UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc. and M.Sci. (Level C/4)

FOUNDATIONS AND PROOF

MATH 10004

(Paper Code MATH-10004)

January 2016 1 hour 30 minutes

This paper contains two sections, Section A and Section B. Please use a separate answer booklet for each section.

*Section A contains **five** short questions. **ALL** answers will be used for assessment. This section is worth 40% of the marks for the paper.*

*Section B contains **two** longer questions. **ALL** answers will be used for assessment. This section is worth 60% of the marks for the paper.*

Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

Section A: Short Questions

A1. Recall that for $a, b \in \mathbb{R}$, $(a,b) = \{x \in \mathbb{R} : \ a \le x \le b \}$. Define

$$f : \left(\frac{5}{2}, \infty\right) \to \left(\frac{3}{2}, \infty\right)$$

by

$$f(x) = \frac{3x}{2x-5}.$$

(a) (2 marks) Show that f is injective.

(b) (2 marks) Show that f is surjective.

(c) (2+2 marks) Find a formula for f^{-1}, and demonstrate that $f \circ f^{-1}$ is the identity function on $\mathbb{R}$.

A2. (a) (3 marks) Negate the following statement:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ so that } (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon).$$

(b) (5 marks) Let P, Q be propositions. Use a truth table to prove that

$$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R).$$

A3. (a) (3 marks) List all the elements in the set $\{1,2\} \times \{a,b\}$.

(b) (3 marks) List all the partitions of the set $\{a,b,c\}$.

(c) (2 marks) Is $2 \equiv 10 \pmod 3$? Justify your answer.

A4. (a) (5 marks) Use Euclid's algorithm to find hcf(87, 33), and integers s, t so that

$$87s + 33t = \text{hcf}(87, 33).$$

(b) (3 marks) Find $x \in \mathbb{Z}$ so that $0 \le x < 11$ and $x \equiv 5^7 \pmod{11}$.

A5. (a) (3 marks) State the contrapositive of the following:

$$\exists N \in \mathbb{Z}_+ \text{ so that } n > N \implies a_n > 3.$$

(b) (2 marks) Which of the following is correct?

$$|\mathbb{Z}| < |\mathcal{P}(\mathbb{Z})|;\ |\mathbb{Z}| = |\mathcal{P}(\mathbb{Z})|;\ |\mathcal{P}(\mathbb{Z})| < |\mathbb{Z}|.$$

(Here $\mathcal{P}(\mathbb{Z})$ denotes the power set of $\mathbb{Z}$.)

(c) (3 marks) Which of the following sets are countable?

$$\mathbb{Z};\ \mathbb{Q} \times \mathbb{Q}_+;\ \mathbb{R}.$$

Section B: Longer Questions

B1. (a) (6 marks) Let A, B be subsets of a set X. Prove that

$$(A \setminus B)^c = A^c \cup B.$$

(b) (5 marks) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x^3|$. Let

$$V = \{x \in \mathbb{R} : \ 27 \leq x < \infty \}.$$

Find $f^{-1}(V)$.

(c) (5 marks) Suppose $g : X \to Y$ is sujective and $W \subseteq Y$. Show that $W \subseteq g(g^{-1}(W))$.

(d) (8 marks) Use induction on n to show that $\forall n \in \mathbb{Z}$ with $n \geq 2$,

$$\sum_{i=2}^{n} \frac{1}{(i-1)i} = 1 - \frac{1}{n}.$$

(e) (6 marks) Suppose $A, B_1, B_2, B_3, \ldots$ are subsets of a set X. Prove that for all $n \in \mathbb{Z}$ with $n \geq 1$, we have

$$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n).$$

B2. (a) (9 marks) Find all primes p so that for some $n \in \mathbb{Z}_+$, $5p + 16 = n^2$. Justify your answer carefully.

(b) (4 marks) Define $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$ by $f(m, n) = 2^m 3^n$. Show that f is injective.

(c) (4 marks) Suppose that $a, b, c \in \mathbb{Z}_+$ with $c = \text{hcf}(a, b)$. Take $x, y \in \mathbb{Z}_+$ so that $a = cx$, $b = cy$. Show that $\text{hcf}(x, y) = 1$. (Suggestion: Argue by contradiction.)

(d) (7 marks) Suppose that $f : X \to Y$ and $g : Y \to Z$ are bijective functions. Show that $g \circ f : X \to \mathbb{Z}$ is bijective.

(e) (6 marks) Suppose $f : X \to Y$ is bijective, and $A \subseteq X$ with A countable. Show that there is some $B \subseteq Y$ so that B is countable.