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# Unipotent flows on the space of branched covers of Veech surfaces

ALEX ESKIN†, JENS MARKLOF‡ and DAVE WITTE MORRIS§

† *Department of Mathematics, University of Chicago, Chicago, IL 60637, USA*
*(e-mail: eskin@math.uchicago.edu)*
‡ *School of Mathematics, University of Bristol, Bristol BS8 1TW, UK*
*(e-mail: j.marklof@bristol.ac.uk)*
§ *Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, Alberta, T1K 3M4, Canada*
*(e-mail: dave.morris@uleth.ca)*

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*Abstract*. There is a natural action of $\mathrm{SL}(2,\mathbb{R})$ on the moduli space of translation surfaces, and this yields an action of the unipotent subgroup $U = \left\{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\right\}$. We classify the $U$-invariant ergodic measures on certain special submanifolds of the moduli space. (Each submanifold is the $\mathrm{SL}(2,\mathbb{R})$-orbit of the set of branched covers of a fixed Veech surface.) For the $U$-action on these submanifolds, this is an analogue of Ratner's theorem on unipotent flows. The result yields an asymptotic estimate of the number of periodic trajectories for billiards in a certain family of non-Veech rational triangles, namely, the isosceles triangles in which exactly one angle is $2\pi/n$, with $n \geq 5$ and $n$ odd.

## 1. *Introduction*

A polygon $P \subset \mathbb{R}^2$ is called rational if all angles of $P$ are rational multiples of $\pi$. Let $N(P, T)$ denote the number of (cylinders of) periodic billiard trajectories of Euclidean length at most $T$. It is a theorem of Masur [**Ma1, Ma2**] that there exist constants $c_1 = c_1(P)$ and $c_2 = c_2(P) > 0$ such that, for $T \gg 1$,

$$c_1 T^2 < N(P, T) < c_2 T^2. \tag{1.1}$$

A natural question is whether equation (1.1) can be converted to an asymptotic formula as $T \to \infty$.

A well-known construction associates a 'translation surface' $S$ to each rational polygon $P$. Essentially the algorithm 'unfolds' the billiard trajectories, by reflecting the polygon instead of reflecting the trajectory. More precisely, let $\Delta \subset O(2)$ denote the group generated by reflections in the sides of the polygon $P$. Since $P$ is rational, $\Delta$ is finite.

The 'translation surface' consists of $\Delta$ copies of $P$, with each copy glued to each of its mirror images along the reflecting side.

For example, if $P$ is the unit square, then $S$ is the torus $\mathbb{R}^2/2\mathbb{Z} \oplus 2\mathbb{Z}$, and if $P$ is the isosceles triangle with angles $\pi/2 - \pi/n$, $\pi/2 - \pi/n$, $2\pi/n$, and $n$ is even, then $S$ is the regular $n$-gon with opposite sides identified.

A translation surface can be defined in one of the following equivalent ways.

(a) A union of polygons $P_1 \cup \cdots \cup P_n$ where each $P_i \subset \mathbb{R}^2$, and the $P_i$ are glued along parallel sides, such that each side is glued to exactly one other, and the total angle in each vertex is an integer multiple of $2\pi$.

(b) An orientable surface with a flat metric and isolated conical singularities that has trivial rotational holonomy. (Note that trivial rotational holonomy means in particular that parallel transport of a vector along a small loop going around a conical point brings a vector back to itself. This implies that all cone angles are integer multiples of $2\pi$.)

(c) A pair $(M, \omega)$, where $M$ is an (orientable) Riemann surface, and $\omega$ is a holomorphic 1-form on $M$. (Note that, away from the zeroes of $\omega$, there is a local coordinate $z$ such that $\omega = dz$, and this coordinate is unique up to translation. Then one can define the metric on $M$ as $|dz|^2$. This metric is flat, with conical singularities appearing at the zeroes of $\omega$.)

The term 'translation surface' comes from the fact that away from the cone points the surface can be covered by charts so that the transition functions are translations ($z \to z+c$). If $\alpha = (\alpha_1, \ldots, \alpha_k)$ is an $n$-tuple of positive integers such that the sum of the $\alpha_i$ is even, we denote by $\mathcal{H}(\alpha)$ the moduli space of translation surfaces $(M, \omega)$ such that the multiplicities of the zeroes of $\omega$ are given by $\alpha_1, \ldots, \alpha_n$ (or equivalently such that the orders of the conical singularities are $2\pi(\alpha_1+1), \ldots, 2\pi(\alpha_n+1)$). (Actually, for technical reasons, the singularities of $(M, \omega)$ should be labeled; thus, an element of $\mathcal{H}(\alpha)$ is a tuple $(M, \omega, p_1, \ldots, p_n)$, where $p_1, \ldots, p_n$ are the singularities of $M$, and the multiplicity of $p_i$ is $\alpha_i$.) The moduli space of translation surfaces is naturally stratified by the spaces $\mathcal{H}(\alpha)$; each is called a *stratum*.

By construction, billiard trajectories on $P$ correspond to 'straight lines' on $S$, which are geodesics not passing through singularities. It is easy to see that any such geodesic is part of a family of freely homotopic parallel geodesics of the same length. Such a family is called a cylinder. Let $N(S, T)$ denote the number of cylinders on $S$ of length at most $T$. (By the length of a cylinder we mean the length of any of the closed geodesics that comprise it.)

1.1. *The* $\mathrm{SL}(2, \mathbb{R})$ *action.* There is an action of $\mathrm{SL}(2, \mathbb{R})$ on the moduli space of translation surfaces that preserves the stratification. For our purpose, it is easiest to see this using definition (1): since $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathbb{R}^2$, for $S = P_1 \cup \cdots \cup P_n$, we can define $gS = gP_1 \cup \cdots \cup gP_n$, where all identifications between the sides of the polygons for $gS$ are the same as for $S$. This action generalizes the action of $\mathrm{SL}(2, \mathbb{R})$ on the space of flat tori $\mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$.

We can visualize this as a composition of 'the usual linear action' with 'cut and paste'. We note that 'cut and paste' is an isometry on the surface (and in fact preserves the horizontal and vertical directions as well). Note that if $S$ is a union of triangles, and $g$

is a large element of $\mathrm{SL}(2,\mathbb{R})$ then $gS$ is a union of long and thin triangles. We may if we wish 'cut and paste' $gS$ and retriangulate to try to present $gS$ as a union of triangles with bounded side lengths.

1.2. *Veech surfaces.* For $S \in \mathcal{H}(\alpha)$, let $\Gamma(S) \subset \mathrm{SL}(2,\mathbb{R})$ denote the stabilizer of $S$. The group $\Gamma(S)$ is called the *Veech group* of $S$. If $\Gamma(S)$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$ then $S$ is called a *Veech surface*. It is a theorem of Veech [**Ve1**] that if $S$ is a Veech surface, then there exists $c = c(S)$ such that

$$N(S,T) \sim cT^2 \tag{1.2}$$

as $T \to \infty$.

1.3. *Counting and Ratner's theorem.* One has the formula [**Ve2**] (reproduced in [**EM**])

$$N(S,T) - N(S,T/2) \approx T^2 \int_0^{2\pi} \hat{f}(a_t r_\theta S)\, d\theta, \tag{1.3}$$

where

$$a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

and $t = \log T$. The left-hand side counts (cylinders of) closed geodesics in an annulus, and the right-hand side is an integral over part of the $\mathrm{SL}(2,\mathbb{R})$ orbit of $S$. Thus, the $\mathrm{SL}(2,\mathbb{R})$ action can be used to count closed geodesics (and thus periodic billiard trajectories).

A closer examination of equation (1.3) shows that the integral is over large circles inside the $\mathrm{SL}(2,\mathbb{R})$ orbit. These large circles can be approximated by horocycles, which are orbits of $u_t = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. Thus the ergodic properties of the action of $U = \{u_t \mid t \in \mathbb{R}\}$ play a key role.

Ratner's theorem [**Ra6**] is the classification of the invariant measures for the action of a unipotent subgroup on the homogeneous space $H/\Gamma$, where $H$ is a Lie group and $\Gamma$ is a lattice in $H$. An important question is whether a similar theorem holds for the $U$-action on a stratum $\mathcal{H}(\alpha)$. One can also ask this question when one restricts the action to any $\mathrm{SL}(2,\mathbb{R})$ invariant submanifold of a stratum. In this paper, we will classify the $U$-invariant measures on a certain family of $\mathrm{SL}(2,\mathbb{R})$-invariant manifolds. Another result in this direction was obtained by McMullen [**Mc**] who, in genus two, classified the measures invariant under all of $\mathrm{SL}(2,\mathbb{R})$.

1.4. *Branched covers of Veech surfaces.* We say that a translation surface $S$ is a branched cover of a translation surface $M$ if the covering map $\pi$ respects the translation structure (i.e. if we identify $S = (L_1, \omega_1)$ and $M = (L_2, \omega_2)$ where the $L_i$ are Riemann surfaces and the $\omega_i$ are holomorphic 1-forms on $L_i$ then we require that $\pi : L_1 \to L_2$ is holomorphic and $\pi^*(\omega_2) = \omega_1$).

Now let $M \in \mathcal{H}(\alpha)$ be a Veech surface. Then the $\mathrm{SL}(2,\mathbb{R})$ orbit of $M$ is a closed subset $D$ of $\mathcal{H}(\alpha)$. Let $\mathcal{H}(\beta)$ be another stratum, and let $\mathcal{M}_D(\beta)$ denote the set of all translations surfaces $S \in \mathcal{H}(\beta)$ that are branched covers of $M \in D$. We will always assume that $\beta$ is such that $\mathcal{M}_D(\beta)$ is non-empty. Then $\mathcal{M}_D(\beta)$ is $\mathrm{SL}(2,\mathbb{R})$ invariant.

There are two types of Veech surfaces: arithmetic and non-arithmetic. A surface $S = (M, \omega)$ is an arithmetic Veech surface if and only if $M$ is a (holomorphic) branched cover of a torus, $\omega$ is the pullback by the covering map of the standard differential $dz$ on the torus, and the branch points project to points of finite order (under the additive group of the torus). Equivalently (see [**GJ**]), $S$ is an arithmetic Veech surface if and only if $\Gamma(S)$ is commensurable to $\mathrm{SL}(2, \mathbb{Z})$. All other Veech surfaces are called non-arithmetic (and their Veech groups, which are always non-uniform lattices, are non-arithmetic lattices in $\mathrm{SL}(2, \mathbb{R})$). The case where $M$ is arithmetic was analyzed in [**EMS**].

In this paper, we assume that $M$ is not arithmetic, which implies that the genus of $M$ is greater than one. Then considering the Euler characteristic, it is easy to see that the degree of $\pi$ is determined by $D$ and $\beta$. This implies that $\mathcal{M}_D(\beta)$ is closed. (In the case where the genus of $M$ is one, one also has to fix the degree of the cover; see [**EMS**] for the details.)

The main result of this paper is a classification of the $U$-invariant ergodic measures on $\mathcal{M}_D(\beta)$. This allows us to prove asymptotic formulas of the form (1.2) for $S \in \mathcal{M}_D(\beta)$ (see Theorem 8.12). In particular we prove the following.

THEOREM 1.4. *Let $P_n$ be a triangle with angles*

$$\frac{n-2}{2n}\pi,\ \frac{n-2}{2n}\pi,\ \frac{4}{2n}\pi,$$

*where $n \geq 5$, $n$ odd. Then, as $T \to \infty$,*

$$N(P_n, T) \sim \frac{\pi}{\zeta(2)} \frac{(n-1)(n^2+n+3)}{144(n-2)} \frac{T^2}{\mathrm{area}(P_n)}.$$

The fact that the surface $S_n$ associated to $P_n$ is not Veech but is a branched cover of degree two of a Veech surface is due to Hubert and Schmidt (see Proposition 4 in [**HS1**] and its proof). We should also note that if $n = 5$ then the Veech group of $S_n$ is infinitely generated (see [**HS2**]). However, the Veech group of $S_n$ plays no direct role in our analysis.

Here is an outline of the paper. Section 2 states our main theorem. Section 3 establishes notation and presents a few basic lemmas. Section 4 explains 'shearing', the foundation of our study of invariant measures. Section 5 proves our main Theorem 2.6 that classifies $U$-invariant measures. Section 6 proves that there are only countably many closed orbits of a certain type. Section 7 uses our main theorem (and the countability result of §6) to prove that large circles in $\mathrm{SL}(2, \mathbb{R})$-orbits become uniformly distributed with respect to certain natural measures. Section 8 applies the equidistribution result of §7 to derive asymptotic estimates for the number of periodic trajectories in branched covers of Veech surfaces.

## 2. *Measure classification*

2.1. *Definitions and notation.* Let $G = \mathrm{SL}(2, \mathbb{R})$. Let $M$ be a Veech surface, which means that $\Gamma = \mathrm{Stab}_G(M)$ is a lattice in $G$. Here, we use $M$ to also denote the isometry class of $M$; this is a single point in the moduli space. For $k \in \mathbb{N}$, we define $\mathcal{X}^k$ to be the natural fiber bundle over $G \cdot M$ whose fiber over $M$ is $M^k$. Thus, a point of $\mathcal{X}^k$ is represented by $(M', p_1, \ldots, p_k)$, where $M' \in GM$ and $p_1, \ldots, p_k \in M'$. In other words, a point in $\mathcal{X}^k$ represents a surface in $M' \in GM$ together with $k$ marked points on $M'$.

We note that the space $\mathcal{M}_D(\beta)$ parameterizing branched covers is itself a finite branched cover of the space $\mathcal{X}^k$ for a suitable $k$. (The covering map just maps $S \in M_D(\beta)$ to the surface in $D$ it covers, and notes the locations of the branch points.) Thus, to classify the $U$-invariant measures on $\mathcal{M}_D(\beta)$ it is enough to classify $U$-invariant measures on $\mathcal{X}^k$ (see Lemma 8.14).

If $M$ is a torus, then $\mathcal{X}^k$ can be identified with the homogeneous space $(G \ltimes (\mathbb{R}^2)^k)/(\mathrm{SL}(2,\mathbb{Z}) \ltimes (\mathbb{Z}^2)^k)$. In this situation, a special case of Ratner's theorem [**Ra6**] classifies all the ergodic $U$-invariant probability measures on $\mathcal{X}^k$. We generalize this to allow $M$ to be any Veech surface. The proof is based heavily on ideas of Ratner [**Ra1–Ra6**] and Margulis and Tomanov [**MaT**]. An introduction to these ideas can be found in [**Mo**].

Let $\Sigma$ be the singular set of $M$. Then for $g \in G$, $g\Sigma$ is the singular set of $gM$. Let $M_0 = M \setminus \Sigma$, and let $\mathcal{X}_0^k \subset \mathcal{X}^k$ denote the set $(gM, p_1, \ldots, p_k)$ where $g \in G$ and $\{p_1, \ldots, p_k\} \cap g\Sigma = \emptyset$. Then $\mathcal{X}_0^k$ is isomorphic to the natural fiber bundle over $GM$ whose fiber over $M$ is $(M_0)^k$.

We have a natural embedding of $\mathbb{R}^2$ in the space $\mathrm{Vect}(M_0)$ of smooth vector fields on $M_0$, so, for each $v \in \mathbb{R}^2$ and $p \in M_0$, we have a trajectory $\gamma_{v,p}(t)$ in $M_0$ that is defined for $t$ in a certain open interval containing 0 (until the trajectory hits the singular set). We are interested only in the forward trajectory, that is, for $t \geq 0$. By including the singular points of $M$, we extend $\gamma_{v,p}$ to a continuous curve $\widehat{\gamma}_{v,p}$ in $M$ that is defined for $t$ in a closed interval (and for all points in $M$):

- let $\widehat{\gamma}_{v,p}(0) = p$ for all $v \in \mathbb{R}^2$ and $p \in M$; and
- if $t > 0$ and $t$ is in the closure of the domain of $\gamma_{v,p}$, let

$$\widehat{\gamma}_{v,p}(t) = \lim_{s \to t^-} \gamma_{v,p}(s) \in \Sigma.$$

Then each $v \in \mathbb{R}^2$ defines a function $\widehat{\phi}_v : M_v \to M$, defined by $\widehat{\phi}_v(p) = \widehat{\gamma}_{p,v}(1)$, where $M_v$ is a dense, open subset of $M$. Note that $\widehat{\phi}_v$ is a local isometry (hence continuous). On the other hand, $\widehat{\phi}_v$ is usually not invertible, because a singular point will typically have several preimages. In addition, $\widehat{\phi}_v$ is usually not uniformly continuous, because of branch cuts.

For $w \in (\mathbb{R}^2)^k$, we have a continuous map $\widehat{\phi}_w^k : \mathcal{X}_w^k \to \mathcal{X}^k$ (where $\mathcal{X}_w^k$ is a certain subset of $\mathcal{X}^k$), defined by

$$\widehat{\phi}_w^k(M, p_1, \ldots, p_k) = (M, \widehat{\phi}_{v_1}(p_1), \ldots, \widehat{\phi}_{v_k}(p_k)).$$

(Thus, $\widehat{\phi}_w^k$ does not change the surface $M$, but moves the marked points in the directions specified by $w$.) Let $\widehat{\Phi}_{(\mathbb{R}^2)^k}^k$ be the pseudosemigroup generated by $\{\widehat{\phi}_w^k \mid w \in (\mathbb{R}^2)^k\}$. (The prefix 'pseudo' simply refers to the fact that these maps are not defined on the entire space $\mathcal{X}^k$, but only on a subset.) Although the maps in $\widehat{\Phi}_{(\mathbb{R}^2)^k}^k$ may not be one-to-one, they are always finite-to-one.

For $w \in (\mathbb{R}^2)^k$, let $\phi_w^k$ be the restriction of $\widehat{\phi}_w^k$ to $(\widehat{\phi}_w^k)^{-1}(\mathcal{X}_0^k)$. Then $\phi_w^k$ is a diffeomorphism (and local isometry) from a dense open subset of $\mathcal{X}_0$ to a dense open subset of $\mathcal{X}_0$. Let $\Phi_{(\mathbb{R}^2)^k}^k$ be the pseudogroup that is generated by $\{\phi_w^k \mid w \in (\mathbb{R}^2)^k\}$. We remark that $\Phi_{(\mathbb{R}^2)^k}^k$ is transitive on $\mathcal{X}_0^k$.

Note that each of $\Phi_{(\mathbb{R}^2)^k}^k$ and $\widehat{\Phi}_{(\mathbb{R}^2)^k}^k$ is normalized by the action of $G = \mathrm{SL}(2,\mathbb{R})$ on $\mathcal{X}^k$, so we have corresponding semidirect products $G \ltimes \Phi_{(\mathbb{R}^2)^k}^k$ and $G \ltimes \widehat{\Phi}_{(\mathbb{R}^2)^k}^k$.

Let

$$\mathrm{Horiz} = \{((x_i, 0))_{i=1}^k \mid x_i \in \mathbb{R}\} \subset (\mathbb{R}^2)^k$$

and

$$\widehat{\mathrm{Horiz}} = \{\widehat{\phi}_w^k \mid w \in \mathrm{Horiz}\}.$$

Note that, for $w_1, w_2 \in$ Horiz, we have $\widehat{\phi}_{w_1+w_2}^k = \widehat{\phi}_{w_1}^k \widehat{\phi}_{w_2}^k$ on the intersection of their domains, so $\widehat{\mathrm{Horiz}}$ is a pseudosemigroup. Also, $\widehat{\mathrm{Horiz}}$ commutes with the action of $U$.

2.2. *Statement of the main results.* Let $\mu$ be an ergodic $U$-invariant probability measure on $\mathcal{X}^k$. The projection of $\mu$ to $G/\Gamma$ is $U$-invariant, so it must be either Lebesgue measure or the arc-length on a closed $U$-orbit [**Da**]. The interesting case is when the projection is Lebesgue. A weak statement of our results is simply to say that, in this case, some horizontal translate of $\mu$ must be $G$-invariant.

THEOREM 2.1. *Suppose $\mu$ is any ergodic $U$-invariant probability measure on $\mathcal{X}^k$, such that the projection of $\mu$ to $G/\Gamma$ is Lebesgue. Then there exists $h \in \widehat{\mathrm{Horiz}}$, such that $h_*\mu$ is $G$-invariant (and the domain of $h$ has full measure).*

To obtain a more precise description of the $U$-invariant measures, one need only describe the $G$-invariant measures on $\mathcal{X}^k$.

*Remark 2.2.*

(1) It is easy to see that the $G$-invariant probability measures on $\mathcal{X}^k$ are in natural one-to-one correspondence with the $\Gamma$-invariant probability measures on $M^k$ (cf., e.g., [**Wi**, proof of Corollary 5.8]).

(2) It is the $\Gamma$-invariant measures on $M_0^k$ that are the most important to understand, because it is easy to see that every ergodic measure on $M^k$ arises from the following construction. Choose some $p_1 \in \Sigma^d$ and some probability measure $\nu$ on $M_0^{k-d}$ that is invariant under a finite-index subgroup of $\Gamma$. The corresponding measure on $\{p_1\} \times M^{k-d}$ is invariant under a finite-index subgroup $\Gamma'$ of $\Gamma$. By averaging over $\Gamma/\Gamma'$, this yields a $\Gamma$-invariant measure supported on the subset $\Sigma^d \times M^{k-d}$ of $M^k$.

We will show that every ergodic measure is carried by a nice subspace of $M^k$. In particular, any ergodic measure carried by $M_0^k$ is the Lebesgue measure on a flat submanifold of $M_0^k$.

*Example 2.3.* The natural Lebesgue measure on the diagonal $\Delta = \{(p, p, p)\}$ of $M_0^3$ is a $\Gamma$-invariant probability measure on $M_0^3$. Note that $W = \{v, v, v\}$ is a $G$-invariant subspace of $(\mathbb{R}^2)^3$, and that the pseudogroup $\Phi_W^3$ of diffeomorphisms it generates is transitive on $\Delta$.

THEOREM 2.4. *Suppose $\mu$ is an ergodic $\Gamma$-invariant probability measure on $M^k$. Then there exist*

- *a point $p \in M^k$; and*
- *a $G$-invariant linear subspace $W$ of $(\mathbb{R}^2)^k$,*

*such that:*

(1) *the orbit $\widehat{\Phi}_W^k(p)$ of $p$ under $\widehat{\Phi}_W^k$ is a closed subset of $M^k$ whose dimension is* $\dim W$*;*

(2) *some finite-index subgroup of $\Gamma$ fixes $\widehat{\Phi}_W^k(p)$ setwise; and*

(3) *$\mu$ is the $\widehat{\Phi}_W^k(p)$-invariant Lebesgue measure on $\Gamma\widehat{\Phi}_W^k(p)$.*

*Remark 2.5.*

(1) Conversely, if $W$ is $G$-invariant, $\widehat{\Phi}^k_W(p)$ is closed, and some finite-index subgroup of $\Gamma$ fixes $\widehat{\Phi}^k_W(p)$, then the $\widehat{\Phi}^k_W$-invariant Lebesgue measure on $\Gamma\widehat{\Phi}^k_W(p)$ is a $\Gamma$-invariant probability measure. However, it may not be ergodic.

(2) We wish to emphasize that conclusion (2) in Theorem 2.4 implies the set $\Gamma\widehat{\Phi}^k_W(p)$ is a finite union of translates of $\widehat{\Phi}^k_W(p)$.

The theorem can be stated in the following equivalent form (see Remark 2.2(1)).

THEOREM 2.4′. *Suppose $\mu$ is an ergodic $G$-invariant probability measure on $\mathcal{X}^k_0$. Then there exist*

- *a point $(M, p) \in \mathcal{X}^k$; and*
- *a $G$-invariant linear subspace $W$ of $(\mathbb{R}^2)^k$,*

*such that:*

(1) *the orbit $(G \ltimes \widehat{\Phi}^k_W)p$ of $p$ under $G \ltimes \widehat{\Phi}^k_W$ is a closed subset of $\mathcal{X}^k$ whose dimension is $\dim(G \ltimes \widehat{\Phi}^k_W)$; and*

(2) *$\mu$ is the $(G \ltimes \widehat{\Phi}^k_W)$-invariant Lebesgue measure on this orbit.*

This results in the following explicit version of Theorem 2.1.

THEOREM 2.6. *Suppose $\mu$ is an ergodic $U$-invariant probability measure on $\mathcal{X}^k_0$. Then there exist*

- *a point $(M, p) \in \mathcal{X}^k$,*
- *a $G$-invariant subspace $W$ of $(\mathbb{R}^2)^k$, and*
- *some $h \in \mathrm{Horiz}$,*

*such that:*

(1) *$\mu(\mathrm{domain}(\widehat{\phi}^k_h)) = 1$;*

(2) *the orbit $(G \ltimes \widehat{\Phi}^k_W)p$ of $p$ under $G \ltimes \widehat{\Phi}^k_W$ is a closed subset of $\mathcal{X}^k$ whose dimension is $\dim G + \dim W$; and*

(3) *$(\widehat{\phi}^k_h)_*\mu$ is the $(G \ltimes \widehat{\Phi}^k_W)$-invariant Lebesgue measure on this orbit.*

We will give an application to counting the number of periodic trajectories on $M$ (see §8).

Theorems 2.1, 2.4, and 2.4′ have been stated only for expository purposes—they are not a part of the logical development. We prove only Theorem 2.6, and the interested reader can easily derive the other theorems as corollaries.

Our results imply that the closure of every $\Gamma$-orbit in $M^k$ is of a nice geometric form. Since $M \setminus M_0 = \Sigma$ is a $\Gamma$-invariant finite set, it suffices to describe the orbits of points in $M^k_0$.

COROLLARY 7.13′. *Suppose $p \in M^k_0$. Then there exists a $G$-invariant linear subspace $W$ of $(\mathbb{R}^2)^k$, such that:*

(1) *the orbit $\widehat{\Phi}^k_W(p)$ of $p$ under $\widehat{\Phi}^k_W$ is a closed subset of $M^k$ (and its dimension is $\dim W$);*

(2) *some finite-index subgroup of $\Gamma$ fixes $\widehat{\Phi}^k_W(p)$ setwise; and*

(3) *$\Gamma\widehat{\Phi}^k_W(p)$ is the closure of the $\Gamma$-orbit of $p$.*

3. *Preliminaries*

We collect all the notation in this section. Some of this repeats the definitions given in the previous sections.

*Notation 3.1.*

- Let $G = \mathrm{SL}(2, \mathbb{R})$.
- There is a natural action of $G$ on the moduli space of translation surfaces. We can visualize this as a composition of 'the usual linear action' with 'cut and paste'. We note that 'cut and paste' is an isometry on the surface (and in fact preserves the horizontal and vertical directions as well).
- Let $M$ be a Veech surface, which means that $\Gamma = \mathrm{Stab}_G(M)$ is a lattice in $G$. Here, we use $M$ to also denote the isometry class of $M$; this is a single point in the moduli space.
- Let $k \in \mathbb{N}$.
- We define $\mathcal{X}^k$ to be the natural fiber bundle over $GM$ whose fiber over $M$ is $M^k$. Thus, a point of $\mathcal{X}^k$ is represented by $(M', p_1, \ldots, p_k)$, where $M' \in GM$ and $p_1, \ldots, p_k \in M'$.
- The metric on $\mathcal{X}^k$ is defined by

$$d_{\mathcal{X}^k}([M', (p_i)_{i=1}^k], [M', (q_i)_{i=1}^k]) = \min_{\substack{g \in G,\\ gM'=M'}} \left( \|g - \mathrm{Id}\| + \sum_{i=1}^{k} d_{M'}(gp_i, q_i) \right).$$

- Note that $\mathcal{X}^k$ is $G$-equivariantly homeomorphic to $(G \times M^k)/\Gamma$, where:
  - $\Gamma$ acts on $G$ by right multiplication;
  - $\Gamma$ acts on $M^k$ componentwise; and
  - $G$ acts on $\mathcal{X}^k$ via $g(h, (p_i)_{i=1}^k) = (gh, g(p_i)_{i=1}^k)$.
- Let $\Sigma$ be the singular set of $M$.
- Let $M_0 = M \setminus \Sigma$.
- Let $\mathcal{X}_0^k = (G \times M_0^k)/\Gamma \subset \mathcal{X}^k$.
- Any $w \in (\mathbb{R}^2)^k$ naturally defines a vector field on $\mathcal{X}_0$. By taking the time-one map of the corresponding flow (where it is defined), we obtain a diffeomorphism $\phi_w^k$ between two dense open subsets of $\mathcal{X}_0$. The collection $\{\phi_w^k \mid w \in (\mathbb{R}^2)^k\}$ generates a transitive pseudogroup $\Phi^k_{(\mathbb{R}^2)^k}$ of local diffeomorphisms of $\mathcal{X}_0^k$.

  We extend $\phi_w^k$ to a (continuous) transformation $\widehat{\phi}_w^k$ that is defined on a slightly larger subset of $\mathcal{X}$, by letting

$$\widehat{\phi}_w^k(x) = \lim_{\substack{x' \to x\\ x' \in \operatorname{domain} \phi_w^k}} \phi_w^k(x')$$

  if the limit exists. (See §2 for a more concrete definition of $\widehat{\phi}_w^k$, in terms of the flow corresponding to $w$.) We let $\widehat{\Phi}^k_{(\mathbb{R}^2)^k}$ be the pseudosemigroup generated by these maps.

  Because the action of $G$ on $\mathcal{X}^k$ normalizes $\Phi^k_{(\mathbb{R}^2)^k}$ and $\widehat{\Phi}^k_{(\mathbb{R}^2)^k}$, we have semidirect products $G \ltimes \Phi_{(\mathbb{R}^2)^k}$ and $G \ltimes \widehat{\Phi}_{(\mathbb{R}^2)^k}$. Note that $G \ltimes \Phi_{(\mathbb{R}^2)^k}$ is transitive on $\mathcal{X}_0^k$.

  It is important to note that, because of the singularities and resulting branch cuts, $\phi_w^k$ is usually not uniformly continuous (even though it is a local isometry).

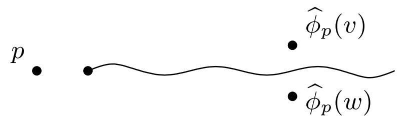


FIGURE 1. In our notation, $v \in \mathbb{R}^2$ and $w \in \mathbb{R}^2$ can be close, but $\widehat{\phi}_v(p)$ and $\widehat{\phi}_w(p)$ may not be close. The wavy line represents a branch cut.

Furthermore, $\phi_w^k(p)$ is not a uniformly continuous function of $w$. See Figure 1. Abusing notation, we may sometimes write $w + p$ instead of $\phi_w(p)$.

- Let $U = \{u^t \mid t \in \mathbb{R}\}$, where $u^t = \left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right] \in G$.
- Let $A = \{a^s \mid s \in \mathbb{R}\}$, where $a^s = \left[\begin{smallmatrix} e^s & 0 \\ 0 & e^{-s} \end{smallmatrix}\right] \in G$.
- Let $V = \{v^r \mid r \in \mathbb{R}\}$, where $v^r = \left[\begin{smallmatrix} 1 & 0 \\ r & 1 \end{smallmatrix}\right] \in G$.
- Let $\mu$ be a $U$-invariant probability measure on $\mathcal{X}^k$, such that $\mu$ projects to the Lebesgue measure on $\Gamma\backslash G$.
- Let $\mathrm{Horiz} = \{((x_i, 0))_{i=1}^k \mid x_i \in \mathbb{R}\} \subset (\mathbb{R}^2)^k$ and $\widehat{\mathrm{Horiz}} = \{\widehat{\phi}_w^k \mid w \in \mathrm{Horiz}\}$. Then $\widehat{\mathrm{Horiz}}$ is a pseudosemigroup.
- Let $\mathrm{Vert} = \{((0, y_i))_{i=1}^k \mid y_i \in \mathbb{R}\} \subset (\mathbb{R}^2)^k$ and

  let $\widehat{\mathrm{Vert}}$ be the pseudosemigroup generated by $\{\widehat{\phi}_w \mid w \in \mathrm{Vert}\}$.
- For $s \in \mathbb{R}$, we define $H_s \colon (\mathbb{R}^2)^k \to \mathrm{Horiz}$ by $H_s(w) = u^s w - w$. Thus,
  $$H_s((x_i, y_i)_{i=1}^k) = (s y_i, 0)_{i=1}^k.$$
- The set
  $$\mathcal{X}_{\mathrm{Horiz}}^k = \{p \in \mathcal{X}_0 \mid \widehat{\mathrm{Horiz}}\, p \subset \mathcal{X}_0\}$$
  is $U$-invariant. Thus, it is either null or conull. Let us assume it is conull. (If not, then by ergodicity, there exists $h \in \widehat{\Phi}_{\mathrm{Horiz}}^k$ such that $h_*\mu$ is supported on $\mathcal{X} \setminus \mathcal{X}_0$. So $h_*\mu$ can be described by a construction similar to Remark 2.2(2). The conclusion of Theorem 2.6 is therefore obtained by induction on $k$.)

  Note that Horiz acts on $\mathcal{X}_{\mathrm{Horiz}}^k$, by $x(p) = \phi_x^k(p)$. Therefore, the group $AU \ltimes \mathrm{Horiz}$ acts on $\mathcal{X}_{\mathrm{Horiz}}^k$.
- Let
  $$\mathcal{X}_{\mathrm{Vert}}^k = \{p \in \mathcal{X}_0 \mid \widehat{\mathrm{Vert}}\, p \subset \mathcal{X}_0\},$$
  note that $AV \ltimes \mathrm{Vert}$ acts on $\mathcal{X}_{\mathrm{Vert}}^k$, but we do not yet know that $\mathcal{X}_{\mathrm{Vert}}^k$ is conull.
- Let $X = \{x \in \mathrm{Horiz} \mid x_*\mu = \mu\}$. Because Horiz acts on $\mathcal{X}_{\mathrm{Horiz}}^k$, we know that $X$ is a closed subgroup of Horiz.
- Let $Y = (v^1 - \mathrm{Id})X \subset \mathrm{Vert}$. Equivalently,
  $$Y = \{y \in \mathrm{Vert} \mid H_s(y) \in X, \text{ for all } s \in \mathbb{R}\}.$$
- Let $W = X + Y$. Note that $W$ is a $G$-invariant subspace of $(\mathbb{R}^2)^k$, so $G \ltimes \widehat{\Phi}_W^k$ is a pseudosemigroup.
- Let $d = \dim X$.
- Let $\mathrm{Horiz} \ominus X = \mathrm{Horiz} \cap (0^d \times (\mathbb{R}^2)^{k-d})$. By permuting coordinates, we may assume $X \cap (\mathrm{Horiz} \ominus X) = 0$.

- Let $\pi_i : \mathcal{X}^k \to \mathcal{X}^i$ (the first $i$ coordinates) be the natural projection.
- For $\omega \in \mathcal{X}^k$, we use $\mu_{\pi_i(\omega)}$ to denote the fiber measure of $\mu$ over the point $\pi_i(\omega)$ of $\mathcal{X}^i$.

The following is obtained by applying the pointwise ergodic theorem to the action of $U$ on $\mathcal{X}^k$.

LEMMA 3.2. (Cf. [**MaT**, Lemma 7.3]) *For any $\rho > 0$, there is a 'uniformly generic set' $\Omega_\rho$ in $\mathcal{X}^k$, such that:*

(1) $\mu(\Omega_\rho) > 1 - \rho$;

(2) *for every $\epsilon > 0$ and every compact subset $K$ of $\mathcal{X}^k$, with $\mu(K) > 1 - \epsilon$, there exists $L_0 \in \mathbb{R}^+$, such that, for all $\omega \in \Omega_\rho$ and all $L > L_0$, we have*

$$\lambda\{s \in [-L, L] \mid d(u^s\omega, K) < \epsilon\} > (1 - \epsilon)(2L),$$

*where $\lambda$ is the Lebesgue measure on $\mathbb{R}$.*

LEMMA 3.3. (Cf. [**Ra4**, Theorem 2.2, **Mo**, Lemma 5.8.6]) *Suppose a Lie group $H$ acts continuously on a Borel subset $M$ of a locally compact metric space. If*

- *$U$ is a one-parameter, normal subgroup of $H$, and*
- *$\mu$ is an ergodic $U$-invariant probability measure on $M$,*

*then:*

(1) *there is a $U$-invariant, Borel subset $\Omega$ of $M$, such that*
 (a) $\mu(\Omega) = 1$, *and*
 (b) $\Omega \cap c\Omega = \emptyset$ *for all* $c \in H \setminus \mathrm{Stab}_H(\mu)$;
 *and*

(2) *for any $\epsilon > 0$, there is a compact subset $K$ of $M$, such that:*
 (a) $\mu(K) > 1 - \epsilon$, *and*
 (b) $K \cap cK = \emptyset$ *for all* $c \in H \setminus \mathrm{Stab}_H(\mu)$.

*Proof.* Ratner's argument in [**Ra4**, Theorem 2.2] shows, for each $h_0 \in H \setminus \mathrm{Stab}_H(\mu)$, that there is a neighborhood $B_{h_0}$ of $h_0$ in $H \setminus \mathrm{Stab}_H(\mu)$ and a conull $U$-invariant subset $\Omega_{h_0}$ of $M$, such that

$$\Omega_{h_0} \cap h\Omega_{h_0} = \emptyset \quad \text{for all } h \in B_{h_0}.$$

For the reader's convenience, we sketch the proof of this fact. Because $h_0$ normalizes $U$ but does not belong to $\mathrm{Stab}_H(\mu)$, we know that $(h_0)_*\mu$ is $U$-invariant and ergodic, but is not equal to $\mu$. Therefore $(h_0)_*\mu$ and $\mu$ are mutually singular, which implies there is a compact subset $K_0$ of $M$, such that $\mu(K_0) > 0.99$ and $K_0 \cap h_0K_0 = \emptyset$. By continuity and compactness, there are open neighborhoods $\mathcal{U}$ and $\mathcal{U}^+$ of $K_0$, and a symmetric neighborhood $B_e$ of $e$ in $H$, such that $\mathcal{U}^+ \cap h_0(\mathcal{U}^+ \cap M) = \emptyset$ and $B_e(\mathcal{U} \cap M) \subset \mathcal{U}^+$. From the pointwise ergodic theorem, we know there is a conull $U$-invariant subset $\Omega_{h_0}$ of $M$, such that the $U$-orbit of every point in $\Omega_{h_0}$ spends 99% of its life in $\mathcal{U} \cap M$. Now suppose there exists $h \in B_eh_0$, such that $\Omega_{h_0} \cap h\Omega_{h_0} \neq \emptyset$. Then there exist $x \in \Omega_{h_0}$, $u \in U$, and $c \in B_e$, such that $ux$ and $ch_0ux$ both belong to $\mathcal{U} \cap M$. This implies that $ux$ and $h_0ux$ both belong to $\mathcal{U}^+$. This contradicts the fact that $\mathcal{U}^+ \cap h_0\mathcal{U}^+ = \emptyset$.

(1) Cover $H \setminus \mathrm{Stab}_H(\mu)$ with countably many balls $B_{h_j}$, and let $\Omega = \bigcap_{j=1}^{\infty} \Omega_{h_j}$.

(2) Let $K$ be any compact subset of $\Omega$ with $\mu(K) > 1 - \epsilon$. □

THEOREM 3.4. (Kerckhoff–Masur–Smillie [**KMS**, Theorem 2]) *For almost every $v \in \mathbb{R}^2$, the foliation by orbits of $\mathbb{R}v$ is uniquely ergodic on $M_0$.*

COROLLARY 3.5. *Suppose $\mu$ is a $U$-invariant probability measure on $\mathcal{X}^k_{\mathrm{Horiz}}$ whose projection to $G/\Gamma$ is Lebesgue.*

*If $\mu$ is* Horiz*-invariant, then $\mu$ is the Lebesgue measure.*

*Proof.* Theorem 3.4 implies that the foliation by orbits of Horiz is uniquely ergodic on $gM_0^k$, for almost every $g \in G$. Thus, almost every fiber of $\mu$ over $G/\Gamma$ is the Lebesgue measure. □

## 4. *Shearing*

In this section, we prove the crucial fact that the direction of fastest transverse divergence between two nearby $U$-orbits is always along the stabilizer of $\mu$. The analogous statement for unipotent flows is a cornerstone of the proof of Ratner's Theorem [**Ra5**, Lemma 3.3, **MaT**, Lemma 7.5, **Mo**, Proposition 5.2.4′].

*Notation 4.1.*

- For any $g \in G$, we may write

$$g = \begin{bmatrix} 1+\mathsf{a} & \mathsf{b} \\ \mathsf{c} & 1+\mathsf{d} \end{bmatrix}$$

  with $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d} \in \mathbb{R}$. For a sequence $\{g_n\} \subset G$, we have $g_n \to e$ if and only if $\mathsf{a}_n, \mathsf{b}_n, \mathsf{c}_n, \mathsf{d}_n \to 0$.
- Suppose $|\mathsf{d}| < 1/4$, say. For $s \in \mathbb{R}$ with $|s| < 1/(4|\mathsf{c}|)$, let
  - $f(s, g) = \dfrac{(1+\mathsf{a})s - \mathsf{b}}{1+\mathsf{d}-\mathsf{c}s} \in \mathbb{R}$;
  - $v_s(g) = \begin{bmatrix} 1 & 0 \\ (1+\mathsf{d}-\mathsf{c}s)\mathsf{c} & 1 \end{bmatrix} \in V$;

    and
  - $a_s(g) = \begin{bmatrix} 1/(1+\mathsf{d}-\mathsf{c}s) & 0 \\ 0 & 1+\mathsf{d}-\mathsf{c}s \end{bmatrix} \in A$.

  Note that $v_s(g) \to e$ if $g \to e$.
- Suppose $\{p_n\}$ and $\{q_n\}$ are two sequences in a metric space. If $d(p_n, q_n) \to 0$, we may write $p_n \approx q_n$.

LEMMA 4.2. *A simple calculation shows that*

$$u^{f(s,g)} g u^{-s} = \begin{bmatrix} 1/(1+\mathsf{d}-\mathsf{c}s) & 0 \\ \mathsf{c} & 1+\mathsf{d}-\mathsf{c}s \end{bmatrix} = v_s(g) a_s(g).$$

*For a sequence $g_n \to e$, we denote $f_n(s_n) = f(s_n, g_n)$, and $a_{n,s_n} = a_{s_n}(g_n)$. Then $u^{f_n(s_n)} g_n u^{-s_n} \approx a_{n,s_n}$ if $g_n \to e$ (and $|s_n| < 1/(4|\mathsf{c}_n|)$).*

*Remark 4.3.* ('Shearing') Let us discuss the action of $U$ on $(\mathbb{R}^2)^k$. For any $s \in \mathbb{R}$ and $w \in (\mathbb{R}^2)^k$, we have

$$u^s(w) = w + H_s(w).$$

Assume, now that

$$w_n, w'_n \to 0 \quad \text{and} \quad H_1(w_n) \neq H_1(w'_n).$$

There is some $s_n \in \mathbb{R}^+$, such that $\|H_{s_n}(w_n - w'_n)\| = 1$. Then

$$u^{s_n}(w_n) - u^{s_n}(w'_n) = (w_n - w'_n) + H_{s_n}(w_n - w'_n) \approx H_{s_n}(w_n - w'_n) \in \text{Horiz}.$$

Thus, under the $U$-flow, $w_n$ and $w'_n$ move apart along a leaf of the Horiz-foliation. In other words, the direction in which two nearby points move apart fastest is along Horiz.

We use Notation 4.1 to state the main result of this section.

PROPOSITION 4.4. *For every $\rho > 0$, there is a compact subset $\Omega_\rho$ of $\mathcal{X}^k_{\text{Horiz}}$, with $\mu(\Omega_\rho) > 1 - \rho$, such that, if*

- *$(M_n, p_n), (M'_n, p'_n)$ are convergent sequences in $\Omega_\rho$,*
- *$(M'_n, p'_n) = g_n w_n (M_n, p_n)$ for some $g_n \in G$ and $w_n \in (\mathbb{R}^2)^k$,*
- *$g_n \to e$ and $w_n \to 0$,*
- *$s_n \in \mathbb{R}$ with*

$$|s_n| \le \frac{1}{\max(4|\mathsf{c}_n|, \|H_1(w_n)\|)},$$

*and*

- *$a_{n,s_n} H_{s_n}(w_n)$ converges,*

*then* $\lim_{n\to\infty} a_{n,s_n} H_{s_n}(w_n) \in \text{Stab}_{A\,\text{Horiz}}(\mu)^\circ$.

*Proof.* Define $\varphi\colon [-1, 1] \to A\,\text{Horiz}$ by

$$\varphi(t) = \lim_{n\to\infty} a_{n,t|s_n|} H_{t|s_n|}(w_n) = a_\infty(t)\, h_\infty(t),$$

where, letting $\mathsf{c}_\infty = \lim_{n\to\infty} \mathsf{c}_n |s_n|$ and $w_\infty = \lim_{n\to\infty} |s_n| H_1(w_n)$, we have

$$a_\infty(t) = \begin{bmatrix} 1/(1 - \mathsf{c}_\infty t) & 0 \\ 0 & 1 - \mathsf{c}_\infty t \end{bmatrix} \quad \text{and} \quad h_\infty(t) = t\, w_\infty.$$

It is clear that $\varphi$ is continuous. We will show $\varphi(t) \in \text{Stab}_{A\,\text{Horiz}}(\mu)$ for all $t$. Then

$$\lim_{n\to\infty} a_{n,s_n} H_{s_n}(w_n) = \varphi(\pm 1) \in \text{Stab}_{A\,\text{Horiz}}(\mu)^\circ,$$

as desired.

Let $\Omega_\rho$ be a uniformly generic set for the action of $U$ on $\mathcal{X}^k$ with $\mu(\Omega_\rho) > 1 - \rho$ (see Lemma 3.2). By passing to a subset, we may assume that $\Omega_\rho \subset \mathcal{X}^k_{\text{Horiz}}$ and that $\Omega_\rho$ is compact. For any $\epsilon > 0$, we know, from Lemma 3.3 (with $H = AU\,\text{Horiz}$), that there is a compact subset $K$ of $\mathcal{X}^k_{\text{Horiz}}$, such that $\mu(K) > 1 - (\epsilon/100)$ and $K \cap hK = \emptyset$, for all $h \in A\,\text{Horiz} \setminus \text{Stab}_{A\,\text{Horiz}}(\mu)$.

When $n$ is large, the definition of $\Omega_\rho$ implies that

$$d(u^s(M_n, p), K) < \epsilon \tag{4.5}$$

for all but $\epsilon\%$ of the values of $s$ in the interval $[-|s_n|/4, |s_n|/4]$ (or longer intervals) (see Lemma 3.2). Note that the Jacobian of $f_n$ is uniformly bounded on $[-|s_n|, |s_n|]$. More precisely, $f_n'(s) = 1/(1 + \mathsf{d}_n - \mathsf{c}_n s)^2$, so $1/4 < f'(s) < 4$. Therefore,

$$d(u^{f(s)}(M_n', p'), K) < \epsilon \tag{4.6}$$

for all but $4\epsilon\%$ of the values of $s$ in the interval $[-|s_n|, |s_n|]$. Thus, equations (4.5) and (4.6) hold *simultaneously* for all but $5\epsilon\%$ of the values of $s$ in the interval $[-|s_n|, |s_n|]$.

Let $(M, p) = \lim_{n\to\infty}(M_n, p_n)$. Because $(M, p) \in \Omega_\rho \subset \mathcal{X}^k_{\text{Horiz}}$, we know that translating $p_n$ by a vector in Horiz cannot move it into $\Sigma$. Hence $d(Cp, \Sigma) > 0$ for any compact subset $C$ of Horiz. Therefore, if $n$ is sufficiently large, and, for convenience, we let $s = t|s_n|$, then

- $M_n$ has no singularities in

$$\left\{ xyp_n \;\middle|\; \begin{array}{c} x \in \text{Horiz},\ y \in \text{Vert}, \\ \|x\| \le 4\|H_{s_n}(w_n)\|, \\ \|y\| \le 2\|w_n\| \end{array} \right\},$$

 so

- $u^s M_n$ has no singularities in

$$\left\{ xy\,u^s p_n \;\middle|\; \begin{array}{c} x \in \text{Horiz},\ y \in \text{Vert}, \\ \|x\| \le 2\|H_{s_n}(w_n)\|, \\ \|y\| \le 2\|w_n\| \end{array} \right\}.$$

This implies that

$$(u^s M_n, u^s p_n + u^s w_n) \approx (u^s M_n, u^s p_n + H_s(w_n)).$$

Therefore,

$$\begin{aligned} u^{f_n(s)}(M_n', p_n') &= u^{f_n(s)} g_n u^{-s}(u^s M_n, u^s p_n + u^s w_n) && \text{(definition of } g_n \text{ and } w_n\text{)} \\ &\approx a_s(u^s M_n, u^s p_n + H_s(w_n)) && \text{(Lemma 4.2)} \\ &= a_s H_s(w_n) u^s(M_n, p_n) && (u^s \text{ commutes with Horiz}). \end{aligned}$$

When (4.5) and (4.6) hold simultaneously, we conclude that

$$d(K, a_s H_s(w_n) K) \to 0.$$

From the definition of $K$, we conclude that $a_s H_s(w_n) \in \text{Stab}_{A\,\text{Horiz}}(\mu)$. That is, $\varphi(t) = a_s H_s(w_n)$ belongs to $\text{Stab}_{A\,\text{Horiz}}(\mu)$ for all but $5\epsilon\%$ of the values of $t$ in $[-1, 1]$. Because $\epsilon$ is arbitrary, $\varphi$ is continuous, and $\text{Stab}_{A\,\text{Horiz}}(\mu)$ is closed subgroup, we conclude that $\varphi(t)$ must actually belong to the stabilizer for all values of $t$, as desired. □

## 5. *Proof of Theorem 2.6*

We assume Notation 3.1. Recall, in particular, that $\mu$ is carried by $\mathcal{X}^k_{\text{Horiz}}$, and that the group $A$ Horiz acts on $\mathcal{X}^k_{\text{Horiz}}$.

PROPOSITION 5.1. *Almost every fiber of $\mu$ over $\mathcal{X}^d$ is supported on finitely many orbits of* Horiz $\ominus X$.

*Proof.* Because $\mu$ is an ergodic probability measure, it suffices to show that almost every fiber is supported on countably many such orbits. For $\Omega_\rho$ as in Proposition 4.4, we know $\bigcup_{n=N}^{\infty} \Omega_{1/n}$ is conull, so it suffices to show, for each $\rho > 0$, that each fiber of $\Omega_\rho$ is contained in the union of countably many orbits of Horiz$\ominus X$. Suppose not. (This will lead to a contradiction.) Because any uncountable set contains one if its accumulation points, there exist $(M', p) \in \Omega_\rho$ and a sequence $\{p_n\}$ in $M'$, such that:

- $(M', p_n) \in \Omega_\rho$;
- $\pi_d(M', p_n) = \pi_d(M', p)$;
- $(M', p_n) \to (M', p)$; and
- $(M', p_n) \notin (\text{Horiz}\ominus X)(M', p)$.

Because $\pi_d(M', p_n) = \pi_d(M', p)$ and $(M', p_n) \to (M', p)$, we may write $(M', p_n) = w_n(M', p)$ for some $w_n \in 0^d \times (\mathbb{R}^2)^{k-d}$ with $w_n \to e$. By assumption, we know $w_n \notin$ Horiz, so $H_1(w_n)$ is a non-zero element of Horiz$\cap(0^d \times (\mathbb{R}^2)^{k-d}) =$ Horiz$\ominus X$. Because

$$\frac{H_1(w_n)}{\|H_1(w_n)\|} = H_{1/\|H_1(w_n)\|}(w_n),$$

and Proposition 4.4 implies that $H_{1/\|H_1(w_n)\|}(w_n)$ converges to an element of $\text{Stab}_{A\,\text{Horiz}}(\mu)^\circ$, we conclude that $\text{Stab}_{\text{Horiz}\ominus X}(\mu)^\circ$ is non-trivial. This contradicts the definition of $X$. □

PROPOSITION 5.2. *After restricting to an appropriate conull subset $\Omega_0$ of $\mathcal{X}^k_{\text{Horiz}}$, each fiber of $\pi_d$ is finite.*

*Proof.* We know, from Proposition 5.1, that almost every fiber of $\pi_d$ is carried by only finitely many orbits of Horiz$\ominus X$. (From Theorem 3.4, we may assume that each of these is an embedded copy of Horiz$\ominus X$.) Letting

$$(\text{Horiz}\ominus X)_i^+ = 0^{d+i-1} \times \mathbb{R}^+ \times 0^{k-d-i},$$

we may define a measurable function $\xi_i : \mathcal{X}^k_{\text{Horiz}} \to [0, 1]$ by $\xi_i(\omega) = \mu_{\pi_d(\omega)}\left((\text{Horiz}\ominus X)_i^+(p)\right)$. This function is essentially $U$-invariant, so it must be essentially constant. Because this is true for all $i$, we conclude that $\pi_d$ is carried by a single point in each orbit of Horiz$\ominus X$. Since there are only finitely many such orbits to consider, we conclude that almost every fiber consists of a finite number of atoms, as desired. □

PROPOSITION 5.3. *We may assume $\mu$ is $A$-invariant.*

*Proof.* Choose $\Omega_\rho$ as in Proposition 4.4, with $\rho = 0.99$. From Corollary 3.5, we know that $\mu$ projects to the Lebesgue measure on $\mathcal{X}^d$. Furthermore, by passing to a conull subset, we may assume $\Omega_\rho$ has finite fibers over $\mathcal{X}^d$ (see Proposition 5.2). Thus, it is easy to see that there exist $(M, p) \in \Omega_\rho$, $\{v_n\} \subset V \setminus \{e\}$, and $\{w_n\} \subset 0^d \times (\mathbb{R}^2)^{k-d}$, such that $v_n w_n(M, p) \in \Omega_\rho$, $v_n \to e$, and $w_n \to e$. Then, following notation of 4.1, with $g_n = v_n$, and choosing $s_n$ appropriately, we have $a_{n,s_n} H_{s_n}(w_n) \in A(\text{Horiz}\ominus X)$ (cf. proof of Proposition 5.1 to see that $H_{s_n}(w_n) \in (\text{Horiz}\ominus X)$). We conclude, from Proposition 4.4, that the identity component of $\text{Stab}_{A\,\text{Horiz}}(\mu)\cap A(\text{Horiz}\ominus X)$ is non-trivial. Because the identity component of $\text{Stab}_{A\,\text{Horiz}}(\mu) \cap (\text{Horiz}\ominus X)$ is trivial (by definition of $X$), we conclude that $\text{Stab}_{A\,\text{Horiz}}(\mu)$ contains a one-parameter subgroup that is not contained in Horiz. Any such subgroup is conjugate to $A$ (via an element of Horiz). Thus, by replacing $\mu$ with a translate under Horiz, we may assume $\mu$ is $A$-invariant. □

LEMMA 5.4. *$\mathcal{X}^k_{\mathrm{Vert}}$ is conull.*

*Proof.* By passing to a quotient, we may assume $k = 1$. For each non-zero vector $w \in \mathbb{R}^2$, let

$$\Sigma^w_g = \{p \in gM \mid (p + \mathbb{R}w) \cap g\Sigma \neq \emptyset\}.$$

Note that $\Sigma^w_{ug} = u\Sigma^{u^{-1}w}_g$.

Suppose there is a subset $E$ of positive measure in $G$, such that $\mu_{gM}(\Sigma^{(0,1)}_g) \neq 0$ for $g \in E\Gamma$. Then the pointwise ergodic theorem implies, for almost every $g_0 \in G$, that we have $ug_0 \in E\Gamma$ for all $u$ in a non-null subset $U_0$ of $U$. Furthermore, because $\mu$ is $U$-invariant, we may assume $\mu_{g_0M} = u_*\mu_{ug_0M}$ for all $u \in U_0$. Therefore,

$$\mu_{g_0M}(\Sigma^{u^{-1}(0,1)}_{g_0}) = \mu_{ug_0M}(u\Sigma^{u^{-1}(0,1)}_{g_0}) = \mu_{ug_0M}(\Sigma^{(0,1)}_{ug_0}) \neq 0$$

for all $u \in U_0$. This contradicts the fact that, because $\Sigma^{w_1}_{g_0} \cap \Sigma^{w_2}_{g_0}$ is countable whenever $\mathbb{R}w_1 \neq \mathbb{R}w_2$, we have $\mu_{gM}(\Sigma^w_g) = 0$ for all but countably many choices of the line $\mathbb{R}w$. □

PROPOSITION 5.5. (Cf. [**MaT**, Corollary 8.4, **Mo**, Corollary 5.5.2]) *There is a conull subset $\Omega$ of $\mathcal{X}^k_{\mathrm{Vert}}$, such that*

$$(V\,\mathrm{Vert}\,\omega) \cap \Omega = (VY\omega) \cap \Omega,$$

*for all $\omega \in \Omega$.*

*Proof.* Let $\Omega$ be a generic set for the action of $A$ on $\mathcal{X}^k_{\mathrm{Vert}}$; thus, $\Omega$ is conull and, for each $\omega \in \Omega$,

$$a^t\omega \in \Omega_\rho \text{ for most } t \in \mathbb{R}^+.$$

Given $(M, p), (M', p') \in \Omega$, such that $(M', p') = vy(M, p)$ with $v \in V$ and $y \in \mathrm{Vert}$, we wish to show $y \in Y$.

Choose a sequence $t_n \to \infty$, such that $a^{t_n}(M, p)$ and $a^{t_n}(M', p')$ each belong to $\Omega_\rho$. Because $t_n \to \infty$ and $V\,\mathrm{Vert}$ is the foliation that is contracted by $a^{\mathbb{R}^+}$, we know that $a^{-t_n}(vy)a^{t_n} \to e$. Furthermore, because $A$ acts on the Lie algebra of $V$ with twice the weight that it acts on the Lie algebra of Vert, we see that $\|a^{-t_n}va^{t_n}\|/\|a^{-t_n}ya^{t_n}\| \to 0$. Thus, letting $s$ be within a constant multiple of $1/\|a^{-t_n}ya^{t_n}\|$, we see, by Notation 4.1, with $g_n = a^{-t_n}va^{t_n}$ and $w_n = a^{-t_n}ya^{t_n}$, that $a_{s_n}(g_n) \to e$, but $H_{s_n}(w_n) \not\to e$. Thus, Proposition 4.4 asserts that $H_{s_n}(w_n)$ converges to a non-trivial element of $\mathrm{Stab}_{\mathrm{Horiz}}(\mu)^\circ = X$. As $H_{s_n}(w_n) = H_{s_n}(a^{-t_n}ya^{t_n})$ is a scalar multiple of $H_1(y)$, we conclude that $H_1(y) \in X$. Therefore $(u^1 - \mathrm{Id})y = H_1(y) \in X$, so $y \in Y$. □

We require the following entropy estimate.

LEMMA 5.6. (Cf. [**MaT**, Theorem 9.7, **Mo**, Proposition 2.5.11]) *Suppose $W$ is a closed connected subgroup of $V$* Vert *that is normalized by $a \in A^+$, and let*

$$J(a^{-1}, W) = \det((\mathrm{Ad}\,a^{-1})|_{\mathfrak{w}})$$

*be the Jacobian of $a^{-1}$ on the Lie algebra $\mathfrak{w}$ of $W$.*

(1) *If $\mu$ is $W$-invariant, then $h_\mu(a) \geq \log J(a^{-1}, W)$.*
(2) *If there is a conull, Borel subset $\Omega$ of $\mathcal{X}^k$, such that $\Omega \cap V \operatorname{Vert}\omega \subset W\omega$, for every $\omega \in \Omega$, then $h_\mu(a) \leq \log J(a^{-1}, W)$.*
(3) *If the hypotheses of* (5.6) *are satisfied, and equality holds in its conclusion, then $\mu$ is $W$-invariant.*

PROPOSITION 5.7. (Cf. [**MaT**, Step 1 of 10.5, **Mo**, Proposition 5.6.1]) *$\mu$ is $VY$-invariant.*

*Proof.* From Lemma 5.6(1), with $a^{-1}$ in the role of $a$, we have

$$\log J(a, UX) \leq h_\mu(a^{-1}).$$

From Proposition 5.5 and Lemma 5.6(2), we have

$$h_\mu(a) \leq \log J(a^{-1}, VY).$$

Combining these two inequalities with the facts that

- $h_\mu(a) = h_\mu(a^{-1})$ and
- $J(a, UX) = J(a^{-1}, VY)$,

we have

$$\log J(a, UX) \leq h_\mu(a^{-1}) = h_\mu(a) \leq \log J(a^{-1}, VY) = \log J(a, UX).$$

Thus, we must have equality throughout, so the desired conclusion follows from Lemma 5.6(3). □

PROPOSITION 5.8. *$\mu$ is the Lebesgue measure on a single orbit $\mathcal{X}_0^k$ of the pseudogroup $G \ltimes \langle \Phi_X^k, \Phi_Y^k \rangle$.*

*Proof.* We know:

- $U$ preserves $\mu$ (by assumption);
- $X$ preserves $\mu$ (by definition);
- $A$ preserves $\mu$ (see Proposition 5.3); and
- $VY$ preserves $\mu$ (see Proposition 5.7).

Therefore, $\mu$ is preserved by the pseudogroup $G \ltimes \langle \Phi_X^k, \Phi_Y^k \rangle$ generated by these maps. Because

- this pseudogroup is transitive on the quotient $\mathcal{X}_0^d$; and
- $\mu$ has finite fibers over $\mathcal{X}_0^d$ (see 5.2),

this implies that some orbit of the pseudogroup has positive measure. By ergodicity of $U$, then this orbit is conull. □

*Remark 5.9.* To obtain the conclusions of Theorem 2.6, we let $W = X + Y$. Then $\mu$ is supported on the $(G \ltimes \widehat{\Phi}_W^k)$-orbit of some point $(p_1, \ldots, p_k)$ in $\mathcal{X}^k$. Note that, by choosing $\dim W$ to be minimal, we can guarantee that whenever $p_i$ is a singular point of $M$, the subspace $W$ projects to 0 in the $i$th coordinate of $(\mathbb{R}^2)^k$. Therefore, the dimension of the orbit is equal to the dimension of the pseudosemigroup.

6. *Countability*

For our application, we need the following analogue of [**Ra7**, Corollary A(2)].

PROPOSITION 6.1. *The set of subspaces $W$ occurring in Theorem 2.6 is countable. For each such $W$, the set of closed orbits of* $\mathrm{SL}(2,\mathbb{R}) \ltimes \widehat{\Phi}^k_W$ *is countable.*

LEMMA 6.2. *The set of $G$-invariant subspaces $W$ of $(\mathbb{R}^2)^k$ such that there exists $p \in M^k$ with $\mathcal{O} = \widehat{\Phi}^k_W p$ closed is countable.*

*Proof.* Let $2d$ be the dimension of $W$. After possibly renumbering the factors, we may assume that

$$W \cap ((0,0)^d \times (\mathbb{R}^2)^{k-d}) = \emptyset.$$

Then, if we denote elements of $(\mathbb{R}^2)^k$ by $(v_1, \dots, v_k)$ where each $v_j \in \mathbb{R}^2$, then $W$ is given by the following equations: for $d+1 \le j \le k$,

$$v_j = \sum_{i=1}^{d} \alpha_{ji} v_i. \tag{6.3}$$

Recall that the linear holonomy map hol : $H_1(M,\mathbb{Z}) \to \mathbb{C} \cong \mathbb{R}^2$ is given by $\mathrm{hol}(\gamma) = \int_\gamma \omega$, where $\omega = dx + i\,dy$ is the holomorphic 1-form that determines the flat structure on $M$. Let $\Delta \subset \mathbb{R}^2$ denote the image of hol, and let $F$ denote the set of real numbers $r$ such that there exist non-zero $v_1 \in \Delta$, $v_2 \in \Delta$ with $v_1 = r v_2$. Then $F$ is clearly a countable set. We will show that each $\alpha_{ji}$ belongs to $F \cup \{0\}$.

Let $\pi_d : M^k \to M^d$ denote projection onto the first $d$ factors. Note that the intersection of $\mathcal{O}$ with each fiber of $\pi_d$ is finite.

Now pick $i$, $1 \le i \le d$, and $j$, $d+1 \le j \le k$. We may assume that $\alpha_{ji} \ne 0$. Choose $p = (p_1, \dots, p_k) \in \mathcal{O}$ such that $p_i$ and $p_j$ are non-singular. Let $\gamma$ be any element of $H_1(M,\mathbb{Z})$ with $\mathrm{hol}(\gamma) \ne 0$. We represent $\gamma$ by a piecewise linear closed curve on $M$ beginning and ending at $p_i$ and not passing through any singularities; we will also denote this representative by $\gamma$. We obtain a closed curve $\gamma_i \in M^d$ by keeping $p_m$ fixed for $1 \le m \le d$, $m \ne i$. Because $\mathcal{O}$ is a branched cover of $M^d$, $\gamma_i$ lifts to a closed curve $\tilde{\gamma}_i$ in $\mathcal{O}$. Let $\gamma'$ denote the projection of $\tilde{\gamma}_i$ to the $j$th factor.

We wish to calculate $\mathrm{hol}(\gamma')$, so let us describe $\gamma'$ more precisely. The curve $\gamma$ is a collection of segments connecting points $p_i = q_0, q_1, \dots, q_{n-1}, q_n = p_i$, with $q_{m+1} = \widehat{\phi}_{w_m}(q_m)$, $w_m \in \mathbb{R}^2$. Then $\gamma'$ is a collection of segments connecting the points $p_j = q'_0, q'_1, \dots, q'_{n-1}, q'_n$, with $q'_{m+1} = \widehat{\phi}_{\alpha_{ji} w_m}(q'_m)$. By perturbing the $w_m$, we can make sure that $\gamma'$ is well defined and is not passing through any singularities.

By construction, the endpoint $q'_n$ of $\gamma'$ belongs to the finite set $\pi_d^{-1}(\pi_d(p)) \cap \mathcal{O}$. After replacing $\gamma$ by an integer multiple, we may assume that $\gamma'$ is closed. But, in view of the explicit description of $\gamma'$, $\mathrm{hol}(\gamma') = \alpha_{ji}\,\mathrm{hol}(\gamma)$, so $\alpha_{ji} \in F$. □

In the rest of this section we will abuse notation by writing $p + v$ for $\widehat{\phi}_v(p)$.

LEMMA 6.4. *Let $M$ be a Veech surface, and let $\Gamma$ be the Veech group of $M$. A point $p$ is called a periodic point if the $\Gamma$ orbit of $p$ is finite. Then the set of periodic points is countable.*

*Remark 6.5.* When $M$ is non-arithmetic, which is the only case that we need to discuss, it is proven in [**GHS**] that the number of periodic points is countable (in fact, *finite*). The following generalization of this statement also follows from the results of [**GHS**], but we include a short proof of as a warm up to the proof of Proposition 6.1.

*Proof of Lemma 6.4.* It is clearly enough to show that for each $n \in \mathbb{N}$, the set $\mathcal{P}_n$ of points of period $n$ is countable. To do this it is enough to show that for each point $p \in \mathcal{P}_n$, there exists a neighborhood $U$ of $p$ that does not contain any other points of $\mathcal{P}_n$. Suppose the last statement is false. Then there exists a sequence of points $p_j \in \mathcal{P}_n$ such that $p_j \to p$. We may assume after passing to a subsequence that the $p_j$ approach $p$ from some given direction $w$ (i.e. that $\lim(p_j - p)/\|p_j - p\| = w$). Let $\Gamma'$ denote the intersection of all the index $n$ subgroups of $\Gamma$. Then, as $\Gamma$ is finitely generated, $\Gamma'$ is of finite index in $\Gamma$ and for each $\gamma' \in \Gamma'$, and all $j, \gamma'(p_j) = p_j$. Then each element of $\Gamma'$ must fix $w$. This contradicts the fact that $\Gamma'$, being a finite index subgroup of $\Gamma$, is Zariski dense in $G$. $\square$

*Proof of Proposition 6.1.* It remains to prove the following assertion. Let $W \subset (\mathbb{R}^2)^k$ be an $G$-invariant subspace. Then the set $\mathcal{H}$ of closed orbits of $\Gamma \ltimes \widehat{\Phi}^k_W$ is countable.

We triangulate $M$, with the vertices at the singular points. This yields a cell decomposition of $M^k$ in which the cells $\Delta_1, \dots, \Delta_m$ of maximal dimension are products of triangles. Let $\Delta_i^0$ denote the interior of $\Delta_i$, and let $M_0^k$ denote the union of the $\Delta_i^0$. For $p \in M^k$, let $\delta(p)$ denote the distance between $p$ and the complement of $M_0^k$ (i.e. the distance to the boundary of the cell containing $p$).

Let $2d = \dim W$, and let $W^\perp$ be any $G$-invariant complement to $W$. We may assume that $W$ is given by the equations (6.3). In view of Lemma 6.4 we may also assume that $W$ has dense projection onto any of the $\mathbb{R}^2$ factors (i.e. for a fixed $j$, not all $\alpha_{ji}$ are zero). Then, for any $\mathcal{O} \in \mathcal{H}$, $\mathcal{O} \cap M_0^k$ is dense in $\mathcal{O}$.

Let $n_1, \dots, n_m$ be an $m$-tuple of non-negative integers, and let $\mathcal{H}(n_1, \dots, n_m)$ denote the set of $\mathcal{O} \in \mathcal{H}$ such that $\mathcal{O} \cap \Delta_i^0$ has exactly $n_i$ connected components.

Now suppose $\mathcal{H}$ is uncountable. Then there exist $n_1, \dots, n_m$ such that $\mathcal{H}(n_1, \dots, n_m)$ is uncountable. Then by compactness, there exist $\mathcal{O}$ in $\mathcal{H}(n_1, \dots, n_m)$ such that for every $\epsilon > 0$ there exists $\mathcal{O}' \in \mathcal{H}(n_1, \dots, n_m)$ such that the Hausdorff distance between $\mathcal{O}$ and $\mathcal{O}'$ is less then $\epsilon$. Let $\rho$ be the minimum over $i$ of the minimal distance between connected components of $\mathcal{O} \cap \Delta_i^0$.

Let $n = n_1 + \cdots + n_m$, and number all the connected components of the intersection of $\mathcal{O}$ with the interiors of the cells as $\mathcal{O}_i$, $1 \le i \le n$. Let $\gamma_1, \dots, \gamma_s$ denote the generators of $\Gamma$. We may choose a point $p_i$ in each $\mathcal{O}_i$ such that for all $j$, $1 \le j \le s$, $\gamma_j p_i$ is in the interior of some component $\mathcal{O}_l$, where $l$ depends on $i$ and $j$.

Let $C = \max_{1 \le j \le s} \|\gamma_j\|$. Now choose $\epsilon > 0$ so that:

- $C\epsilon < \rho/3$;
- for any $i$, $1 \le i \le m$, we have $\delta(p_i) > 2C\epsilon$;
- for each $i$, $1 \le i \le m$ and each $j$, $1 \le j \le s$, we have $\delta(\gamma_j p_i) > 2C\epsilon$.

Now choose $\mathcal{O}' \in \mathcal{H}(n_1, \dots, n_m)$ so that the Hausdorff distance between $\mathcal{O}'$ and $\mathcal{O}$ is less then $\epsilon$. Note that if $q \in \mathcal{O}_i$ with $\delta(q) \ge 2C\epsilon$ there exists a unique $v_i \in W^\perp$ such that $\|v_i\| \le C\epsilon$ and $q + v_i \in \mathcal{O}'$. Also $v_i$ does not depend on the choice of $q$, and $\|v_i\| \le \epsilon$.

Furthermore $v_i \neq 0$ since $\mathcal{O}$ and $\mathcal{O}'$ cannot share a point in $M_0^k$. Let $V$ denote the finite set $\{v_1, \dots, v_n\}$.

We now claim that each generator $\gamma_j$ preserves the set $V$. Indeed consider the points $p_i \in \mathcal{O}$ and $p_i + v_i \in \mathcal{O}'$. Since both $\mathcal{O}$ and $\mathcal{O}'$ are $\Gamma$-invariant, we must have $\gamma_j p_i \in \mathcal{O}$ and $\gamma_j(p_i + v_i) \in \mathcal{O}'$. By construction, $\gamma_j p_i \in \mathcal{O}_l$, and $\delta(\gamma_j p_i) > 2C\epsilon$. Recall that $v_l$ is the only vector in $W^\perp$ of norm at most $C\epsilon$ such that $\gamma_j p_i + v_l \in \mathcal{O}'$. But $\gamma_j(p_i + v_i) = \gamma_j p_i + \gamma_j v_i \in \mathcal{O}'$, and $\|\gamma_j v_i\| \leq \|\gamma_j\| \|v_i\| \leq C\epsilon$. Also $\gamma_j v_i \in W^\perp$, since $W^\perp$ is $G$-invariant. Thus $\gamma_j v_i = v_l$.

We have proved that for each generator $\gamma_j$, we have $\gamma_j V \subseteq V$. This immediately implies that $\Gamma V = V$. Then a finite index subgroup of $\Gamma$ will fix a single vector in $V$, which contradicts the fact that $\Gamma$ is Zariski dense in $G$. $\square$

## 7. *Averages over large circles*

Let $m_K$ denote the Haar measure on SO(2) $\subset G$. For $x \in \mathcal{X}^k$ and $t > 0$, let

$$\nu_t = \nu_{t,x} = a_t m_K \delta_x$$

where $\delta_x$ is the atomic probability measure supported at $x$, and $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. Then each $\nu_t$ is a probability measure on $\mathcal{X}^k$. We can think of $\nu_t$ as the measure supported on a circle of radius $t$ inside the $G$-orbit through $x$. In this section we prove the following theorem.

THEOREM 7.1. *Suppose $x \in \mathcal{X}_0^k$. Then there exists a $G$-invariant subspace $W$ of* $(\mathbb{R}^2)^k$ *such that:*

(1) *the $G \ltimes \widehat{\Phi}_W^k$ orbit through $x$ is closed; and*

(2) $\lim_{t\to\infty} \nu_t = \mu$*, where $\mu$ is Lebesgue measure on this orbit.*

*Remark 7.2.* If $W = (\mathbb{R}^2)^k$, then $\mu$ is the Lebesgue measure on $\mathcal{X}^k$.

LEMMA 7.3. (Invariance under a unipotent) *Suppose $t_i \to \infty$. Then there is a subsequence $t_{i_j}$ such that the measures $\nu_{t_{i_j}}$ converge to a probability measure $\nu_\infty$ that is invariant under the unipotent element $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $G$.*

*Proof.* It follows from [**EM**, Corollary 5.3] that there is a subsequence $t_{i_j}$ such that the measures $\nu_{t_{i_j}}$ converge to a probability measure $\nu_\infty$. We can find $\theta_j \to 0$ such that $a_{t_{i_j}} r_{\theta_j} a_{t_{i_j}}^{-1}$ converges to $u$. (Recall that $r_\theta$ is the $(2 \times 2)$ matrix representing rotation by $\theta$.) Now the measures $\nu_{t_{i_j}} = (a_{t_{i_j}} \nu_K a_{t_{i_j}}^{-1}) a_{t_{i_j}} \delta_x$ are $a_{t_{i_j}} r_{\theta_j} a_{t_{i_j}}^{-1}$-invariant, hence $\nu_\infty$ is $u$-invariant. $\square$

*Assumption 7.4.* Assume $\nu_\infty$ is not the Lebesgue measure on $\mathcal{X}^k$.

### 7.1. *Application of the measure classification theorem.*

Note that we do not know at this point whether $\nu_\infty$ is ergodic. However, standard results (using $u$-invariance) imply that $\nu_\infty$ projects to Lebesgue measure in $G/\Gamma$.

*Notation 7.5.* For convenience, if $B \subset (\mathbb{R}^2)^k$ and $X \subset \mathcal{X}^k$, let
$$B^{-1}X = \bigcup_{v \in B} (\widehat{\phi}^k_v)^{-1}(X).$$

By Theorem 2.6, and by Proposition 6.1, there exists a $G$-invariant proper subspace $W$ of $(\mathbb{R}^2)^k$ and an orbit $\mathcal{O}$ of $G \ltimes \widehat{\Phi}^k_W$ such that
$$\nu_\infty(\mathrm{Horiz}^{-1}\, \mathcal{O}) > 0.$$
We will show that this implies that $x \in \mathcal{O}$. In that case, the entire $G$-orbit of $x$ lies in $\mathcal{O}$, so $\nu_\infty(\mathcal{O}) = 1$. Furthermore, we show that as long as $W$ was chosen as small as possible, $\nu_\infty$ must be Lebesgue measure on $\mathcal{O}$.

7.2. *Projection and fiber measures.* We choose $W$ to be of minimal dimension. From the structure of the $G$-invariant subspaces on $(\mathbb{R}^2)^k$, we see that $\dim W = 2d$, $0 \le d < k$, and after renumbering the factors, we can make sure that
$$W \text{ projects surjectively to } (\mathbb{R}^2)^d \times 0^{k-d}.$$
Thus, $(0,0)^d \times (\mathbb{R}^2)^{k-d}$ is complementary to $W$, and $0^d \times \mathbb{R}^{k-d}$ is complementary to $W \cap \mathrm{Horiz}$ in Horiz.

LEMMA 7.6. *There exists $\epsilon > 0$ and a box*
$$B = \{0\}^d \times [\alpha_{d+1}, \beta_{d+1}] \times \cdots \times [\alpha_k, \beta_k] \subset \mathrm{Horiz}$$
*such that $\nu_\infty(B^{-1}\mathcal{O}) > 2\epsilon$.*

*Proof.* Let $\mathcal{O}_0$ be the (unique) orbit of $G \ltimes \Phi^k_W$ that is open and dense in $\mathcal{O}$. (In other words, $\mathcal{O}_0$ consists of the elements in $\mathcal{O}$ of which as few coordinates as possible are singular points.) Note that
$$W^{-1}\mathcal{O}_0 = \mathcal{O}_0. \tag{7.7}$$
By the minimality of $\dim W$, we see that $\nu_\infty(\mathrm{Horiz}^{-1}(\mathcal{O} \setminus \mathcal{O}_0)) = 0$. Hence $\nu_\infty(\mathrm{Horiz}^{-1}\, \mathcal{O}_0) > 0$. By combining this with (7.7) and the fact that $\mathrm{Horiz} = W + (0^d \times \mathbb{R}^{k-d})$, we conclude that there is a box $B \subset 0^d \times \mathbb{R}^{k-d}$, such that $\nu_\infty(B^{-1}\mathcal{O}_0) > 0$. As $\mathcal{O}_0 \subset \mathcal{O}$, then $\nu_\infty(B^{-1}\mathcal{O}) > 0$, as desired. □

As in the previous sections, let $\pi_d \colon \mathcal{X}^k \to \mathcal{X}^d$ be the natural projection onto the first $d$ coordinates. For $z \in \mathcal{X}^d$, we let $F_z = \pi_d^{-1}(z) \cap \mathcal{O}$. Note that $F_z$ is a finite set.

We claim that
$$\nu_\infty \text{ projects to the Lebesgue measure on } \mathcal{X}^d. \tag{7.8}$$
To see this, note that, because $W$ is a proper subspace of $(\mathbb{R}^2)^k$, we have $d < k$. Hence, by induction on $k$, we may assume there is a $G$-invariant subspace $W_d$ of $(\mathbb{R}^2)^d$, such that the projection of $\nu_\infty$ to $\mathcal{X}^d$ is the Lebesgue measure on the $G \ltimes \widehat{\Phi}^d_{W_d}$ orbit $\mathcal{O}_d$ through $\pi_d(x)$. Then $\pi_d^{-1}(\mathrm{Horiz}^{-1}\, \mathcal{O}_d)$ is conull for $\nu_\infty$, so
$$\nu_\infty(\pi_d^{-1}(\mathrm{Horiz}^{-1}\, \mathcal{O}_d) \cap \mathcal{O}) = \nu_\infty(\mathcal{O}) \ne 0.$$
From the minimality of $\dim W$, we conclude that $W_d = (\mathbb{R}^2)^d$. Therefore, $\mathcal{O}_d = \mathcal{X}^d$, which establishes the claim.

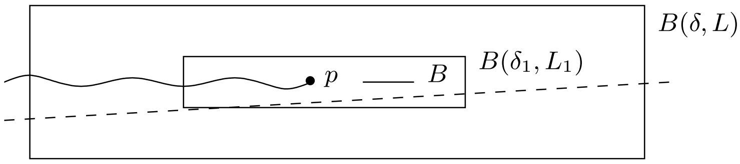


FIGURE 2. The time the ellipse (drawn here as a dashed line) spends inside the small box $B(\delta_1, L_1)$ is at most $\epsilon$ times the time the ellipse spends in the larger box $B(\delta, L)$. In Lemma 7.10, this is proved as a result in $(\mathbb{R}^2)^k$. Because of Lemma 7.12, it can be transferred to $\mathcal{X}^k$, even if the ellipse crosses the branch cut starting at the possibly singular point $p$.

*Assumption 7.9.* We may assume $x \notin \mathcal{O}$. (Otherwise, from the fact that $\mathcal{O}$ is a branched cover of $\mathcal{X}^d$ (and Lemma 8.14 below), we would immediately conclude that $\nu_\infty$ is the Lebesgue measure on $\mathcal{O}$, as desired.) This will lead to a contradiction.

7.3. *The key estimate.* For $L_1 > 0$, $\delta_1 > 0$, let

$$B(\delta_1, L_1) = \left\{ (x, y) \in (\mathbb{R}^2)^k \;\middle|\; \begin{array}{ll} x_i = y_i = 0, & \text{for } 1 \le i \le d, \\ |x_i| \le L_1 \text{ and } |y_i| \le \delta_1, & \text{for } d+1 \le i \le k. \end{array} \right\}$$

LEMMA 7.10. (The key estimate) *Suppose $B \subset B(\delta_1, L_1) \subset B(\delta, L)$, where $B$ is as defined in Lemma 7.6. Suppose also that $\rho > 0$, $\epsilon < 1$, $\delta_1 < \epsilon\delta/5$, and $L_1 < \epsilon L/5$. Then there exists $t_0$ depending only on $\rho, \delta, L$ such that for any $t > t_0$ and any $v \in B(\delta_1, L_1)$ with*

$$d(v, \text{Horiz}) > \frac{k}{5} e^{-t} \rho, \tag{7.11}$$

*we have*

$$|\{\theta \mid a_t r_\theta a_t^{-1} v \in B(\delta_1, L_1)\}| \le \frac{\epsilon}{2} |\{\theta \mid a_t r_\theta a_t^{-1} v \in B(\delta, L)\}|,$$

*where*

$$r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

*Proof.* If we write $v = (v_1, \ldots, v_k)$, with $v_j \in \mathbb{R}^2$, and also write $v_j = \binom{x_j}{y_j}$ then the condition (7.11) implies that there exists at least one $j$, $m+1 \le j \le k$ with $|y_j| > \frac{1}{5} e^{-t} \rho$. The rest of the argument will take place in the $j$th factor (see Figure 2).

We note that the components of the map $\theta \to a_t r_\theta a_t^{-1} v$ are trigonometric polynomials of degree one. In other words, the path $\theta \to a_t r_\theta a_t^{-1} v_j$ parametrizes an ellipse. Let $t_0 = \max(\log(5L/\rho), 0)$. Then if $t > t_0$ and $\theta = \pi/2$ then

$$a_t r_\theta a_t^{-1} v_j = \begin{pmatrix} e^{2t} y_j \\ -e^{-2t} x_j \end{pmatrix} \notin B(\delta, L).$$

Thus, the ellipse $\theta \to a_t r_\theta a_t^{-1} v_j$ leaves $B(\delta, L)$. Then in view of the dimensions of the boxes, the portion of the ellipse in $B(\delta_1, L_1)$ is at most $\epsilon/2$ times the portion of the ellipse in $B(\delta, L)$. □

LEMMA 7.12. *For any $L > 0$, there exist $\delta > 0$ and a compact subset $E$ of $\mathcal{X}^d$ with $\nu_\infty(\pi_d^{-1}(E)) > 1 - \epsilon/4$, such that:*

(1) *$B(\delta, L)^{-1}(\mathcal{O} \cap \pi_d^{-1}(E))$ does not contain any singular points, other than perhaps points in $\mathcal{O}$; and*

(2) *for each $p \in B(\delta, L)^{-1}(\mathcal{O} \cap \pi_d^{-1}(E))$, there is a unique $b \in B(\delta, L)$, such that $\widehat{\phi}_v^k(p) \in \mathcal{O}$.*

*Proof.* For $z \in \mathcal{X}^d$, let $S_z$ be the surface corresponding to $z$ (so we may write $\pi_d^{-1}(z)$ as $S_z^{k-d}$). Let $\Sigma_z$ denote the singular set of $S_z$. For any $L > 0$ on the fixed surface $S_z$, there exist only finitely many horizontal trajectories of length at most $L$ connecting points of $F_z \cup \Sigma_z$ to points of $F_z \cup \Sigma_z$. Therefore, we can find a large compact subset $E$ of $\mathcal{X}^d$ such that for any $z \in E$, $S_z$ has no horizontal trajectories of length at most $2L$ connecting points of $F_z \cup \Sigma_z$ to other points of $F_z \cup \Sigma_z$. As $\nu_\infty$ projects to Lebesgue measure on $\mathcal{X}^d$ (see equation 7.8), we may choose $E$ so that $\nu_\infty(\pi_d^{-1}(E)) > 1 - \epsilon/4$. Now we can choose $\delta > 0$ by compactness.

Note that, because $B(\delta, L) \subset (0, 0)^d \times (\mathbb{R}^2)^{k-d}$, we have $B(\delta, L)^{-1} F_z \subset \pi^{-1}(z)$. Therefore, $B(\delta, L)^{-1} F_z \cap \mathcal{O} = F_z$. □

*Completion of the proof of Theorem 7.1.* Because $x \notin \mathcal{O}$ (see Assumption 7.9), we may choose $\rho > 0$ so that $d(x, \mathcal{O}) > k\rho$. We may also assume that on the surface corresponding to $x$, the distance between any two singular points is at least $k\rho$. Let $B$, $[\alpha_i, \beta_i]$ and $\epsilon$ be as in Lemma 7.6. Choose $L_1$ so that for all $d + 1 \le i \le k$, we have $[\alpha_i, \beta_i] \subset [-L_1, L_1]$. Let $L = 10L_1/\epsilon$. Now choose $E \subset \mathcal{X}^d$ and $\delta > 0$ so that Lemma 7.12 holds. Finally, choose $\delta_1 = \epsilon\delta/10$. Assume $t > \log(5L/\rho)$. We will abuse notation by writing $p + v$ for $\widehat{\phi}_v^k(p)$.

We claim that if $a_t r_{\theta_0} x + v \in \mathcal{O}$, with $v \in B(\delta_1, L_1)$, then (7.11) holds. Indeed, we then have

$$r_{\theta_0} x + a_t^{-1} v = a_t^{-1}(a_t r_{\theta_0} x + v) \in a_t^{-1}\mathcal{O} = \mathcal{O},$$

so

$$|a_t^{-1} v| \ge d(r_{\theta_0} x, \mathcal{O}) = d(x, \mathcal{O}) > k\rho.$$

Also,

$$|a_t^{-1} v| \le e^t \cdot d(v, \text{Horiz}) + \frac{L_1}{e^t}.$$

Therefore,

$$d(v, \text{Horiz}) \ge e^{-t}\left(|a_t^{-1} v| - \frac{L_1}{e^t}\right) > e^{-t}\left(k\rho - \frac{L_1}{5L/\rho}\right) > e^{-t}\frac{k}{5}\rho.$$

Now let

$$R = \{\theta \mid a_t r_\theta x \in B(\delta_1, L_1)^{-1}(\mathcal{O} \cap \pi_d^{-1}(E))\}.$$

Suppose $\theta \in R$. Let $v$ be the unique element of $B(\delta_1, L_1)$ with $a_t r_\theta x + v \in \mathcal{O}$, and let

$$I'_\theta = \{\theta' \mid a_t r_{\theta'} r_\theta^{-1} a_t^{-1} v \in B(\delta, L)\}.$$

Note that $\theta \in I'_\theta$, so we may let $I_\theta$ be the component of $I'_\theta$ that contains $\theta$. By (the proof of) Lemma 7.10, $|I_\theta \cap R| \le (\epsilon/2)|I_\theta|$.

We claim that if $I_{\theta_1} \neq I_{\theta_2}$, then $I_{\theta_1} \cap I_{\theta_2}$ is disjoint from $R$. To see this, note that if $\theta' \in I_{\theta_1} \cap I_{\theta_2}$, then there exist $v_1, v_2 \in B(\delta_1, L_1)$, such that, letting

$$v_i' = a_t r_{\theta'} r_{\theta_i}^{-1} a_t^{-1} v_i,$$

we have $v_i' \in B(\delta, L)$ and

$$a_t r_{\theta'} x + v_i' = a_t r_{\theta'} r_{\theta_i}^{-1} a_t^{-1} (a_t r_{\theta_i} x + v_i) \in a_t r_{\theta'} r_{\theta_i}^{-1} a_t^{-1} \mathcal{O} = \mathcal{O}.$$

Now if $I_{\theta_1} \neq I_{\theta_2}$, then $r_\theta^{-1} a_t^{-1} v_1 \neq r_\theta^{-1} a_t^{-1} v_2$, so $v_1' \neq v_2'$. Lemma 7.12 therefore implies that $a_t r_{\theta'} x \notin B(\delta, L)^{-1}(\mathcal{O} \cap \pi_d^{-1}(E))$, so $\theta' \notin R$.

Since each point of $R \cap I_\theta$ is contained in a unique interval, the circle is covered at most twice by the intervals $I_\theta$. It follows that $|R| < \epsilon$. Equivalently, this means that

$$\nu_t((B(\delta_1, L_1)^{-1}\mathcal{O}) \cap \pi_d^{-1}(E)) < \epsilon.$$

As this holds for all sufficiently large $t$, we get

$$\nu_\infty((B(\delta_1, L_1)^{-1}\mathcal{O}) \cap \pi_d^{-1}(E)) \leq \epsilon.$$

As $\nu_\infty$ projects to Lebesgue measure, we know that $\nu_\infty(\pi_d^{-1}(E)) > 1 - \epsilon/4$. Hence $\nu_\infty(B(\delta_1, L_1)^{-1}\mathcal{O}) < 5\epsilon/4$. This contradicts Lemma 7.6. □

COROLLARY 7.13. *Suppose $x \in \mathcal{X}_0^k$. Then there exists a $G$-invariant subspace $W$ of $(\mathbb{R}^2)^k$, such that the closure of $Gx$ is $(G \ltimes \widehat{\Phi}_W^k)(x)$.*

*Proof.* Let $W$ be as in the conclusion of Theorem 7.1. Because $(G \ltimes \widehat{\Phi}_W^k)(x)$ is closed and $G$-invariant, it contains the closure of $Gx$. On the other hand, the support of $\nu_t$ is a subset of $Gx$, so $Gx$ is dense in the support of $\lim_{t\to\infty} \nu_t$; that is, $Gx$ is dense in $(G \ltimes \widehat{\Phi}_W^k)(x)$. □

Corollary 7.13′ (stated at the end of §2) follows from Corollary (7.13) by a standard argument (inducing the action of $\Gamma$ to an action of $G$).

## 8. *Application to counting*

We now give the general set-up for the counting problems we are considering. For additional background and more detailed definitions, see the introduction to [**EMZ**].

*Notation 8.1.*

- Let $S$ be a translation surface. A *saddle connection* on $S$ is a straight line segment connecting two singularities. Since a saddle connection has a well-defined length and direction, each saddle connection is associated with a non-zero vector in $\mathbb{R}^2$. Let $V_{\mathrm{sc}}(S) \subset \mathbb{R}^2$ denote the set of vectors in $\mathbb{R}^2$ that are associated to saddle connections in $S$.
- By a *regular* closed geodesic on $S$, we mean a closed geodesic that does not pass through singularities.
- As mentioned in the introduction, any regular closed geodesic is part of a family of freely homotopic parallel closed geodesics of the same length. Such a family is called a *cylinder*. All the geodesics comprising a cylinder have the same length and direction; thus we can associate to a cylinder a non-zero vector in $\mathbb{R}^2$.

Note that each boundary component of a cylinder is a union of saddle connections. Let $V_{\mathrm{cyl}}(S) \subset \mathbb{R}^2$ denote the set (with multiplicity) of vectors in $\mathbb{R}^2$ that are associated to cylinders in $S$. In particular, if $S$ is a standard torus, then $V_{\mathrm{cyl}}(S)$ is the set of primitive vectors in $\mathbb{Z}^2$.

- For any $T > 0$, let $B(T)$ denote the ball in $\mathbb{R}^2$ of radius $T$ centered at 0.
- Let $V(S)$ be a subset of $\mathbb{R}^2 - (0, 0)$ with multiplicity, i.e. a set of vectors with positive weights. The weights are usually positive integers (e.g., we may consider saddle connections with multiplicity), but need not be (e.g., we may weight each cylinder by the reciprocal of its area).
- Let $N_V(S, T)$ denote the cardinality (with weights) of $V(S) \cap B(T)$. We are interested in the asymptotics of $N_V(S, T)$ as $T \to \infty$. If $V(S) = V_{\mathrm{sc}}(S)$, we will denote $N_V(S, T)$ by $N_{\mathrm{sc}}(S, T)$, and if $V(S) = V_{\mathrm{cyl}}(S)$ then, as in the introduction, we will denote $N_V(S, T)$ simply by $N(S, T)$.
- Recall from the introduction that $\mathcal{H}(\beta)$ denotes a *stratum* of translation surfaces.
- Let $\mathcal{H}_1(\beta)$ denote the subset of $\mathcal{H}(\beta)$ consisting of the surfaces of area 1 (where area is taken using the associated translation metric).
- As in §7, let $m_K$ denote the Haar measure on $\mathrm{SO}(2) \subset \mathrm{SL}(2, \mathbb{R})$.
- For $S \in \mathcal{H}_1(\beta)$ and $t > 0$, let

$$\nu_{t,S} = a_t m_K \delta_S$$

where $\delta_S$ is the atomic probability measure supported at $S$, and $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. Then $\nu_{t,S}$ is a probability measure on $\mathcal{H}_1(\beta)$.

- Finally, for a bounded compactly supported function $f : \mathbb{R}^2 \to \mathbb{R}$, let

$$\hat{f}_V(S) = \sum_{v \in V(S)} f(v).$$

The function $\hat{f}_V$ is called the *Siegel–Veech* transform of $f$.

*The general counting problem.* We now summarize the relevant results from [**Ve2, EM, EMS**] that will be used in §9.

THEOREM 8.2. *Let $S \in \mathcal{H}_1(\beta)$ be a translation surface, and suppose the following hold (using Notation 8.1).*

(A) *$V(\cdot)$ varies linearly under the $\mathrm{SL}(2, \mathbb{R})$ action, i.e. for all $g \in \mathrm{SL}(2, \mathbb{R})$ and all $S \in \mathcal{H}_1(\beta)$, we have $V(gS) = gV(S)$.*

(B) *There exists a constant $C$ such that, for all $S \in \mathcal{H}_1(\beta)$, we have $N_V(S, 2) \leq C N_{\mathrm{sc}}(S, 2)$.*

(C) *As $t \to \infty$, the measures $\nu_{t,S}$ converge to an $\mathrm{SL}(2, \mathbb{R})$-invariant (probability) measure $\mu$.*

(D) *Let $h : \mathbb{R}^2 \to \mathbb{R}$ denote the characteristic function of the trapezoid whose vertices are at $(1, 1)$, $(0, 1)$, $(0, 1/2)$, and $(1/2, 1/2)$. Let $\widetilde{\mathcal{O}}$ denote the closure of the $\mathrm{SL}(2, \mathbb{R})$ orbit of $S$. Then for any $\epsilon > 0$ and any compact subset $K$ of $\mathcal{H}_1(\beta)$, there exist continuous functions $\phi_+ : \widetilde{\mathcal{O}} \to \mathbb{R}$ and $\phi_- : \widetilde{\mathcal{O}} \to \mathbb{R}$ such that, for all $S \in \widetilde{\mathcal{O}} \cap K$,* *we have*

$$\phi_-(S) \le \hat{h}_V(S) \le \phi_+(S) \quad \textit{and} \quad \int_{\mathcal{H}_1(\beta)} (\phi_+ - \phi_-)\, d\mu < \epsilon.$$

*Then, the following hold.*

(i) *There exists a constant $c = c(S, V)$ such that, as $T \to \infty$,*

$$N_V(S, T) \sim \pi c T^2.$$

(ii) *We have the Siegel–Veech formula: there exists a constant $c$ such that for any continuous compactly supported $f : \mathbb{R}^2 \to \mathbb{R}$,*

$$\int_{\mathcal{H}_1(\beta)} \hat{f}_V \, d\mu = c \int_{\mathbb{R}^2} f. \tag{8.3}$$

(iii) *The constant $c$ in (i) is the same as the constant $c$ in (ii).*

*Remark 8.4.* Conclusion (ii) depends only on Assumptions (A) and (some version of) (B). It was proved by Veech in [**Ve2**], where this approach to counting on translation surfaces was originated. The proof is reproduced in [**EM**, Theorem 2.2].

*Remark 8.5.* Assumption (B) may be replaced by the following.

(B′) There exist constants $C > 0$ and $0 < s < 2$ such that for all $S \in \mathcal{H}_1(\beta)$, $N_V(S, 2) \le C/\ell(S)^s$, where $\ell(S)$ is the length of the shortest saddle connection on $S$.

In fact, (B′) is used in the proof of Theorem 8.2 instead of (B). The assertion that (B) implies (B′) follows from [**EM**, Theorem 5.1].

*Remark 8.6.* It follows from (B′) and [**EM**, Theorem 5.2] that any limit measure of the probability measures $\nu_{t,S}$ must be a probability measure (see [**EM**, Corollary 5.3]). Thus, the measure $\mu$ of (C) is automatically a probability measure.

*Remark 8.7.* The assertion (D) is a technical assumption needed since the Siegel–Veech transform $\hat{f}$ may not be continuous even if $f$ is.

*Outline of proof of Theorem 8.2.* Let $h$ be the characteristic function of the trapezoid as in (D). We have the following lemma from calculus (cf. [**EM**, Lemma 3.4]): for any $v \in \mathbb{R}^2$,

$$\int_0^{2\pi} h(a_t r_\theta v)\, d\theta \approx \begin{cases} e^{-2t} & \text{if } e^t/2 \le \|v\| \le e^t, \\ 0 & \text{otherwise.} \end{cases} \tag{8.8}$$

If we multiply both sides of (8.8) by $T^2 = e^{2t}$ and sum over all $v \in V(S)$, we get, under Assumption (A),

$$T^2 \int_0^{2\pi} \hat{h}_V(a_t r_\theta S)\, d\theta \approx N_V(S, T) - N_V(S, T/2)$$

or, equivalently,

$$N_V(S, T) - N_V(S, T/2) \approx 2\pi T^2 \int_{\mathcal{H}_1(\beta)} \hat{h}_V \, d\nu_{t,S}. \tag{8.9}$$

(The fact that we only have approximate equality and not equality in equation (8.9) does not affect the asymptotics. See [**EM**, §3] for the details.)

The Assumption (C) means that for any bounded continuous function $\phi$ on $\mathcal{H}_1(\beta)$,

$$\lim_{t\to\infty}\int_{\mathcal{H}_1(\beta)}\phi\, d\nu_{t,S} = \int_{\mathcal{H}_1(\beta)}\phi\, d\mu. \tag{8.10}$$

We would like to apply equation (8.10) to $\hat{h}_V$, which is neither bounded nor continuous. The fact that $\hat{h}_V$ is not continuous is handled by Assumption (D). To handle the fact that $\hat{h}_V$ is not bounded, we decompose $\hat{h}_V = h_1 + h_2$, where $h_1$ is bounded and $h_2$ is supported outside of a large compact set. Then the contribution of $h_2$ can be shown to be negligible using [**EM**, Theorem 5.2], in view of Assumption (B$'$). The details of this argument are given in [**EMS**, §2].

Now applying equation (8.10) with $\phi = \hat{h}_V$ and substituting into equation (8.9), we get

$$\lim_{T\to\infty}\frac{N(S,T) - N(S,T/2)}{T^2} = 2\pi\int_{\mathcal{H}_1(\beta)}\hat{h}_V\, d\mu.$$

By iterating (replacing $T$ with $T/2, T/4, T/8, \ldots$), and summing the resulting geometric series, we get

$$\lim_{T\to\infty}\frac{N(S,T)}{T^2} = \frac{8\pi}{3}\int_{\mathcal{H}_1(\beta)}\hat{h}_V\, d\mu. \tag{8.11}$$

This implies (i). Now by equation (8.3),

$$\int_{\mathcal{H}_1(\beta)}\hat{h}_V\, d\mu = c\int_{\mathbb{R}^2} h = \frac{3c}{8}.$$

This, together with equation (8.11), implies (iii). □

As a corollary of Theorem 8.2 and Theorem 7.1 we have the following.

THEOREM 8.12. *Suppose $S$ is a branched cover of a Veech surface $M$. Let $N(S,T)$ denote the number of cylinders of periodic trajectories in $S$ of length at most $T$. Then there exists a constant $c = c(S)$ such that, as $T\to\infty$,*

$$N(S,T) \sim cT^2. \tag{8.13}$$

*Proof.* We use Theorem 8.2, with $V(\cdot) = V_{\mathrm{cyl}}(\cdot)$. Assumption (A) clearly holds, and (B) also holds since the boundary of every cylinder contains a saddle connection.

Now let $\mathcal{M}$ be the connected component of $\mathcal{M}_D(\beta)$ that contains $S$ (where $\mathcal{M}_D(\beta)$ is as in the introduction). Since $S\in\mathcal{M}$ and $\mathcal{M}$ is closed and $\mathrm{SL}(2,\mathbb{R})$-invariant, the support of any of the measures $\nu_{t,S}$ is contained in $\mathcal{M}$. Also, as $\mathcal{M}$ is a branched cover of the space $\mathcal{X}^k$, a measure classification theorem on $\mathcal{X}^k$ automatically yields a measure classification theorem on $\mathcal{M}$ (see Lemma 8.14 below). Thus Assumption (C) of Theorem 8.2 follows from Theorem 7.1.

Finally, in our setting (D) is automatically satisfied, since the orbit closure $\widetilde{\mathcal{O}}$ is a proper submanifold of $\mathcal{H}_1(\beta)$, the measure $\mu$ is Lebesgue measure on $\widetilde{\mathcal{O}}$, and (after intersecting with any compact set) the set of discontinuities of $\hat{h}_V$ is contained in a finite union of submanifolds of positive codimension in $\widetilde{\mathcal{O}}$. Thus Theorem 8.12 follows from (i) of Theorem 8.2. □

LEMMA 8.14. *Suppose $W$ is a $G$-invariant subspace of $(\mathbb{R}^2)^k$, $\mathcal{O}$ is a closed orbit of $G \ltimes \widehat{\Phi}^k_W$ in $\mathcal{X}^k$, and $\widetilde{\mathcal{O}}$ is a (connected) branched cover of $\mathcal{O}$, such that the action of $G$ on $\mathcal{O}$ lifts to $\widetilde{\mathcal{O}}$.*

*If $\nu$ is any $u$-invariant probability measure on $\widetilde{\mathcal{O}}$ that projects to the Lebesgue measure on $\mathcal{O}$, then $\nu$ is the Lebesgue measure on $\widetilde{\mathcal{O}}$.*

*Proof.* Let $\mu$ and $\widetilde{\mu}$ be the Lebesgue measures on $\mathcal{O}$ and $\widetilde{\mathcal{O}}$, respectively. Then, because it projects to $\mu$, the measure $\nu$ must be absolutely continuous with respect to $\widetilde{\mu}$; thus, we may write $\nu = f\widetilde{\mu}$, for some Borel function $f$ on $\widetilde{\mathcal{O}}$.

It is not difficult to see that $\widetilde{\mu}$ is ergodic for $G$, so (by decay of matrix coefficients [**Zi**, Theorem 2.4.2, p. 29], or by the Mautner phenomenon [**Zi**, Theorem 2.2.15, p. 21]) it is ergodic for $u$. This implies that $f$ is constant. So $\nu = \widetilde{\mu}$ (up to a normalizing scalar multiple). □

## 9. *Triangular billiards*

Let $n \geq 5$ be an odd integer. As in the introduction, let

$$P_n \text{ denote the triangle with angles } \frac{(n-2)\pi}{2n}, \frac{(n-2)\pi}{2n}, \frac{2\pi}{n}$$

and let $S_n$ denote the corresponding translation surface. In the rest of this section, we complete the proof of Theorem 1.4 by computing the constant $c$ in Theorem 8.12 for the case of the surface $S_n$. Our general strategy is to use (ii) and (iii) of Theorem 8.2. To pass from $S_n$ to $P_n$, note that $N(P_n, T) = N(S_n, T)$, and as $S_n$ consists of $4n$ triangles, $\text{area}(S_n) = 4n \,\text{area}(P_n)$.

The surface $S_n$ can be drawn as in Figure 3. As shown in [**HS1**] and as one can see from the figure, $S_n$ is a double cover of a surface $X_n$ consisting of a double $n$-gon with opposite sides identified. The surface $X_n$ is a Veech surface (see [**Ve1**]), but $S_n$ is not (see [**HS1**]).

### 9.1. *The Veech surface.*

Most of the information in this section comes from [**Ve1**]. Let

$$Q_n \text{ denote the triangle with angles } \frac{\pi}{n}, \frac{\pi}{n}, \frac{(n-2)\pi}{n}$$

(realized with the two equal sides having length 1, and one of the equal sides horizontal). Then the surface corresponding to $Q_n$ can easily be seen to be (isomorphic to) $X_n$. The cylinder decomposition in the vertical direction consists of $(n-1)/2$ cylinders $V_j$, and for $1 \leq j \leq (n-1)/2$, we have

$$h_j = \text{ height } V_j = 4\sin\frac{\pi(2j-1)}{n}\cos\frac{\pi}{n}, \tag{9.1}$$

$$w_j = \text{ width } V_j = 2\sin\frac{\pi(2j-1)}{n}\sin\frac{\pi}{n} \tag{9.2}$$

(the closed trajectories in the cylinder $V_j$ have length $h_j$). Since for all $1 \leq j \leq (n-1)/2$, $h_j/w_j = 2\cot(\pi/n)$, the unipotent

$$u_n = \begin{pmatrix} 1 & 0 \\ 2\cot(\pi/n) & 1 \end{pmatrix}$$

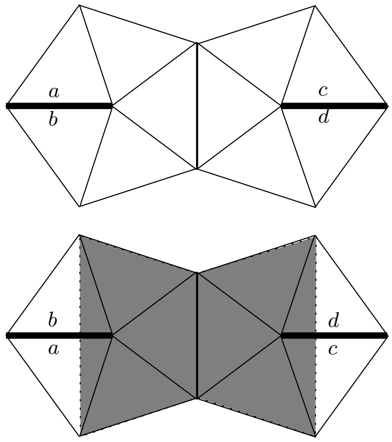


FIGURE 3. We draw the surface $S_n$ (for $n = 5$), tessellated by (reflections of) the triangle $P_n$. In each of the double $n$-gon shapes, the opposite parallel sides are identified. The bottom double $n$-gon can be identified with the surface $X_n$. The covering map from $S_n$ to $X_n$ is specified by the two slits (drawn as thick lines), with identifications as shown. For $n = 5$, the shaded region in the bottom double pentagon is one of the cylinders in the vertical cylinder decomposition for $X_n$; the unshaded region in the bottom double pentagon is the other cylinder.

belongs to the Veech group $\Gamma_n$ of $X_n$. Note that

$$u_n \begin{pmatrix} w_j \\ 0 \end{pmatrix} = \begin{pmatrix} w_j \\ h_j \end{pmatrix}. \tag{9.3}$$

The unipotent $u_n$, together with the rotation by $2\pi/n$ generate $\Gamma_n$. It is shown in [**Ve1**] that

$$\mathrm{Vol}(\Gamma_n \backslash \mathbb{H}) = \frac{n-2}{n}\pi$$

where Vol denotes the Poincaré volume on the hyperbolic plane $\mathbb{H}$.

The following lemma is from [**GJ**].

LEMMA 9.4. *Suppose $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ is a lattice, and suppose that $\Gamma$ intersects non-trivially the stabilizer $N$ in $\mathrm{SL}(2,\mathbb{R})$ of $v$. (The above condition is equivalent to the discreteness of the orbit $\Gamma v$.) Let $\gamma \in \Gamma$ be either of the two generators of $\Gamma \cap N$. Let $B(T)$ denote the ball in $\mathbb{R}^2$ of radius $T$ centered at the origin. Then, as $T \to \infty$,*

$$|\Gamma v \cap B(T)| \sim \mathrm{Vol}(\Gamma_n \backslash \mathbb{H})^{-1} \frac{|\langle \gamma v^\perp, v\rangle|}{\|v\|^3 \|v^\perp\|} T^2,$$

*where $v^\perp$ is any vector perpendicular to $v$.*

We also record the following trivial consequence (of the existence of the asymptotics).

Suppose $v$ is as in Lemma 9.4, and suppose $v'$ is a scalar multiple of $v$. Then, as $T \to \infty$.

$$|\Gamma v' \cap B(T)| \sim \frac{\|v\|^2}{\|v'\|^2} |\Gamma v \cap B(T)|. \tag{9.5}$$

We now apply Lemma 9.4 with $\Gamma = \Gamma_n$, $v = \binom{0}{h_1}$, $v^\perp = \binom{w_1}{0}$, $\gamma = u_n$, and using equation (9.3) we get that the number of cylinders in the ball of radius $T$ that are in the orbit of the cylinder $V_1$ is asymptotic to

$$\mathrm{Vol}(\Gamma_n\backslash\mathbb{H})^{-1}\frac{h_1^2}{h_1^3 w_1}T^2 = \frac{n}{(n-2)\pi}\frac{1}{h_1 w_1}T^2. \tag{9.6}$$

We now use equation (9.5) to see that for $1 \le j \le (n-1)/2$, the number of cylinders in the ball of radius $T$ that are in the $\Gamma_n$ orbit of $V_j$ is asymptotic to

$$\frac{n}{(n-2)\pi}\frac{1}{h_j w_j}T^2 \tag{9.7}$$

(in the above, we used the identity $(1/h_1 w_1)(h_1^2/h_j^2) = 1/h_j w_j$). As every cylinder is in the orbit of some $V_j$, we get (after summing over $j$),

$$N(X_n, T) \sim \frac{n}{(n-2)\pi}\left(\sum_{j=1}^{(n-1)/2}\frac{1}{h_j w_j}\right)T^2. \tag{9.8}$$

Using the identity [**Ve1**, Lemma 6.3]

$$\sum_{j=1}^{(n-1)/2}\frac{1}{\sin^2[\pi(2j-1)/n]} = \frac{n^2-1}{6}$$

and using the expressions (9.1), we get

$$N(X_n, T) \sim \frac{n}{(n-2)\pi}\frac{n^2-1}{48}\frac{T^2}{\sin(\pi/n)\cos(\pi/n)}.$$

As

$$\sin\frac{\pi}{n}\cos\frac{\pi}{n} = \frac{1}{2}\sin\frac{2\pi}{n} = \text{area } Q_n = \frac{1}{2n}\text{area } X_n,$$

we have

$$N(X_n, T) \sim \frac{n^2(n^2-1)}{24(n-2)\pi}\frac{T^2}{\mathrm{area}(X_n)} = \frac{\pi}{\zeta(2)}\frac{n^2(n^2-1)}{144(n-2)}\frac{T^2}{\mathrm{area}(X_n)}. \tag{9.9}$$

This is the formula in [**Ve1**].

9.2. *The Siegel–Veech formula applied to $X_n$.* It is useful for the following to compare the result of (9.7) with the result of the corresponding Siegel–Veech formula. Let $D_n = \mathrm{SL}(2,\mathbb{R})X_n$ denote the orbit of $X_n$. This is a closed submanifold of the stratum, which is also called a Teichmüller curve. For $1 \le j \le (n-1)/2$, define $U_j : D_n \to$ subsets of $\mathbb{R}^2$ by the formula $U_j(gX_n) = g\Gamma_n\binom{0}{h_j}$ (where $g \in \mathrm{SL}(2,\mathbb{R})$). Let $f_\epsilon$ denote the characteristic function of the $\epsilon$-ball in $\mathbb{R}^2$ centered at the origin and, for $M \in D_n$, define the Siegel–Veech transform $\hat{f}_{j,\epsilon}(M) = \sum_{v\in U_j(M)} f_\epsilon(v)$.

LEMMA 9.10. *If $\epsilon$ is sufficiently small, then $\hat{f}_{j,\epsilon} : D_n \to \mathbb{R}$ takes on only the values* 0 *and* 1*; we have $\hat{f}_{j,\epsilon}(M) = 1$ if and only if $M$ has a cylinder decomposition such that the $j$th cylinder from the left has height at most $\epsilon$. Given $M$, such a cylinder decomposition is unique if it exists.*

*Proof.* This is straightforward. (The uniqueness of the decomposition follows from the fact, proved by Veech [**Ve1**], that $\mathbb{H}^2/\Gamma_n$ has only one cusp.) □

Let $\nu$ denote the normalized $\mathrm{SL}(2,\mathbb{R})$-invariant measure on $D_n$. Then, we have the Siegel–Veech formula:

$$\int_{D_n} \hat{f}_{j,\epsilon}\, d\nu = c_j \int_{\mathbb{R}^2} f_\epsilon. \tag{9.11}$$

We now apply Theorem 8.2 with $V(\cdot) = U_j(\cdot)$, and $S = X_n$. The validity of Assumption (C) can be deduced from the mixing property of the geodesic flow, see [**Mar**] for a general proof in variable negative curvature, or [**EMc**] for a simplified exposition in the constant curvature case. We obtain that

$$|U_j(X_n) \cap B(T)| \sim \pi c_j T^2,$$

where $c_j$ is as in (9.11). Comparing with (9.7), we see that

$$c_j = \frac{n}{(n-2)\pi^2} \frac{1}{h_j w_j}.$$

Substituting into (9.11) we get

$$\frac{1}{\pi\epsilon^2} \int_{D_n} \hat{f}_{j,\epsilon}\, d\nu = \frac{n}{(n-2)\pi^2} \frac{1}{h_j w_j}. \tag{9.12}$$

*Remark 9.13.* It is possible to prove (9.12) directly, and thus to compute the asymptotics in (9.6) without using Lemma 9.4. We chose this indirect derivation of (9.12) to minimize the amount of computation.

9.3. *The branched cover.* We now return to our surface $S_n$, which is a branched cover of $X_n$ (see Figure 3). $X_n$ is a union of two $n$-gons, and the two branch points $p$ and $p'$ are at the centers of the $n$-gons. We now wish to apply Theorem 7.1 to the point $(X_n, p, p') \in \mathcal{X}^2$.

It is important to note that $X_n$ is hyperelliptic, and that our two branch points are interchanged by the hyperelliptic involution. As the hyperelliptic involution commutes with the $\mathrm{SL}(2,\mathbb{R})$ action, this it true for any point in the orbit of $(X_n, p, p')$. Thus, the $\mathrm{SL}(2,\mathbb{R})$ orbit of $(X_n, p, p')$ is *not* dense in the space $\mathcal{X}^2$, and indeed we have in Theorem 7.1 a proper $W \subset (\mathbb{R}^2)^2$ of real dimension two. Let $L$ denote the subspace $\{(v, -v) \mid v \in \mathbb{R}^2\}$. The above argument shows that $W \subseteq L$. But since we know that $S_n$ is not Veech, $\dim W > 0$. Hence $\dim W = 2$ and $W = L$. Let $\mathcal{O} = (\mathrm{SL}(2,R) \ltimes \Phi_W^2)(X_n, p, p')$. Then $\mathcal{O} \subset \mathcal{X}^2$ consists of points of the form $(M, q, q')$ where $M \in D_n$, $q \in M$, $q' \in M$ and $q$ and $q'$ are interchanged by the hyperelliptic involution of $M$. By Theorem 7.1,

$$\lim_{t\to\infty} \nu_{t,(X_n,p,p')} = \mu, \tag{9.14}$$

where $\mu$ is Lebesgue measure on $\mathcal{O}$.

Now let $\tilde{\mathcal{O}}$ denote the orbit closure $\overline{\mathrm{SL}(2,\mathbb{R})S_n}$. As $S_n$ is a double cover of $X_n$, branched over $p$ and $p'$, for any $g \in \mathrm{SL}(2,\mathbb{R})$, $gS_n$ is a double cover of $gX_n$ branched over $gp$ and $gp'$, and $(gX_n, gp, gp') \in \mathcal{O}$. Thus, in particular, every surface in $\tilde{\mathcal{O}}$ is a double cover of a surface in $D_n$. Thus we have a natural map $\tilde{\pi} : \tilde{\mathcal{O}} \to \mathcal{O}$ that maps each surface $S \in \tilde{\mathcal{O}}$ to the surface in $D_n$ of which it is a double cover, and notes the locations of the branch points. Now in view of (9.14) and Lemma 8.14,

$$\lim_{t\to\infty} \nu_{t,S_n} = \tilde{\mu},$$

where $\tilde{\mu}$ is the normalized Lebesgue measure on $\tilde{\mathcal{O}}$. Hence, by Theorem 8.2, we have a quadratic asymptotic formula

$$N(S_n, T) = |V_{\mathrm{cyl}}(S_n) \cap B(T)| \sim \pi c T^2,$$

with the constant $c$ given by

$$c = \frac{1}{\pi\epsilon^2} \int_{\tilde{\mathcal{O}}} \hat{f}_\epsilon \, d\tilde{\mu} \tag{9.15}$$

where, as above, $f_\epsilon : \mathbb{R}^2 \to \mathbb{R}$ is the characteristic function of the ball of radius $\epsilon$ centered at the origin, and $\hat{f}_\epsilon(S) = \sum_{v\in V_{\mathrm{cyl}}(S)} f_\epsilon(v)$.

Let $v$ be some periodic direction for $S_n$, hence for $X_n$. We may use an element $\gamma$ of the Veech group $\Gamma_n$ of $X_n$ to map $v$ to the vertical direction. Note that $\gamma S_n$ is a double cover of $\gamma X_n = X_n$. In the vertical direction, $X_n$ has the cylinder decomposition $V_1, \ldots, V_{(n-1)/2}$ described above.

LEMMA 9.16. *For any $\gamma \in \Gamma_n$, the branch points of $\gamma S_n$ will project to two points in the same cylinder, say $V_k$. The cylinder decomposition of $\gamma S_n$ in the vertical direction is the following:*

(a) *for each $j \neq k$, there are two cylinders on $\gamma S_n$ of the same length as $V_j$ (one on each 'sheet');*

(b) *on $\gamma S_n$ there are two cylinders of the same length as $V_k$ and two cylinders of twice the length of $V_k$.*

*Proof.* The fact that both branch points project to the same cylinder of $X_n$ follows from the fact that each cylinder of $X_n$ is preserved by the hyperelliptic involution $\sigma$ of $X_n$ (since different cylinders have different lengths) and the fact that the branch points are interchanged by $\sigma$. From Figure 3, the cover $S_n$ is determined by two slits (drawn as the thick lines in the figure), which are interchanged by $\sigma$. As $\sigma$ commutes with the SL(2, $\mathbb{R}$) action, the cover $\gamma S_n$ of $X_n$ is also determined by two slits, which are interchanged by $\sigma$. For each cylinder $V_j$ of $X_n$, let $\lambda_j$ denote the closed trajectory in the center of $V_j$. Note that for any $j$, $\lambda_j$ is mapped to itself under $\sigma$. Also, as $\sigma$ exchanges the slits, we see that $\lambda_j$ intersects each slit the same number of times. Thus, $\lambda_j$ breaks up into two closed paths of the same length when lifted from $X_n$ to $\gamma S_n$. This proves (a) and the first assertion of (b). It is easy to see that the closed vertical trajectories on $V_k$ between the boundary of $V_k$ and one of the branch points double in length when lifted from $X_n$ to $\gamma S_n$. This proves the second assertion of (b). □

COROLLARY 9.17. *The function $\hat{f}_\epsilon : \tilde{\mathcal{O}} \to \mathbb{R}$ is constant on the fibers of $\tilde{\pi}$ almost everywhere, and thus descends to a function $\bar{f}_\epsilon : \mathcal{O} \to \mathbb{R}$. The latter function, for $\epsilon$ sufficiently small, is given almost everywhere by the formula*

$$\bar{f}_\epsilon(M, q, q') = 2\hat{f}_{k,\epsilon}(M) + 2\hat{f}_{k,\epsilon/2}(M) + 2\sum_{j\neq k} \hat{f}_{j,\epsilon}(M), \tag{9.18}$$

*where for* $1 \leq j \leq (n-1)/2$, $\hat{f}_{j,\epsilon} : D_n \to \mathbb{R}$ *is as in Lemma 9.10, and* $k$ *is such that* $q$ *(and* $q'$*) belong to the* $k$*th cylinder from the left in the unique cylinder decomposition that contains a cylinder of height at most* $\epsilon$.

*Proof* (sketch). Choose a fundamental domain for $\Gamma_n$ in the upper half-plane. Since $X_n$ has a vertical cylinder decomposition, we may assume that the cusp of the fundamental domain approaches $\infty$, rather than approaching a point on the real axis. This means that as $h$ goes to infinity in the fundamental domain, the unique short cylinder decomposition of $hX_n$ is the image under $h$ of the vertical cylinder decomposition of $X_n$.

We first prove that (9.18) is correct for all $M$ in the $\mathrm{SL}(2,\mathbb{R})$ orbit of $S_n$. To do this, let $g \in \mathrm{SL}(2,\mathbb{R})$, and write $g = h\gamma$, where $\gamma \in \Gamma_n$, and $h$ is in the fundamental domain. Note that $\hat{f}_\epsilon(M)$ is zero unless $M$ has a short cylinder. Thus, if $g = h\gamma$, and $h$ is in a compact part of the fundamental domain, then (in view of Lemma 9.16), we have $\hat{f}_\epsilon(h\gamma S_n) = 0$. Therefore, we may assume that $h$ is in the cusp, and hence the unique short cylinder decomposition of $gS_n = h\gamma S_n$ is the image, under the linear action of $h$, of the vertical cylinder decomposition of $\gamma S_n$. Then it is clear from Lemma 9.16 that (9.18) holds for $M = gS_n$.

To complete the proof, note that both sides of (9.18) are continuous off a closed set of measure zero, namely, the set where a branch point projects to an edge of a cylinder (in a cylinder decomposition containing a cylinder of height at most $\epsilon$). Then use the fact that the $\mathrm{SL}(2,\mathbb{R})$ orbit of $S_n$ is dense. □

*Remark 9.19.* The analogue of the first assertion of Corollary 9.17 fails in the context of [**EMS**], in part since there we are dealing with covers of high degree. This is responsible for most of the combinatorial complexity of the argument in [**EMS**].

In view of Corollary 9.17, (9.15) becomes

$$\begin{aligned} c &= \frac{1}{\pi\epsilon^2}\int_{\mathcal{O}} \bar{f}_\epsilon \, d\mu \\ &= \frac{1}{\pi\epsilon^2}\int_{D_n}\int_M \bar{f}_\epsilon(M, q, \sigma(q))\, d\lambda_M(q)\, d\nu(M), \end{aligned}$$

where $\lambda_M$ is the Lebesgue measure on the translation surface $M$, and $\sigma$ denotes the hyperelliptic involution. Performing the integral over $M$, we get

$$c = \sum_{k=1}^{(n-1)/2} p_k\left(\frac{1}{\pi\epsilon^2}\int_{D_n}\left(2\hat{f}_{k,\epsilon}(M) + 2\hat{f}_{k,\epsilon/2}(M) + 2\sum_{j\neq k}\hat{f}_{j,\epsilon}(M)\right) d\nu(M)\right),$$

where $p_k = h_k w_k/A$ and $A = \mathrm{area}(X_n)$ (so that $p_k$ denotes the relative area of the $k$th cylinder from the left in any cylinder decomposition).

Now, using (9.12), we get

$$\begin{aligned} N(S_n, T) \sim c\,T^2 &= \frac{n}{(n-2)\pi}\sum_{k=1}^{(n-1)/2} p_k\left(\frac{2}{h_k w_k} + \frac{2}{4h_k w_k} + \sum_{j\neq k}\frac{2}{h_j w_j}\right)T^2 \\ &= \frac{n}{(n-2)\pi}\sum_{k=1}^{(n-1)/2} p_k\left(\frac{2}{4h_k w_k} + \sum_{j=1}^{(n-1)/2}\frac{2}{h_j w_j}\right)T^2. \end{aligned}$$

As the second term in the parenthesis is independent of $k$, and $\sum p_k = 1$, this can be rewritten as

$$N(S_n, T) \sim \left( \frac{n}{(n-2)\pi} \sum_{j=1}^{(n-1)/2} \frac{2}{h_j w_j} + \frac{n}{(n-2)\pi} \sum_{k=1}^{(n-1)/2} \frac{2p_k}{4h_k w_k} \right) T^2. \tag{9.20}$$

The first term in the parenthesis is in view of (9.8) twice the limit of $N(X_n, T)/T^2$. The second term in the parenthesis is, since $p_k = h_k w_k / (\operatorname{area} X_n)$, equal to

$$\frac{n}{(n-2)\pi} \frac{(n-1)}{4 \operatorname{area}(X_n)}.$$

In view of (9.9), we get

$$N(S_n, T) \sim \left( \frac{n^2(n^2-1)}{12(n-2)\pi} \frac{1}{\operatorname{area}(X_n)} + \frac{n}{(n-2)\pi} \frac{(n-1)}{4 \operatorname{area}(X_n)} \right) T^2.$$

Simplifying, we get

$$N(S_n, T) \sim \frac{n(n-1)(n^2+n+3)}{12(n-2)\pi} \frac{T^2}{\operatorname{area}(X_n)}.$$

Alternatively,

$$N(S_n, T) \sim \frac{n(n-1)(n^2+n+3)}{6(n-2)\pi} \frac{T^2}{\operatorname{area}(S_n)} = \frac{\pi}{\zeta(2)} \frac{n(n-1)(n^2+n+3)}{36(n-2)} \frac{T^2}{\operatorname{area}(S_n)}.$$

To pass from $S_n$ to $P_n$, note that $N(P_n, T) = N(S_n, T)$, and as $S_n$ consists of $4n$ triangles, $\operatorname{area}(S_n) = 4n \operatorname{area}(P_n)$.

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