Algebraic Geometry Lecture 20 – Projective Recappery

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Projective Space

For an algebraically closed field k recall that affine n-space is

$$\mathbb{A}^n = \{(a_1, \ldots, a_n) \mid a_i \in k, 1 \leqslant i \leqslant n\}.$$

Now consider an equivalence relation $\sim$ on $\mathbb{A}^{n+1} \setminus \{0\}$ given by

$$(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n) \quad \Leftrightarrow \quad \text{there exists } \lambda \in k^\times \text{ such that } a_i = \lambda b_i \text{ for } 0 \leqslant i \leqslant n.$$

The space $(\mathbb{A}^{n+1} \setminus \{0\})/\sim$ is called n-dimensional projective space, $\mathbb{P}^n$. So

$$\mathbb{P}^n = \{[a_0, \ldots, a_n] \mid a_i \in k, 0 \leqslant i \leqslant n\},$$

where the a_i aren't all zero, and each 'point' $[a_0, \ldots, a_n]$ is really an equivalence class under $\sim$.

Projective Varieties

We defined an affine algebraic set U as the set of points in $\mathbb{A}^n$ that vanished on an ideal $J \subset k[X]$. For example, over $\mathbb{C}$,

$$\begin{aligned} V((x^2 - y)) &= \{(x, y) \in \mathbb{C}^2 \mid x^2 - y = 0\} \\ &= \{(x, x^2) \mid x \in \mathbb{C}\} \\ &= \{(a + bi, a^2 - b^2 + 2abi) \mid a, b \in \mathbb{R}\}. \end{aligned}$$

In $\mathbb{P}^n$ this doesn't necessarily make sense. For example, $x^2 - y$ is zero for $[x, y] = [1, 1] \in \mathbb{P}^1$. But in $\mathbb{P}^1$ we have $[1, 1] = [2, 2]$, and $2^2 - 2 = 2 \neq 0$. So instead we consider homogeneous polynomials. These satisfy

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n)$$

for any $\lambda \in k$ and some $d \in \mathbb{Z}$, $d \geqslant 0$, where d is called the degree of the polynomial. Homogeneous polynomials are just those polynomials, all of whose monomials have the same total degree – which coincides with the degree just defined. For example

$$X^4 - 7X^2Y^2 + XY^3 - 19Y^4$$

is a homogeneous polynomial of degree 4.

So if we let $S \subset k[X]$ be a set of homogeneous polynomials then, utterly analogous to the affine case, we define

$$V(S) = \{P \in \mathbb{P}^n \mid f(P) = 0 \text{ for all } f \in S\}.$$

Then a projective algebraic set is a subset $U \subseteq \mathbb{P}^n$ that can be written $U = V(S)$ for some set of homogeneous polynomials $S \subset k[X]$. We also define the ideal of a set $U \subset \mathbb{P}^n$ as

$$I(U) = \{f \in k[X] \mid f \text{ homogeneous}, f(P) = 0 \text{ for all } P \in U\}.$$

The ideal is called a homogeneous ideal since it only contains homogeneous polynomials. A projective variety is then just an irreducible projective algebraic set. Equivalently it's a projective algebraic set whose homogeneous ideal is prime.

Functions

Given a projective variety we want to know about interesting functions on it. Recall that a function $f : U \to k$ was called regular in the affine case if there was a polynomial $F(x) \in k[x]$ such that $f(x) = F(x)$ for every $x \in U$. This is rather pointless in the projective case. Suppose we have a regular function f on a projective variety U and that it isn't everywhere zero. So

$$f(P) = \alpha \neq 0$$

at some $P \in U$. Then

$$f(\lambda P) = \lambda^d f(P) = \lambda^d \alpha$$

for any $\lambda \in k$. Clearly, then, $d = 0$, and so f has degree zero, i.e. it's a constant function. So the only 'regular' functions are constant ones, thus using last weeks notation, $\mathcal{O}(U) = k$.

We may still define the coordinate ring on U as

$$k[U] = k[X]/I(U)$$

where, as usual, $k[X]$ is the set of homogeneous polynomials. The fact that $\mathcal{O}(U) \ncong k[U]$ is a consequence of projective varieties having more intrinsic structure than their affine counterparts[1].

This is still an integral domain so we take the function field on U to be the field of fractions of $k[U]$, however we have to add the proviso that the numerator and denominator in our rational functions have the same degree to ensure that $f(\lambda P)/g(\lambda P) = f(P)/g(P)$. So

$$k(U) = \{f/g \mid f, g \in k[X], \text{ homogeneous of same degree}, g \in I(U)\} / \sim$$

where $\sim$ is the expected equivalence relation for a field of fractions, that is

$$\frac{f_1}{g_1} \sim \frac{f_2}{g_2} \quad \Leftrightarrow \quad f_1 g_2 - f_2 g_1 \in I(U).$$

We then define the dimension of a projective variety U as

$$\dim U := \operatorname{trdeg}_k k(U).$$

E.g.. Consider $U = \{X - Y = 0\} \subset \mathbb{P}^1$. Then

$$\begin{aligned} \mathbb{C}[U] &= \mathbb{C}[X, Y]/(X - Y) \\ &\cong \mathbb{C}[X]. \end{aligned}$$

Hence $\mathbb{C}(U) \cong \mathbb{C}(X)$, which has transcendence degree 1 over $\mathbb{C}$.

[1] Apparently.

Zariski Topology

A topology T on a set X is a collection of subsets of X such that:

(1) $\emptyset, X \in T$;
(2) if $A_1, A_2, \ldots \in T$ then $\bigcup A_i \in T$;
(3) if $A, B \in T$ then $A \cap B \in T$.

The subsets of X in T are called the open sets of X, and their complements are called the closed sets of X.

The Zariski topology is a topology on $\mathbb{A}^n$ (or $\mathbb{P}^n$) where the closed sets are defined to the be algebraic sets. And a quasi-affine (or -projective) variety is an open subset of an affine (respectively projective) variety.