UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc. and M.Sci. (Level C/4)

FOUNDATIONS AND PROOF
MATH 10004
(Paper Code MATH-10004)

January 2015 1 hour 30 minutes

This paper contains two sections, Section A and Section B. Please use a separate answer booklet for each section.
Section A contains ***five*** *short questions.* ***ALL*** *answers will be used for assessment. This section is worth 40% of the marks for the paper.*
Section B contains ***two*** *longer questions.* ***ALL*** *answers will be used for assessment. This section is worth 60% of the marks for the paper.*

Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

Section A: Short Questions

A1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 5x + 1$.

(a) (2 marks) Show that f is injective.

(b) (2 marks) Show that f is surjective.

(c) (2+2 marks) Find a formula for f^{-1}, and demonstrate that $f^{-1} \circ f$ is the identity function on $\mathbb{R}$.

A2. (a) (3 marks) Negate the following statement:

$$\exists M \in \mathbb{R} \text{ so that } \forall n \in \mathbb{Z}_+, |a_n| \leq M.$$

(b) (5 marks) Let P, Q be propositions. Use a truth table to prove that

$$\neg(P \implies Q) \iff (P \wedge \neg Q).$$

A3. (a) (3 marks) List all the elements in the set $\{1, 2\} \times \{3, 4\}$.

(b) (3 marks) List all the partitions of the set $\{1, 2, 3\}$.

(c) (2 marks) For $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, when is $a \equiv b \pmod{n}$?

A4. (a) (5 marks) Use Euclid's algorithm to find $\text{hcf}(76, 13)$, and integers s, t so that $76s + 13t = \text{hcf}(76, 13)$.

(b) (1 mark) Find $x \in \mathbb{Z}$ so that $x \equiv 2 \pmod{76}$ and $x \equiv 5 \pmod{13}$. (You may write x as a sum of products.)

(c) (2 marks) Find $x \in \mathbb{Z}$ so that $0 \leq x < 7$ and $x \equiv 3^5 \pmod{7}$.

A5. (a) (2 marks) Define what it means for sets A and B to have the same cardinality.

(b) (2 marks) Define what it means for a set C to be countable.

(c) (4 marks) Which of the following sets are countable?

$$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z};\ \mathbb{Q};\ \mathbb{R};\ \mathcal{P}(\mathbb{Z})$$

(where $\mathcal{P}(\mathbb{Z})$ denotes the power set of $\mathbb{Z}$).

Section B: Longer Questions

B1. (a) (8 marks) Let A, B, C be subsets of a set X. Prove that

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

(b) (4 marks) Define $f : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ by $f(m,n) = (m,0)$. Let $C = \{(m,0) :\ m \in \mathbb{Z}_+ \}$. Find $f^{-1}(C)$.

(c) (4 marks) Suppose $f : X \to Y$ and $U \subseteq X$. Show that $U \subseteq f^{-1}(f(U))$.

(d) (6 marks) Suppose $f : X \to Y$, $g : Y \to X$, and $g \circ f$ is the identity map on X. Also suppose g is injective. Show that $f \circ g$ is the identity map on Y.

(e) (8 marks) Use induction on n to show that for all $n \in \mathbb{Z}_+$

$$\sum_{i=1}^{n} (3i-2)^2 = \frac{n(6n^2-3n-1)}{2}.$$

B2. (a) (4 marks) State the Fundamental Theorem of Arithmetic.

(b) (9 marks) Find all primes p so that for some $n \in \mathbb{Z}_+$, $3p+1 = n^2$. Justify your answer carefully.

(c) (6 marks) Suppose p is an odd prime with $p \neq 3$. Prove that either $p \equiv 1 \pmod 6$ or $p \equiv 5 \pmod 6$.

(d) (4 marks) Let A, B be sets. Show that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

(e) (7 marks) Let X be a set and $\sim$ a relation on X. Let

$$N = \{x \in X :\ \neg(x \sim x) \},$$

$$B = \{b \in X :\ (\forall n \in N,\ b \sim n) \wedge (\forall n \notin N,\ \neg(b \sim n)) \}.$$

Show that $B = \emptyset$.