Continued Fractions

OR

What's the Best Way to Write Numbers?

When it comes to writing down an actual number, most people know at least two ways to do it. The first way is as a **fraction**. Fractions are great, they reduce complicated sounding numbers like "that number that when you times it by seven gives you one," to the simple expression

$$\frac{1}{7}$$

But fractions have one big drawback – not every number can be written as one. It's been known for over two thousand years that the square root of two can not be written as a fraction. We can get very close, for example the fraction

$$\frac{577}{408}$$

is the same as $\sqrt{2}$ to six decimal places. But no fraction is exactly equal to $\sqrt{2}$.

The other well-known way of writing down numbers is as a **decimal**. Using this method we can write down *any* number we want, although we often have to limit our accuracy since decimals usually go on forever. So we can write down $\sqrt{2}$, which to twelve decimal places is

$$1.414213562373$$

This isn't exactly $\sqrt{2}$, but by adding more decimal places we can get as close as we want to the number in question. The first downside to decimals is that they are over-reliant on the number 10. The word 'decimal' comes from the medieval Latin word *decimus*, which means tenth. When we write a number in decimal we are really writing down the number in terms of hundreds, tens, and units (remember primary school?) and after the decimal place in terms of tenths, hundredths, thousandths, and so on. But sometimes other numbers instead of ten are a better choice, computer scientists in particular replace ten with two, and write numbers in what is called *binary*. The other drawback of decimals is that even though a number may be simple, it's decimal representation may not be. The number $1/7$ for example is easy to imagine, just slice a cake into seven parts and eat six of them. And yet if we write this number as a decimal it goes on forever:

$$\frac{1}{7} = 0.142857142857142857...$$

In fact most numbers that can be written as fractions go on for ever as decimals.

What would be nice is a way of writing down numbers that has the benefits of both methods. This method is known as **continued fractions**.

Continued fractions typically look like this:

$$11+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{17+\cfrac{1}{4}}}} \qquad \text{OR} \qquad \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\dots}}}}}$$

In general they look like

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\dots}}}}$$

where a_0 is a whole number and $a_1, a_2, a_3, a_4, \dots$ are all positive whole numbers.

This notation is somewhat page-filling, so people often abbreviate continued fractions as follows:

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\dots}}}} = [a_0; a_1, a_2, a_3, a_4, \dots]$$

One of the nice things about continued fractions is that if we write a normal fraction out as a continued fraction then it eventually stops, that is we don't have to keep writing numbers forever. And if the continued fraction stops then the number it represents can be written out as a normal fraction. This is a handy way to see if a number can be written as a fraction or not. Later you'll show that $\sqrt{2}$ cannot be written as a fraction by showing that it's continued fraction goes on forever. First though we'll see how to write down a number as a continued fraction.

Suppose we want to write the number 19/7 as a continued fraction. We'd start by noticing that this number is a little more than two, in fact we have

$$\frac{19}{7} = 2 + \frac{5}{7} = 2 + \frac{1}{7/5}$$

Now we look at 7/5. That's a little bit more than one, specifically:

$$\frac{7}{5} = 1 + \frac{2}{5} = 1 + \frac{1}{5/2}$$

And what about 5/2? Well that's a bit more than two, and we have:

$$\frac{5}{2} = 2 + \frac{1}{2}$$

There's no point carrying on anymore, as we now have a number of the form $a + \frac{1}{b}$. So let's put all this back together:

$$\begin{aligned}\frac{19}{7} &= 2 + \frac{1}{7/5} \\ &= 2 + \cfrac{1}{1 + \cfrac{1}{5/2}} \\ &= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}\end{aligned}$$

Or, using our abbreviated notation, we have:

$$\frac{19}{7} = [2; 1, 2, 2]$$

How about the opposite problem, that is figuring out what number a given continued fraction represents? For example, what's the following number?

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}} = [1; 1, 1, 1, 1, 1, \dots]$$

Let's call this number x. The trick is to notice that if you cover up the first part of the continued fraction, that is the opening part $1 + \frac{1}{}$ then what's left is x again. So we have

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}} = 1 + \frac{1}{x}$$

Multiplying both sides of this equation by x gives us a quadratic equation to solve

$$x^2 = x + 1$$

This has two solutions, one of which is negative. The number we're looking at is clearly positive so we take the positive solution and find that

$$x = \frac{1 + \sqrt{5}}{2}$$

Now you figure out what number is represented by the following continued fractions.

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}}}=[1;2,2,2,2,2,\dots]$$

and

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\dots}}}}}}=[1;1,2,1,2,1,2,\dots]$$