Θ HAND-OUT⋆

JENS MARKLOF

1. Schrödinger and Shale-Weil representation

1.1. Let ω be the standard symplectic form on $\mathbb{R}^{2k}$, i.e.,

$$\omega(\boldsymbol{\xi},\boldsymbol{\xi}') = \boldsymbol{x}\cdot\boldsymbol{y}' - \boldsymbol{y}\cdot\boldsymbol{x}',$$

where

$$\boldsymbol{\xi} = \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}, \qquad \boldsymbol{\xi}' = \begin{pmatrix}\boldsymbol{x}'\\ \boldsymbol{y}'\end{pmatrix}, \qquad \boldsymbol{x},\boldsymbol{y},\boldsymbol{x}',\boldsymbol{y}' \in \mathbb{R}^k.$$

The *Heisenberg group* $\mathbb{H}(\mathbb{R}^k)$ is then defined as the set $\mathbb{R}^{2k}\times\mathbb{R}$ with multiplication law [2]

$$(\boldsymbol{\xi},t)(\boldsymbol{\xi}',t') = (\boldsymbol{\xi}+\boldsymbol{\xi}', t+t'+\tfrac{1}{2}\omega(\boldsymbol{\xi},\boldsymbol{\xi}')).$$

Note that we have the decomposition

$$(\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix},t) = (\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{0}\end{pmatrix},0)(\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{y}\end{pmatrix},0)(\boldsymbol{0},t-\tfrac{1}{2}\boldsymbol{x}\cdot\boldsymbol{y}).$$

1.2. The *Schrödinger representation* of $\mathbb{H}(\mathbb{R}^k)$ on $f\in \mathrm{L}^2(\mathbb{R}^k)$ is given by (cf. [2], p. 15)

$$[W(\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{0}\end{pmatrix},0)f](\boldsymbol{w}) = e(\boldsymbol{x}\cdot\boldsymbol{w})\, f(\boldsymbol{w}), \quad \text{with } \boldsymbol{x},\boldsymbol{w}\in\mathbb{R}^k,$$

$$[W(\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{y}\end{pmatrix},0)f](\boldsymbol{w}) = f(\boldsymbol{w}-\boldsymbol{y}), \quad \text{with } \boldsymbol{y},\boldsymbol{w}\in\mathbb{R}^k,$$

$$W(\boldsymbol{0},t) = e(t)\,\mathrm{id}, \quad \text{with } t\in\mathbb{R}.$$

We have therefore for a general element $(\boldsymbol{\xi},t)$ in $\mathbb{H}(\mathbb{R}^k)$

$$[W(\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix},t)f](\boldsymbol{w}) = e(t-\tfrac{1}{2}\boldsymbol{x}\cdot\boldsymbol{y})\, e(\boldsymbol{x}\cdot\boldsymbol{w})\, f(\boldsymbol{w}-\boldsymbol{y}).$$

1.3. For every element M in the symplectic group $\mathrm{Sp}(k,\mathbb{R})$ of $\mathbb{R}^{2k}$, we can define a new representation W_M of $\mathbb{H}(\mathbb{R}^k)$ by

$$W_M(\boldsymbol{\xi},t) = W(M\boldsymbol{\xi},t).$$

All such representations are irreducible and, by the Stone-von Neumann Theorem, unitarily equivalent (see [2] for details). That is, for each $M\in\mathrm{Sp}(k,\mathbb{R})$ there exists a unitary operator $R(M)$ such that

$$R(M)\, W(\boldsymbol{\xi},t)\, R(M)^{-1} = W(M\boldsymbol{\xi},t).$$

The $R(M)$ is determined up to a unitary phase factor and defines the projective *Shale-Weil representation* of the symplectic group. *Projective* means that

$$R(MM') = c(M,M')R(M)R(M')$$

with cocycle $c(M,M')\in\mathbb{C}$, $|c(M,M')|=1$, but $c(M,M')\neq 1$ in general.

1.4. For our present purpose it suffices to consider the group $\mathrm{SL}(2,\mathbb{R})$ which is embedded in $\mathrm{Sp}(k,\mathbb{R})$ by

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} \mapsto \begin{pmatrix} a\,1_k & b\,1_k\\ c\,1_k & d\,1_k\end{pmatrix}$$

where 1_k is the $k\times k$ unit matrix.

The action of $M\in\mathrm{SL}(2,\mathbb{R})$ on $\boldsymbol{\xi}\in\mathbb{R}^{2k}$ is then given by

$$M\boldsymbol{\xi} = \begin{pmatrix} a\boldsymbol{x}+b\boldsymbol{y}\\ c\boldsymbol{x}+d\boldsymbol{y}\end{pmatrix}, \quad \text{with } M = \begin{pmatrix} a & b\\ c & d\end{pmatrix}, \quad \boldsymbol{\xi} = \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}.$$

1.5. For $M\in\mathrm{SL}(2,\mathbb{R})\hookrightarrow\mathrm{Sp}(k,\mathbb{R})$ we have the explicit representations (see [2], p. 61f.)

$$[R(M)f](\boldsymbol{w}) = \begin{cases} |a|^{k/2} & e(\tfrac{1}{2}\|\boldsymbol{w}\|^2 ab) f(a\boldsymbol{w}) \\ & (c=0) \\ |c|^{-k/2} & \int_{\mathbb{R}^k} e\left[\frac{\frac{1}{2}(a\|\boldsymbol{w}\|^2+d\|\boldsymbol{w}'\|^2)-\boldsymbol{w}\cdot\boldsymbol{w}'}{c}\right] f(\boldsymbol{w}')\,d\boldsymbol{w}' \\ & (c\neq 0). \end{cases}$$

Here $\|\cdot\|$ denotes the euclidean norm in $\mathbb{R}^k$,

$$\|\boldsymbol{x}\| = \sqrt{x_1^2+\cdots+x_k^2}.$$

1.6. If

$$M_1 = \begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}, \quad M_2 = \begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}, \quad M_3 = \begin{pmatrix} a_3 & b_3\\ c_3 & d_3\end{pmatrix},$$

$\in\mathrm{SL}(2,\mathbb{R})$ with $M_1M_2=M_3$, the corresponding cocycle is

$$c(M_1,M_2) = \mathrm{e}^{-\mathrm{i}\pi k\,\mathrm{sign}(c_1c_2c_3)/4},$$

where

$$\mathrm{sign}(x) = \begin{cases} -1 & (x<0)\\ 0 & (x=0)\\ 1 & (x>0).\end{cases}$$

1.7. In the special case when

$$M_1 = \begin{pmatrix} \cos\phi_1 & -\sin\phi_1\\ \sin\phi_1 & \cos\phi_1\end{pmatrix}, \quad M_2 = \begin{pmatrix} \cos\phi_2 & -\sin\phi_2\\ \sin\phi_2 & \cos\phi_2\end{pmatrix},$$

we find

$$c(M_1,M_2) = \mathrm{e}^{-\mathrm{i}\pi k(\sigma_{\phi_1}+\sigma_{\phi_2}-\sigma_{\phi_1+\phi_2})/4}$$

where

$$\sigma_\phi = \begin{cases} 2\nu & \text{if } \phi=\nu\pi,\\ 2\nu+1 & \text{if } \nu\pi<\phi<(\nu+1)\pi.\end{cases}$$

1.8. Every $M\in\mathrm{SL}(2,\mathbb{R})$ admits the unique Iwasawa decomposition

$$M = \begin{pmatrix} 1 & u\\ 0 & 1\end{pmatrix}\begin{pmatrix} v^{1/2} & 0\\ 0 & v^{-1/2}\end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix} = (\tau,\phi),$$

where $\tau = u+\mathrm{i}v\in\mathfrak{H}$, $\phi\in[0,2\pi)$. This parametrization leads to the well known action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathfrak{H}\times[0,2\pi)$,

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}(\tau,\phi) = (\frac{a\tau+b}{c\tau+d}, \phi+\arg(c\tau+d) \bmod 2\pi).$$

We will sometimes use the convenient notation $(M\tau,\phi_M) := M(\tau,\phi)$ and $u_M := \mathrm{Re}(M\tau)$, $v_M := \mathrm{Im}(M\tau)$.

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⋆This hand-out is an extract from [1] secs 3,4.

1.9. The (projective) Shale-Weil representation of $\mathrm{SL}(2,\mathbb{R})$ reads in these coordinates

$$[R(\tau,\phi)f](\boldsymbol{w}) = [R(\tau,0)R(\mathrm{i},\phi)f](\boldsymbol{w}) = v^{k/4} e(\tfrac{1}{2}\|\boldsymbol{w}\|^2 u)[R(\mathrm{i},\phi)f](v^{1/2}\boldsymbol{w})$$

and

$$[R(\mathrm{i},\phi)f](\boldsymbol{w}) = \begin{cases} f(\boldsymbol{w}) & (\phi = 0 \bmod 2\pi) \\ f(-\boldsymbol{w}) & (\phi = \pi \bmod 2\pi) \\ |\sin\phi|^{-k/2} \int_{\mathbb{R}^k} e\left[\frac{\frac{1}{2}(\|\boldsymbol{w}\|^2+\|\boldsymbol{w}'\|^2)\cos\phi - \boldsymbol{w}\cdot\boldsymbol{w}'}{\sin\phi}\right] f(\boldsymbol{w}')\, d\boldsymbol{w}' & (\phi \neq 0 \bmod \pi). \end{cases}$$

Note that $R(\mathrm{i},\pi/2) = \mathcal{F}$ is the Fourier transform.

2. Theta sums

2.1. The *Jacobi group* is defined as the semidirect product

$$\mathrm{Sp}(k,\mathbb{R}) \ltimes \mathbb{H}(\mathbb{R}^k)$$

with multiplication law

$$(M;\boldsymbol{\xi},t)(M';\boldsymbol{\xi}',t') = (MM';\boldsymbol{\xi}+M\boldsymbol{\xi}',t+t'+\tfrac{1}{2}\omega(\boldsymbol{\xi},M\boldsymbol{\xi}')).$$

This definition is motivated by the fact that, since

$$R(M)W(\boldsymbol{\xi}',t') = W(M\boldsymbol{\xi}',t')R(M),$$

(recall 1.3) we have

$$W(\boldsymbol{\xi},t)R(M)\, W(\boldsymbol{\xi}',t')R(M') = W(\boldsymbol{\xi},t)W(M\boldsymbol{\xi}',t')\, R(M)R(M') = c(M,M')^{-1}\, W(\boldsymbol{\xi}+M\boldsymbol{\xi}',t+t'+\tfrac{1}{2}\omega(\boldsymbol{\xi},M\boldsymbol{\xi}'))\, R(MM').$$

Hence

$$R(M;\boldsymbol{\xi},t) = W(\boldsymbol{\xi},t)R(M)$$

defines a projective representation of the Jacobi group, with cocycle $c(M,M')$ as above, the so-called *Schrödinger-Weil representation.*

Let us also put

$$\tilde{R}(\tau,\phi;\boldsymbol{\xi},t) = W(\boldsymbol{\xi},t)\tilde{R}(\tau,\phi).$$

2.2. **Jacobi's theta sum.** We define Jacobi's theta sum for $f \in \mathcal{S}(\mathbb{R}^k)$ by

$$\Theta_f(\tau,\phi;\boldsymbol{\xi},t) = \sum_{\boldsymbol{m}\in\mathbb{Z}^k} [\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f](\boldsymbol{m}).$$

More explicitly, for $\tau = u+\mathrm{i}v$, $\boldsymbol{\xi} = \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}$,

$$\Theta_f(\tau,\phi;\boldsymbol{\xi},t) = v^{k/4} e(t-\tfrac{1}{2}\boldsymbol{x}\cdot\boldsymbol{y}) \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f_\phi((\boldsymbol{m}-\boldsymbol{y})v^{1/2}) e(\tfrac{1}{2}\|\boldsymbol{m}-\boldsymbol{y}\|^2 u + \boldsymbol{m}\cdot\boldsymbol{x}),$$

where

$$f_\phi = \tilde{R}(\mathrm{i},\phi)f.$$

It is easily seen that if $f \in \mathcal{S}(\mathbb{R}^k)$ then $f_\phi \in \mathcal{S}(\mathbb{R}^k)$ for ϕ fixed, and thus also $\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f \in \mathcal{S}(\mathbb{R}^k)$ for fixed $(\tau,\phi;\boldsymbol{\xi},t)$. This guarantees rapid convergence of the above series. We have the following uniform bound.

2.3. **Lemma.** *Let $f_\phi = \tilde{R}(\mathrm{i},\phi)f$, with $f \in \mathcal{S}(\mathbb{R}^k)$. Then, for any $R > 1$, there is a constant c_R such that for all $\boldsymbol{w} \in \mathbb{R}^k$, $\phi \in \mathbb{R}$, we have*

$$|f_\phi(\boldsymbol{w})| \le c_R(1+\|\boldsymbol{w}\|)^{-R}.$$

2.4. The following transformation formulas are crucial for our further investigations:

Jacobi 1.

$$\Theta_f(-\frac{1}{\tau},\phi+\arg\tau;\begin{pmatrix}-\boldsymbol{y}\\ \boldsymbol{x}\end{pmatrix},t) = \mathrm{e}^{-\mathrm{i}\pi k/4}\Theta_f(\tau,\phi;\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix},t).$$

Proof. The Poisson summation formula states that for any $f \in \mathcal{S}(\mathbb{R}^k)$ we have

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^k}[\mathcal{F}f](\boldsymbol{m}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f(\boldsymbol{m})$$

where $\mathcal{F}$ is the Fourier transform. Because

$$\mathcal{F} = R(\mathrm{i},\pi/2) = R(S), \quad S = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

and secondly $\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f \in \mathcal{S}(\mathbb{R}^k)$ for fixed $(\tau,\phi;\boldsymbol{\xi},t)$, the Poisson summation formula yields

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^k}[R(S)\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f](\boldsymbol{m}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^k}[\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f](\boldsymbol{m}).$$

We have

$$R(S)\tilde{R}(\tau,\phi;\boldsymbol{\xi},t) = R(S)W(\boldsymbol{\xi},t)\tilde{R}(\tau,0)\tilde{R}(\mathrm{i},\phi) = W(S\boldsymbol{\xi},t)R(S)R(\tau,0)\tilde{R}(\mathrm{i},\phi);$$

furthermore

$$R(S)R(\tau,0) = R(-\frac{1}{\tau},\arg\tau) = R(-\frac{1}{\tau},0)R(\mathrm{i},\arg\tau),$$

since $(\tau,0)$ and $(-\frac{1}{\tau},0)$ are upper triangular matrices, and hence the corresponding cocycles are trivial, i.e., equal to 1 (recall 1.6). Finally, since $0 < \arg\tau < \pi$ for $\tau \in \mathfrak{H}$,

$$R(\mathrm{i},\arg\tau)\tilde{R}(\mathrm{i},\phi) = \mathrm{e}^{\mathrm{i}\pi k/4}\tilde{R}(\mathrm{i},\arg\tau)\tilde{R}(\mathrm{i},\phi) = \mathrm{e}^{\mathrm{i}\pi k/4}\tilde{R}(\mathrm{i},\phi+\arg\tau).$$

Collecting all terms, we find

$$R(S)\tilde{R}(\tau,\phi;\boldsymbol{\xi},t) = \mathrm{e}^{\mathrm{i}\pi k/4}\tilde{R}(-\frac{1}{\tau},\phi+\arg\tau;S\boldsymbol{\xi},t),$$

and hence

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^k}[\tilde{R}(-\frac{1}{\tau},\phi+\arg\tau;S\boldsymbol{\xi},t)f](\boldsymbol{m}) = \mathrm{e}^{-\mathrm{i}\pi k/4}\sum_{\boldsymbol{m}\in\mathbb{Z}^k}[\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f](\boldsymbol{m}).$$

which proves the claim. □

Jacobi 2.

$$\Theta_f(\tau+1,\phi;\begin{pmatrix}\boldsymbol{s}\\ \boldsymbol{0}\end{pmatrix}+\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix},t+\tfrac{1}{2}\boldsymbol{s}\cdot\boldsymbol{y}) = \Theta_f(\tau,\phi;\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix},t),$$

with

$$\boldsymbol{s} = {}^{\mathrm{t}}(\tfrac{1}{2},\tfrac{1}{2},\ldots,\tfrac{1}{2}) \in \mathbb{R}^k.$$

Proof. Clearly for any $f \in \mathcal{S}(\mathbb{R}^k)$

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^k} [\tilde{R}(\mathrm{i}+1,0;\begin{pmatrix}\boldsymbol{s}\\ \boldsymbol{0}\end{pmatrix},0)f](\boldsymbol{m}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f(\boldsymbol{m}),$$

and hence also (replace f with $\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f$)

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^k} [\tilde{R}(\mathrm{i}+1,0;\begin{pmatrix}\boldsymbol{s}\\ \boldsymbol{0}\end{pmatrix},0)\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f](\boldsymbol{m}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^k} [\tilde{R}(\tau,\phi;\boldsymbol{\xi},t)f](\boldsymbol{m}).$$

We conclude by noticing

$$\tilde{R}(\mathrm{i}+1,0;\begin{pmatrix}\boldsymbol{s}\\ \boldsymbol{0}\end{pmatrix},0)\tilde{R}(\tau,\phi;\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix},t) = \tilde{R}(\tau+1,\phi;\begin{pmatrix}\boldsymbol{s}\\ \boldsymbol{0}\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix},t+\tfrac{1}{2}\boldsymbol{s}\cdot\boldsymbol{y}),$$

where we have used that $c((\mathrm{i},0),(\tau,\phi))=1$ since $(\mathrm{i},0)$ is an upper triangular matrix, cf. 1.6. □

Jacobi 3.

$$\Theta_f(\tau,\phi;\begin{pmatrix}\boldsymbol{k}\\ \boldsymbol{l}\end{pmatrix}+\boldsymbol{\xi},r+t+\tfrac{1}{2}\omega(\begin{pmatrix}\boldsymbol{k}\\ \boldsymbol{l}\end{pmatrix},\boldsymbol{\xi})) = (-1)^{\boldsymbol{k}\cdot\boldsymbol{l}}\,\Theta_f(\tau,\phi;\boldsymbol{\xi},t)$$

for any $\boldsymbol{k},\boldsymbol{l}\in\mathbb{Z}^k$, $r\in\mathbb{Z}$.

Proof. By virtue of 1.2 we have for all f

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^k} [W(\begin{pmatrix}\boldsymbol{k}\\ \boldsymbol{l}\end{pmatrix},r)f](\boldsymbol{m}) = e(-\tfrac{1}{2}\boldsymbol{k}\cdot\boldsymbol{l})\sum_{\boldsymbol{m}\in\mathbb{Z}^k} f(\boldsymbol{m}),$$

and therefore, replacing f with $W(\boldsymbol{\xi},t)\tilde{R}(\tau,\phi)f$,

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^k} [W(\begin{pmatrix}\boldsymbol{k}\\ \boldsymbol{l}\end{pmatrix},r)W(\boldsymbol{\xi},t)\tilde{R}(\tau,\phi)f](\boldsymbol{m}) = e(-\tfrac{1}{2}\boldsymbol{k}\cdot\boldsymbol{l})\sum_{\boldsymbol{m}\in\mathbb{Z}^k} [W(\boldsymbol{\xi},t)\tilde{R}(\tau,\phi)f](\boldsymbol{m}),$$

which gives the desired result. □

2.5. In what follows, we shall only need to consider products of theta sums of the form

$$\Theta_f(\tau,\phi;\boldsymbol{\xi},t)\overline{\Theta_g(\tau,\phi;\boldsymbol{\xi},t)},$$

where $f,g\in\mathcal{S}(\mathbb{R}^k)$. Clearly such combinations do not depend on the t-variable. Let us therefore define the semi-direct product group

$$G^k = \mathrm{SL}(2,\mathbb{R})\ltimes\mathbb{R}^{2k}$$

with multiplication law

$$(M;\boldsymbol{\xi})(M';\boldsymbol{\xi}') = (MM';\boldsymbol{\xi}+M\boldsymbol{\xi}'),$$

and put

$$\Theta_f(\tau,\phi;\boldsymbol{\xi}) = v^{k/4}\sum_{\boldsymbol{m}\in\mathbb{Z}^k} f_\phi((\boldsymbol{m}-\boldsymbol{y})v^{1/2})e(\tfrac{1}{2}\|\boldsymbol{m}-\boldsymbol{y}\|^2u+\boldsymbol{m}\cdot\boldsymbol{x}).$$

By virtue of Lemma 2.3 and the Iwasawa parametrization 1.8, $\Theta_f\overline{\Theta_g}$ is a continuous $\mathbb{C}$-valued function on G^k.

2.6. A short calculation yields that the set

$$\Gamma^k = \left\{(\begin{pmatrix}a&b\\c&d\end{pmatrix};\begin{pmatrix}ab\boldsymbol{s}\\cd\boldsymbol{s}\end{pmatrix}+\boldsymbol{m}) : \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z}),\ \boldsymbol{m}\in\mathbb{Z}^{2k}\right\},$$

with $\boldsymbol{s}={}^{\mathrm{t}}(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})\in\mathbb{R}^k$, is closed under multiplication and inversion, and therefore forms a subgroup of G^k. Note also that the subgroup

$$N = \{1\}\ltimes\mathbb{Z}^{2k}$$

is normal in Γ^k.

2.7. **Lemma.** Γ^k *is generated by the elements*

$$(\begin{pmatrix}0&-1\\1&0\end{pmatrix};\boldsymbol{0}),\quad (\begin{pmatrix}1&1\\0&1\end{pmatrix};\begin{pmatrix}\boldsymbol{s}\\ \boldsymbol{0}\end{pmatrix}),\quad (\begin{pmatrix}1&0\\0&1\end{pmatrix};\boldsymbol{m}),\quad \boldsymbol{m}\in\mathbb{Z}^{2k}.$$

Proof. The map

$$\mathrm{SL}(2,\mathbb{Z})\to N\backslash\Gamma^k,\qquad \begin{pmatrix}a&b\\c&d\end{pmatrix}\mapsto(\begin{pmatrix}a&b\\c&d\end{pmatrix};\begin{pmatrix}ab\boldsymbol{s}\\cd\boldsymbol{s}\end{pmatrix}+\mathbb{Z}^{2k})$$

defines a group isomorphism. The matrices $(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix})$ and $(\begin{smallmatrix}1&1\\0&1\end{smallmatrix})$ generate $\mathrm{SL}(2,\mathbb{Z})$, hence the lemma. □

2.8. **Proposition.** *The left action of the group* Γ^k *on* G^k *is properly discontinuous. A fundamental domain of* Γ^k *in* G^k *is given by*

$$\mathcal{F}_{\Gamma^k} = \mathcal{F}_{\mathrm{SL}(2,\mathbb{Z})}\times\{\phi\in[0,\pi)\}\times\{\boldsymbol{\xi}\in[-\tfrac{1}{2},\tfrac{1}{2})^{2k}\}.$$

where $\mathcal{F}_{\mathrm{SL}(2,\mathbb{Z})}$ *is the fundamental domain in* $\mathfrak{H}$ *of the modular group* $\mathrm{SL}(2,\mathbb{Z})$*, given by* $\{\tau\in\mathfrak{H} : u\in[-\frac{1}{2},\frac{1}{2}), |\tau|>1\}$.

Proof. As mentioned before, the matrices $(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix})$ and $(\begin{smallmatrix}1&1\\0&1\end{smallmatrix})$ generate $\mathrm{SL}(2,\mathbb{Z})$, which explains $\mathcal{F}_{\mathrm{SL}(2,\mathbb{Z})}$. Note furthermore that $(\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix})$ generates the shift $\phi\mapsto\phi+\pi$. □

2.9. **Proposition.** *For* $f,g\in\mathcal{S}(\mathbb{R}^k)$, $\Theta_f(\tau,\phi;\boldsymbol{\xi})\overline{\Theta_g(\tau,\phi;\boldsymbol{\xi})}$ *is invariant under the left action of* Γ^k.

Proof. This follows directly from Jacobi 1–3, since the left action of the generators from 2.7 is

$$(\tau,\phi;\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix})\mapsto(-\frac{1}{\tau},\phi+\arg\tau;\begin{pmatrix}-\boldsymbol{y}\\ \boldsymbol{x}\end{pmatrix}),$$

$$(\tau,\phi;\boldsymbol{\xi})\mapsto(\tau+1,\phi;\begin{pmatrix}\boldsymbol{s}\\ \boldsymbol{0}\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}),$$

and

$$(\tau,\phi;\boldsymbol{\xi})\mapsto(\tau,\phi;\boldsymbol{\xi}+\boldsymbol{m}),$$

respectively. □

We find the following uniform estimate.

2.10. **Proposition.** *Let* $f,g\in\mathcal{S}(\mathbb{R}^k)$*. For any* $R>1$*, we have*

$$\Theta_f(\tau,\phi;\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix})\overline{\Theta_g(\tau,\phi;\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix})} = v^{k/2}\sum_{\boldsymbol{m}\in\mathbb{Z}^k} f_\phi((\boldsymbol{m}-\boldsymbol{y})v^{1/2})\overline{g_\phi((\boldsymbol{m}-\boldsymbol{y})v^{1/2})}+O_R(v^{-R})$$

uniformly for all $(\tau,\phi;\boldsymbol{\xi})\in G^k$ *with* $v>\frac{1}{2}$.

References

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School of Mathematics, University of Bristol, Bristol BS8 1TW, United Kingdom

j.marklof@bristol.ac.uk