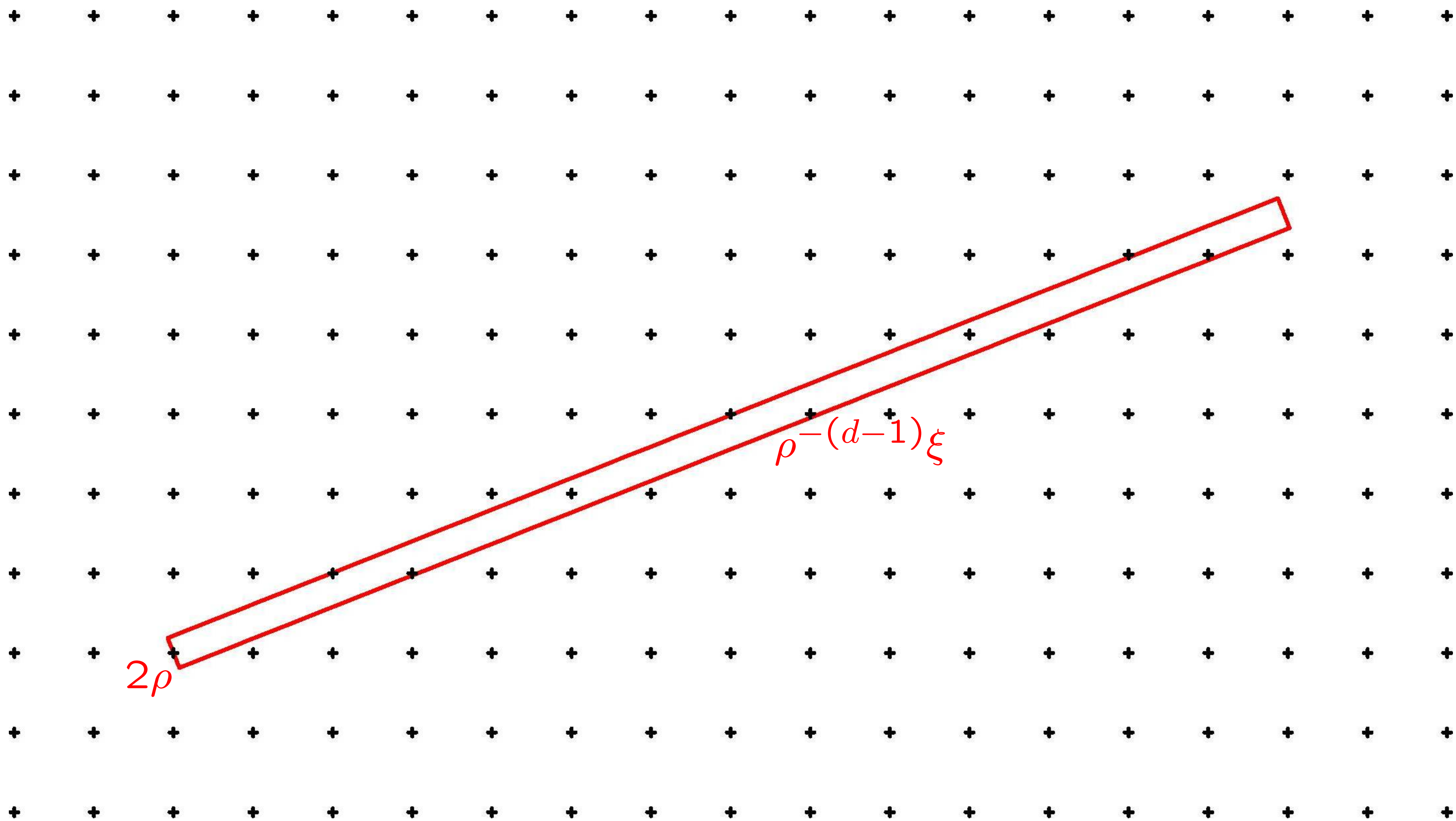
(in the case $\boldsymbol{q} \in L$ and $L = \mathbb{Z}^d$)



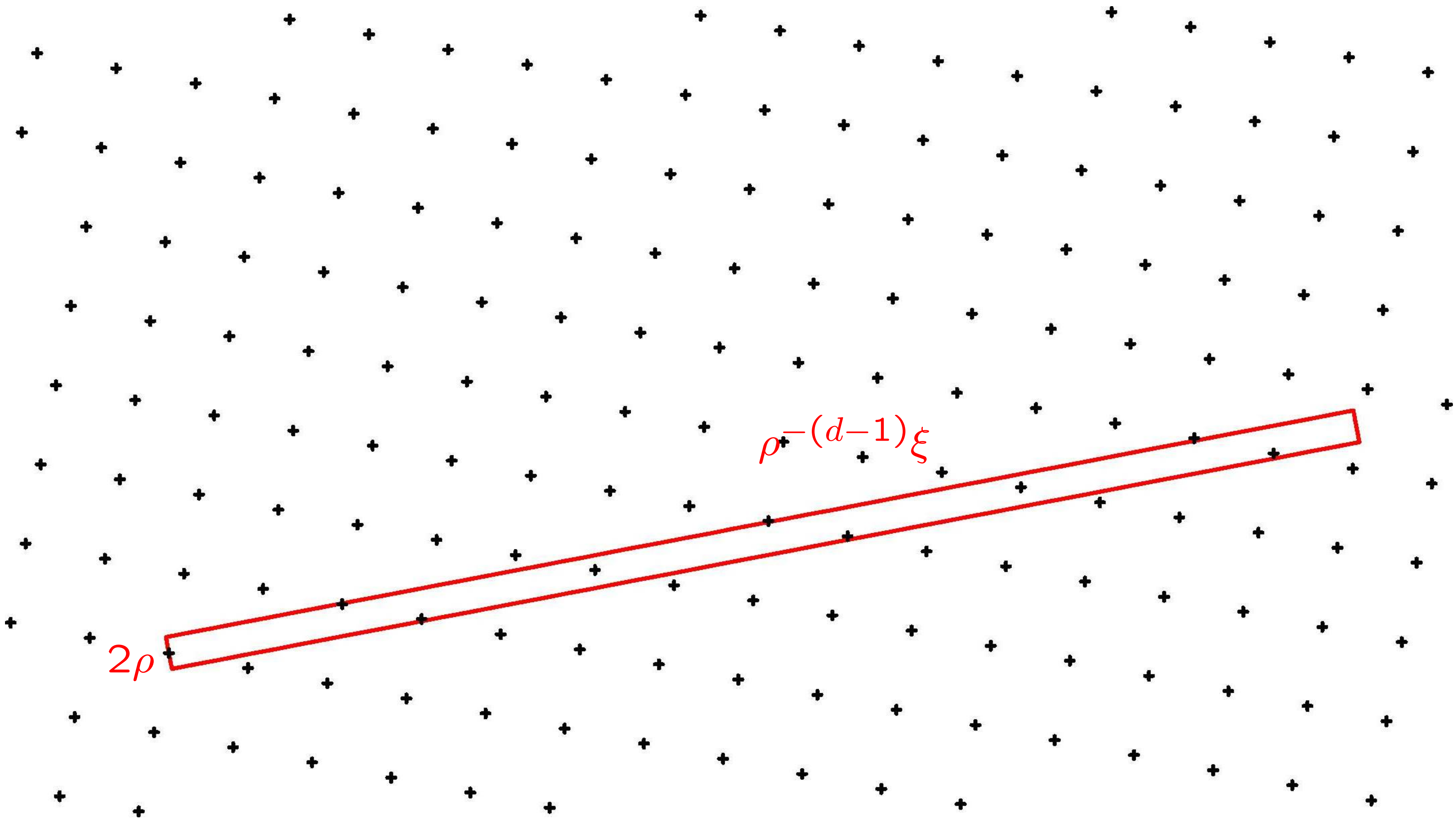
$$\lambda\Big(\Big\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \rho^{d-1}\tau_1 \geq \xi\Big\}\Big) = \ldots$$



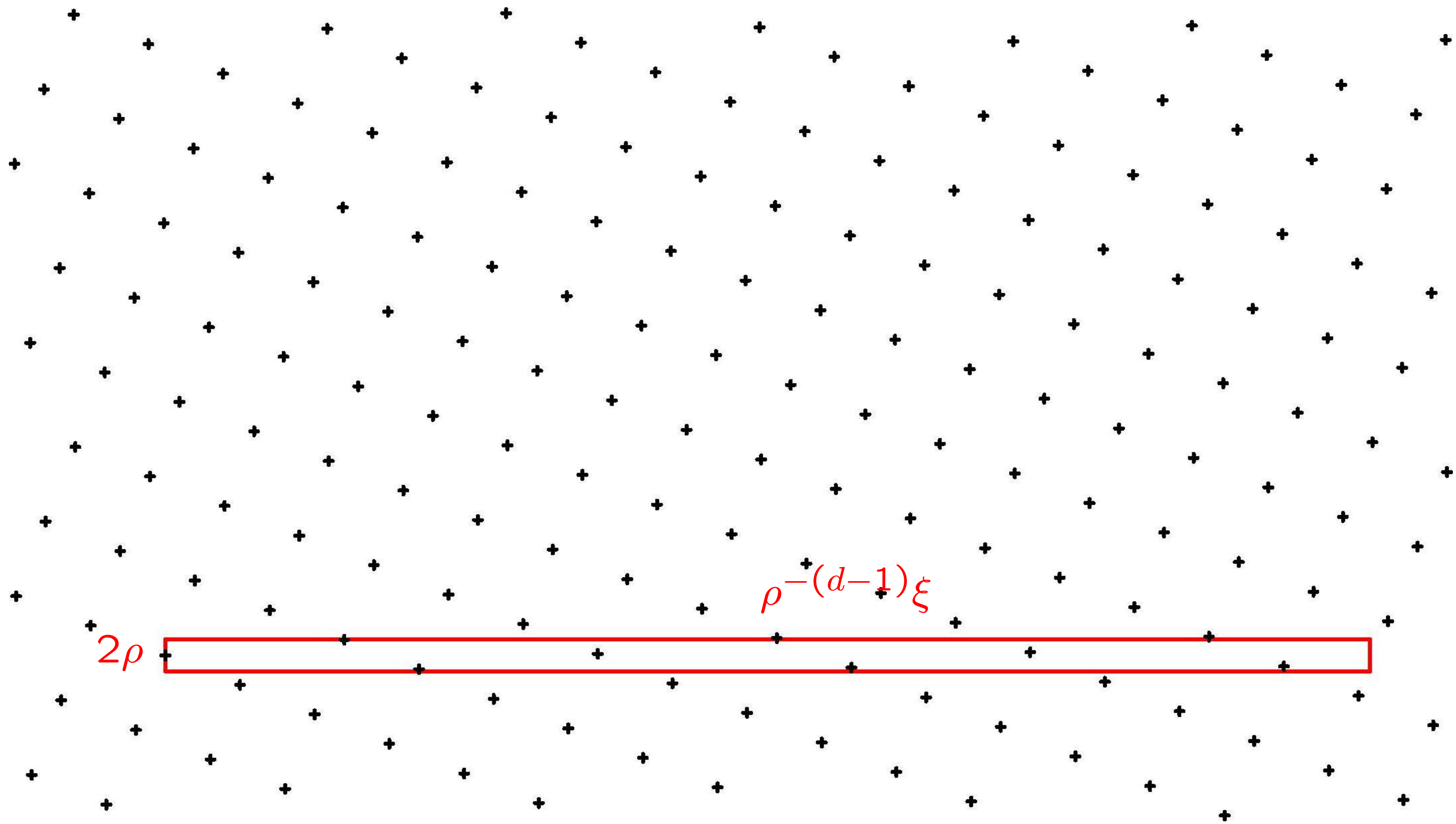
$$= \lambda\Big(\Big\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \text{ no scatterer intersects } \mathsf{ray}(\boldsymbol{v}, \rho^{-(d-1)}\xi)\Big\}\Big)$$



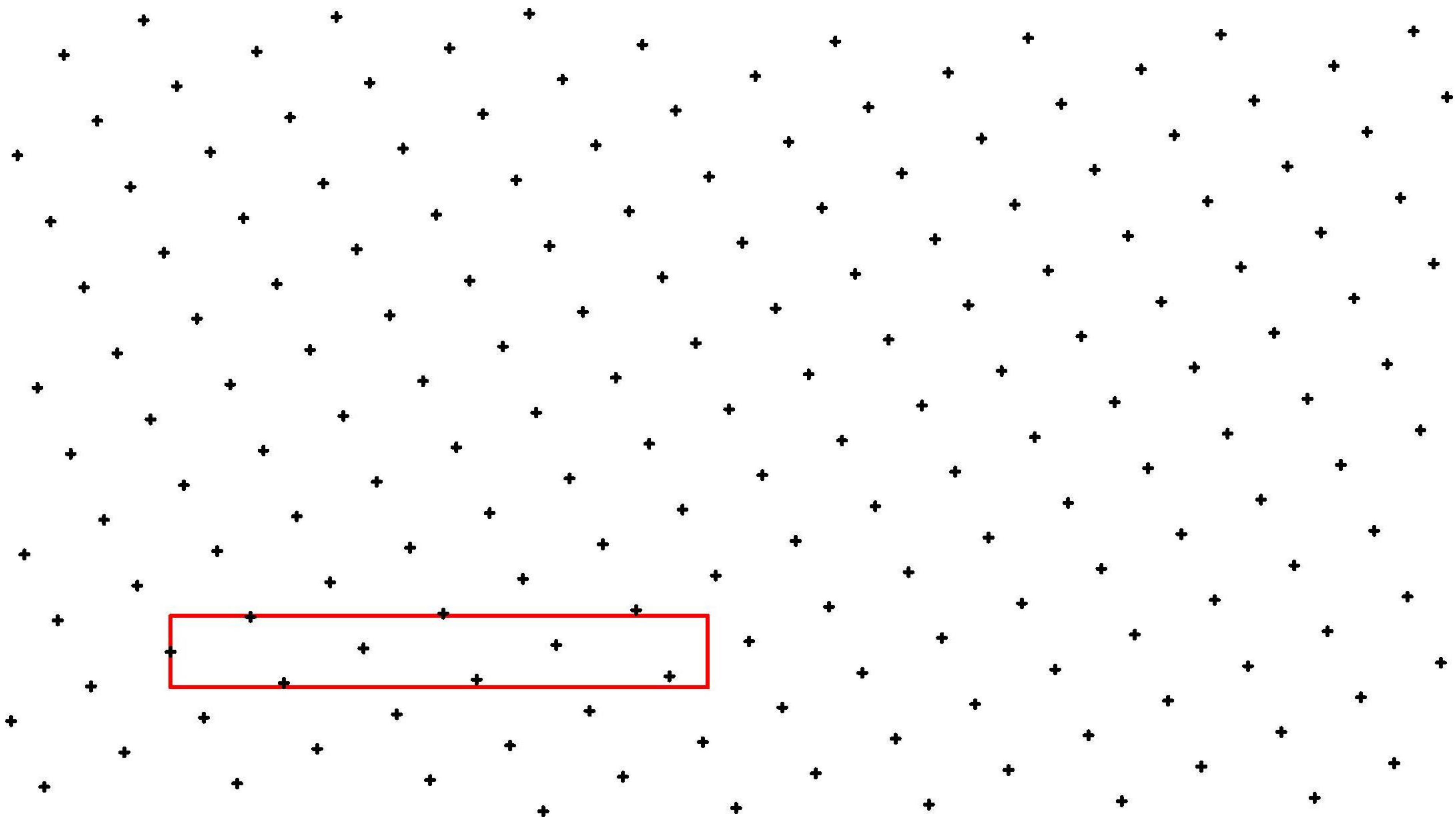
$$\approx \lambda\Big(\big\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \mathbb{Z}^d \cap Z(\boldsymbol{v}, \rho^{-(d-1)}\xi, \rho) = \emptyset\big\}\Big)$$



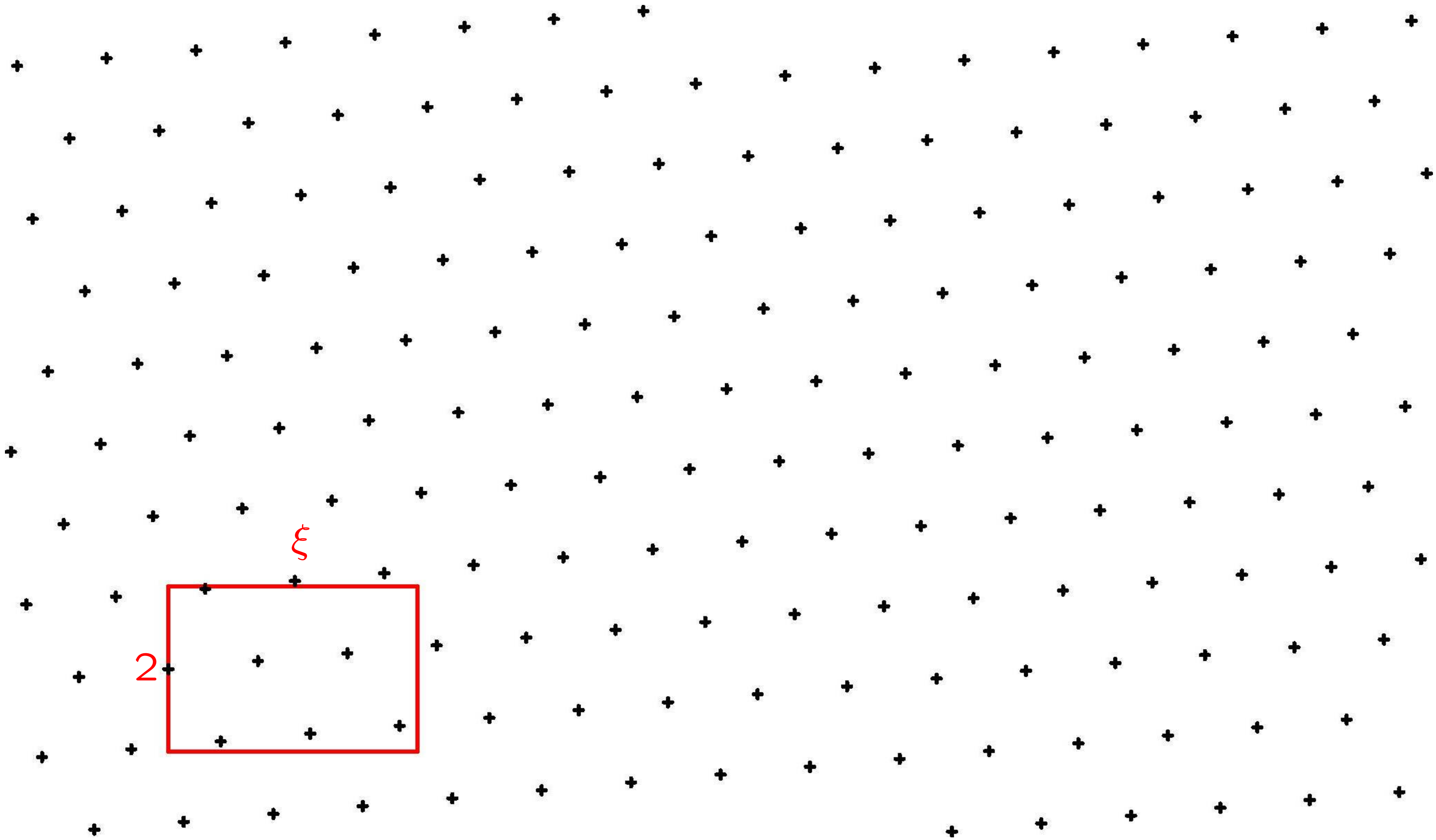
$\Big($Rotate by $K(\boldsymbol{v}) \in \mathsf{SO}(d)$ such that $\boldsymbol{v} \mapsto \boldsymbol{e}_1\Big)$



$$\lambda\Big(\Big\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \mathbb{Z}^d K(\boldsymbol{v}) \cap Z(\boldsymbol{e}_1, \rho^{-(d-1)}\xi, \rho) = \emptyset\Big\}\Big)$$



$$\left(\text{Apply } D_\rho = \mathsf{diag}(\rho^{d-1}, \rho^{-1}, \ldots, \rho^{-1}) \in \mathsf{SL}(d, \mathbb{R})\right)$$



$$\lambda\Big(\Big\{\boldsymbol{v} \in \mathsf{S}_1^{d-1} : \mathbb{Z}^d K(\boldsymbol{v}) D_\rho \cap Z(\boldsymbol{e}_1, \xi, 1) = \emptyset\Big\}\Big)$$

The following Theorem shows that in the limit $\rho \to 0$ the lattice

$$\mathbb{Z}^d K(\boldsymbol{v}) \begin{pmatrix} \rho^{d-1} & \mathbf{0} \\ {}^{\mathrm{t}}\mathbf{0} & \rho^{-1} 1 \end{pmatrix}$$

behaves like a random lattice with respect to Haar measure μ_1.

Define a flow on $X_1 = \mathrm{SL}(d,\mathbb{Z}) \backslash \mathrm{SL}(d,\mathbb{R})$ via right translation by

$$\Phi^t = \begin{pmatrix} \mathrm{e}^{-(d-1)t} & \mathbf{0} \\ {}^{\mathrm{t}}\mathbf{0} & \mathrm{e}^{t} 1 \end{pmatrix}.$$

Theorem D. Fix any $M_0 \in \mathrm{SL}(d,\mathbb{R})$. Let λ be an a.c. Borel probability measure on S_1^{d-1}. Then, for every bounded continuous function $f : X_1 \to \mathbb{R}$,

$$\lim_{t\to\infty} \int_{\mathrm{S}_1^{d-1}} f(M_0 K(\boldsymbol{v}) \Phi^t) d\lambda(\boldsymbol{v}) = \int_{X_1} f(M) d\mu_1(M).$$

Theorem D is a direct consequence of the mixing property for the flow Φ^t.

This concludes the proof of Theorem C when $\boldsymbol{q} \in L = \mathbb{Z}^d M_0$.

The generalization of Theorem D required for the full proof of Theorem C uses Ratner's classification of ergodic measures invariant under a unipotent flow (measure rigidity). We exploit a close variant of a theorem by N. Shah (Proc. Ind. Acad. Sci. 1996) on the uniform distribution of translates of unipotent orbits.

Other scatterer scatterer configurations: unions of lattices

- Consider now scatterer locations at the point set

$$\mathcal{P}_0 = \bigcup_{i=1}^{N} \mathcal{L}_j, \qquad \mathcal{L}_i = \overline{n}_i^{-1/d}(\mathbb{Z}^d + \boldsymbol{\omega}_i)M_i$$

with $\boldsymbol{\omega}_i \in \mathbb{R}^d$, $M_i \in \mathrm{SL}(d,\mathbb{R})$ and $\overline{n}_i > 0$ such that $\overline{n}_1 + \ldots + \overline{n}_N = 1$

- Let $\mathcal{S}$ be the commensurator of $\mathrm{SL}(d,\mathbb{Z})$ in $\mathrm{SL}(d,\mathbb{R})$:

$$\mathcal{S} = \{(\det T)^{-1/d} T : T \in \mathrm{GL}(d,\mathbb{Q}),\ \det T > 0\}.$$

- We say that the matrices $M_1, \ldots, M_N \in \mathrm{SL}(d,\mathbb{R})$ *are pairwise incommensurable* if $M_i M_j^{-1} \notin \mathcal{S}$ for all $i \neq j$. A simple example is

$$M_i = \zeta^{-i/d} \begin{pmatrix} \zeta^i & 0 \\ 0 & 1_{d-1} \end{pmatrix}, \qquad i = 1, \ldots, N,$$

where ζ is any positive number such that $\zeta, \zeta^2, \ldots, \zeta^{N-1} \notin \mathbb{Q}$.

Free path lengths for unions of lattices

Theorem E (JM & Strömbergsson, 2013). Let $\mathcal{P}_0$ as on the previous slide with $M_i \in \mathrm{SL}(d,\mathbb{R})$ pairwise incommensurable. Let λ be any a.c. Borel probability measure on S_1^{d-1}. Then, for every $\xi > 0$, the limit

$$F_{\mathcal{P}_0,\boldsymbol{q}}(\xi) := \lim_{\rho\to 0} \lambda(\{\boldsymbol{v} \in \mathrm{S}_1^{d-1} : \rho^{d-1}\tau_1 \geq \xi\})$$

exists. If for instance $\boldsymbol{\omega}_i - \overline{n}_i^{1/d}\boldsymbol{q}M_i^{-1} \notin \mathbb{Q}^d$ for all i, then

$$F_{\mathcal{P}_0,\boldsymbol{q}}(\xi) = \prod_{i=1}^{N} F(\overline{n}_i\xi)$$

where $F(\xi)$ is the distribution of free path length corresponding to a single lattice and generic initial point (as in Theorem C).

Recall $F(\xi) \sim C\xi^{-1}$ for $\xi \to \infty$. Thus

$$F_{\mathcal{P}_0,\boldsymbol{q}}(\xi) \sim C^N \xi^{-N}.$$

Key step in the proof: equidistribution in products

...is as before an equidistribution theorem that follows from Ratner's measure classification (again via Shah's theorem):

Theorem F (JM & Strömbergsson, 2013). Assume that $M_1, \ldots, M_N \in \mathrm{SL}(d, \mathbb{R})$ are pairwise incommensurable, and $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_N \notin \mathbb{Q}^d$ (for simplicity). Let λ be an a.c. Borel probability measure on S_1^{d-1}. Then

$$\lim_{t\to\infty} \int_{\mathrm{S}_1^{d-1}} f\Big((1_d, \boldsymbol{\alpha}_1)(M_1 K(\boldsymbol{v})\Phi^t, \mathbf{0}), \ldots,$$
$$\ldots, (1_d, \boldsymbol{\alpha}_N)(M_N K(\boldsymbol{v})\Phi^t, \mathbf{0})\Big)\, d\lambda(\boldsymbol{v})$$
$$= \int_{X^N} f(g_1, \ldots, g_N)\, d\mu(g_1)\cdots d\mu(g_N).$$