UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc. and M.Sci. (Level M)

MATH 2700
(Paper Code MATHM-2700)

January 2016, 2 hours 30 minutes

This paper contains **four** *questions*
All **FOUR** *questions should be attempted.*

Calculators are **not** *permitted in this examination.*

Do not turn over until instructed.

1. (a) (2+2+3+3=10 marks) Suppose that $K \subseteq L$ are fields.
   (i) Define $[L : K]$, the degree of the field extension $L : K$.
   (ii) Define what it means for $L : K$ to be an algebraic extension.
   (iii) Show that when $1 < [L : K] < \infty$, then there exists $\beta \in L$ for which
   $$[L : K(\beta)] < [L : K].$$
   (iv) Suppose that $[L : K] < \infty$. Show that there exist elements $\alpha_1, \ldots \alpha_n \in L$ for which $L = K(\alpha_1, \ldots, \alpha_n)$.

   (b) (2+5=7 marks) (i) Define what it means for a field extension $L : K$ to be an algebraic closure.
   (ii) Suppose that M is an algebraically closed field. Show that every irreducible polynomial in $M[t]$ has degree 1.

   (c) (5+3=8 marks) (i) Suppose that $L : M : K$ is a tower of field extensions. Prove that whenever $L : K$ is separable, then both $L : M$ and $M : K$ are separable.
   (ii) Suppose that $L : E : K$ and $L : F : K$ are towers of field extensions with $E : K$ and $F : K$ separable. Show that $E \cap F : K$ is a separable extension.
   Solution: The field $E \cap F$ is contained in E, so $E : E \cap F : K$ is a tower of field extensions with $E : K$ separable. Then it follows from (i) that $E \cap F : K$ is separable.

2. (a) (3+3 marks)] (i) State Eisenstein's criterion for irreducibility of polynomials in $\mathbb{Z}[t]$.
   (ii) Determine whether or not the polynomial $5t^5 - 40t - 2$ is irreducible over $\mathbb{Z}$, and explain your reasoning.

   (b) (4+2=6 marks) Let $f \in \mathbb{Q}[t]$ be irreducible, suppose that L is a splitting field for f over $\mathbb{Q}$, and suppose that $L : M : \mathbb{Q}$ is a tower of field extensions. Prove that the field extension $L : M$ is normal, and deduce that it is Galois.

   (c) (3+3=6 marks) Let K be a field. Suppose that $f \in K[t]$ is irreducible, and suppose that L is a splitting field for f over K. Briefly explain why, whenever $\alpha, \beta \in L$ and $f(\alpha) = 0 = f(\beta)$, then $K(\alpha) \simeq K(\beta)$. Hence deduce that there exists $\tau \in \mathrm{Gal}(L : K)$ such that $\tau(\alpha) = \beta$.

   (d) (2+5=7 marks) Throughout, let $f = t^5 - 8t - \frac{2}{5}$, let L be a splitting field for f over $\mathbb{Q}$, and let M be a field with $\mathbb{Q} \subsetneq M \subsetneq L$.
   (i) Show that, for any $\sigma \in \mathrm{Gal}(L : \mathbb{Q})$, and for any $\alpha \in M$, the polynomial $\sigma(m_{\alpha,\mathbb{Q}})$ is monic and irreducible. Here $m_{\alpha,\mathbb{Q}}$ denotes the minimal polynomial of α over $\mathbb{Q}$.
   (ii) Suppose that $M : \mathbb{Q}$ is Galois and that f factors as a product of monic irreducibles $f_1, \ldots, f_r$ over $M[t]$. Show that $\deg(f_i) = \deg(f_1)$ for each i, and hence deduce that f remains irreducible over M.

3. (a) (10 marks) Define what it means for a field extension $L : K$ to be

(i) normal:

(ii) a splitting field extension:

(iii) separable:

(iv) finite:

(v) Galois:

(b) (15 marks) Indicate whether each of the following statements is always true or can be false. For those that are false, please provide a short (one or two sentence) justification. Each fully correct answer is worth 1 mark.

(i) Every algebraic extension of $\mathbb{Q}$ is separable.

(ii) If $L : K$ is an extension of fields, then the algebraic closure $\overline{L}$ of L is equal to the algebraic closure $\overline{K}$ of K.

(iii) Every field extension of finite degree is normal.

(iv) There is an isomorphism $\varphi : \mathbb{Q}(\sqrt[4]{11}) \to \mathbb{Q}(i\sqrt[4]{11})$, where we write $i = \sqrt{-1}$.

(v) Suppose that K is a field of characteristic p. If $L : K$ is a field extension and $\tau \in L$ is transcendental over K, then τ^p is transcendental over L.

(vi) Let $L : \mathbb{Q}$ be a field extension, and suppose that K_1 and K_2 are subfields of L with the property that $[K_1 : \mathbb{Q}]$ and $[K_2 : \mathbb{Q}]$ are coprime. Then $K_1 \cap K_2 = \mathbb{Q}$.

(vii) Suppose that $f \in K[t] \setminus \{0\}$, and that $\beta \in \overline{K}$ has the property that $f(\beta) = 0$. Then f is an element of the ideal generated by the minimal polynomial of β over K.

(viii) When p is an odd prime, any splitting field of a degree p irreducible polynomial in $\mathbb{F}_p[t]$ has degree $p!$ over $\mathbb{F}_p$.

(ix) Let $f \in \mathbb{Q}[x]$ be cubic and irreducible, and suppose that f has a root α in a splitting field L for f over $\mathbb{Q}$. Then there exists $\beta \in L$ with $\mathbb{Q}(\alpha, \alpha^2, \alpha^3) = \mathbb{Q}(\beta)$.

(x) Let $f \in \mathbb{Z}[x]$ be a polynomial having prime degree p, and let θ be any root of f in a splitting field extension for f over $\mathbb{Q}$. Then $[\mathbb{Q}(\theta) : \mathbb{Q}] = p$.

(xi) Suppose that $L : M$ and $M : K$ are field extensions, and the field extension $L : K$ is normal. Then $M : K$ is normal.

(xii) When p is a prime number, then for every multiple n of p there exists a field extension $K : \mathbb{F}_p$ such that $\mathrm{card}(K) = n$.

(xiii) If $L : \mathbb{Q}$ is a cyclic field extension, then the Galois group $\mathrm{Gal}(L : \mathbb{Q})$ is cyclic.

(xiv) Suppose that $M : L$ and $L : K$ are finite Galois extensions. Then $M : K$ is a Galois extension.

(xv) Suppose $M : L$ and $L : K$ are finite Galois extensions. Then

$$\mathrm{Gal}(M : L) \lhd \mathrm{Gal}(M : K).$$

4. (a) (4 marks) State the Fundamental Theorem of Galois Theory.

   (b) (3+4+2=9 marks) Let p be a prime number , let K be the finite field having p elements, and let L be a field extension of K with $|L| = p^n$.

   (i) Show that $a^{p^n} = a$ for all elements $a \in L$, and deduce that $L : K$ is a splitting field extension for $x^{p^n} - x$.

   (ii) Define the Frobenius map ϕ on L, and deduce that $\mathrm{Gal}(L : K) = \langle \phi \rangle$.

   (iii) Noting that $\langle \phi \rangle \simeq \mathbb{Z}/n\mathbb{Z}$, show that there can exist a subfield of L having p^d elements only when $d|n$.

   (c) (4+4+4 marks) Let $L : \mathbb{Q}$ be a splitting field extension for $f(X) = X^3 - 3$.

   (i) Determine the degree of the extension $L : \mathbb{Q}$, justifying your answer.

   (ii) Describe the Galois group $\mathrm{Gal}(L : \mathbb{Q})$ (that is, give generators and relations for the Galois group).

End of examination.