SOLUTIONS

UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc. and M.Sci. (Level C/4)

FOUNDATIONS AND PROOF

MATH 10004

(Paper Code MATH-10004J)

January 2017 1 hour 30 minutes

This paper contains two sections, Section A and Section B. Please use a separate answer booklet for each section.
Section A contains ***five*** *short questions.* ***ALL*** *answers will be used for assessment. This section is worth 40% of the marks for the paper.*
Section B contains ***two*** *longer questions.* ***ALL*** *answers will be used for assessment. This section is worth 60% of the marks for the paper.*

Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

Section A: Short Questions

A1. Define $f = \{(x^2, x) : x \in \mathbb{R}\}$ and $g = \{(x, x^2) : x \in \mathbb{N}\}$.

(i) (3 marks) Is f a function? Justify.

Solution:
No, f is not a function. For example, we have that $(4, 2) \in f$, but we also have that $(4, -2) \in f$.

(ii) (3 marks) Is g a function? Justify.

Solution:
Let $x^2, y^2 \in \mathbb{N}$ with $x^2 \neq y^2$. Then $x \neq y$ since both x and y are positive. Hence, for each distinct x, there exists a distinct $g(x) = x^2$ such that $(x, g(x)) \in g$. Therefore, g is a function.

(iii) (3 marks) If f is a function, please list the domain, co-domain and range of f. Similarly, if g is a function, please list the domain, co-domain and range of g.

Solution:
We have that f is not a function. The domain of the function g is $\mathbb{N}$, the co-domain of g is $\mathbb{N}$, and the range of g is equal to $g(\mathbb{N}) = \{g(x) : x \in \mathbb{N}\} = \{x^2 : x \in \mathbb{N}\}$.

A2. (i) (2 marks) Negate the following statement:

$$(\forall\, x \in X,\, x \in Y \Rightarrow X \subseteq Y)$$

Solution:
$(\forall\, x \in X,\, x \in Y \text{ and } X \not\subseteq Y)$

(ii) (2 marks) State the contrapositive of the following statement:

(If x is even, then $x = 2n$ for some $n \in \mathbb{N}$)

Solution:
(If $x \neq 2n$ for all $n \in \mathbb{N}$, then x is odd)

(iii) (5 marks) Let P, Q be two statements. Using a truth table, show that

$$\neg(P \vee Q) \iff ((\neg P) \wedge (\neg Q)).$$

Solution:
We have the following truth table:

P	Q	P ∨ Q	¬(P ∨ Q)	¬P	¬Q	(¬P) ∧ (¬Q)
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Since the columns for $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$ are the same, we have that the two statements $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$ are the same.

A3. (i) (2 marks) Let $X = \{a, b, c\}$. Let $Y = \emptyset$. Find $X \times Y$.

Solution:
We have that $X \times Y = \emptyset$.

(ii) Let $A = \{\sqrt{2}, e, \pi, \frac{\sqrt{8}}{2}\}$.

(a) (2 mark) What is the cardinality of A?

Solution:
First, we note that $\frac{\sqrt{8}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$. Therefore, we have that the cardinality of A is given by

$$|A| = \left|\left\{\sqrt{2}, e, \pi\right\}\right| = 3.$$

(b) (3 marks) List all possible partitions of the set A.

Solution:
We have the following partitions of A:

$$\{\{\sqrt{2}\}, \{e, \pi\}\}, \quad \{\{\sqrt{e}\}, \{\sqrt{2}, \pi\}\} \quad \{\{\pi\}, \{e, \sqrt{2}\}\},$$
$$\{\{\sqrt{2}\}, \{e\}, \{\pi\}\}, \quad \{\{\sqrt{2}, e, \pi\}\}$$

A4. (i) (5 marks) Use Euclid's algorithm to find $\gcd(72, 51)$, and integers s, t such that

$$72s + 51t = \gcd(72, 51).$$

Solution:
We have

$$\begin{aligned} 72 &= 1 \cdot 51 + 21 \\ 51 &= 2 \cdot 21 + 9 \\ 21 &= 2 \cdot 9 + 3 \\ 9 &= 3 \cdot 3 + 0. \end{aligned}$$

Then $\gcd(72, 51) = 3$. Further, we have that

$$\begin{aligned} 3 &= 21 - 2 \cdot 9 \\ &= 21 - 2(51 - 2 \cdot 21) = 21 - 2 \cdot 51 + 4 \cdot 21 \\ &= 5 \cdot 21 - 2 \cdot 51 \\ &= 5(72 - 51) - 2 \cdot 51 = 5 \cdot 72 - 5 \cdot 51 - 2 \cdot 51 \\ &= 5 \cdot 72 - 7 \cdot 51. \end{aligned}$$

Then setting $s = 5$ and $t = -7$, we have that $\gcd(72, 51) = 5 \cdot 72 - 7 \cdot 51 = 3$.

(ii) (3 marks) Find $x \in \mathbb{Z}$ (with $0 \leq x \leq 4$) such that $x \equiv 7^{22} \pmod 5$.

Solution:

We have that $7^2 \equiv 49 \equiv 4 \equiv -1 \pmod 5$. Then

$$7^{22} \equiv \left(7^2\right)^{11} \equiv (-1)^{11} = -1 \equiv 4 \pmod 5.$$

Therefore, $x = 4$.

(iii) (2 marks) For natural numbers a and b, when do we have that $a \equiv b \pmod 7$?

Solution:

We have that $a \equiv b \pmod 7$ means that $7 \mid (a - b)$.

A5. (i) (2 marks) Define what it means for a set A to be countable.

Solution:

A set A is countable if there exists a bijective function $f : \mathbb{N} \to A$ [or equivalently, if there is a bijective map $g : A \to \mathbb{N}$].

(ii) (3 marks) Which of the following sets are countable?

(a) $\mathbb{Z} \times \mathbb{Z}$

Solution:

We have that $\mathbb{Z} \times \mathbb{Z}$ is countable [since the Cartesian product of countable sets is countable].

(b) $\mathbb{Z} \times \mathbb{Q}^+$

Solution:

We have that $\mathbb{Z} \times \mathbb{Q}^+$ is countable [since the Cartesian product of countable sets is countable].

(c) $\mathbb{N}$

Solution:

We have that $\mathbb{N}$ is countable.

Section B: Longer Questions

B1.

(i) (10 marks) Define $f : \mathbb{R}^+ \to \mathbb{R}^+$ by $f(x) = \frac{x^2}{4}$, for all $x \in \mathbb{R}^+$. Prove or disprove that f is bijective.

Solution:

We will prove that f is bijective.

First, we will show that f is injective. So suppose $x, y \in \mathbb{R}^+$ are such that $f(x) = f(y)$. Then $\frac{x^2}{4} = \frac{y^2}{4}$, so we have that $x^2 = y^2$. Since both x and y are non-negative [since $\mathbb{R}^+$ denotes all the non-negative real numbers], it follows that $x = y$. Hence, f is injective.

Second, we will show that f is surjective. So choose $c = 2\sqrt{x}$. Then $c \in \mathbb{R}^+$, and we have that

$$f(c) = f\left(2\sqrt{x}\right) = \frac{\left(2\sqrt{x}\right)^2}{4} = \frac{4x}{4} = x,$$

for all $x \in \mathbb{R}^+$. [So for every x in the co-domain $\mathbb{R}^+$, we found a c in the domain $\mathbb{R}^+$ such that $f(c) = x$] Hence, f is surjective. Since f is injective and surjective, we have that f is bijective.

(ii) (10 marks) Let X, Y be two non-empty sets and suppose that $f : X \to Y$, $g : Y \to X$ and $h : Y \to X$ with both g and h being inverses of f. Show that $g = h$.

Solution:

Let $y \in Y$. Then since both $f \circ g$ and $f \circ h$ are the identity maps on Y, we have that

$$f(g(y)) = y = f(h(y)),$$

for all $y \in Y$. Now, suppose that $g(y) = x$ and $h(y) = x'$ with $x \neq x'$. Then

$$f(g(y)) = f(x) = y = f(h(y)) = f(x').$$

which is a contradiction since f is injective [since f has an inverse, so it is bijective]. Hence, we must have that $x = x'$, that is $g(y) = h(y)$. Since y is arbitrary, it follows that $g(y) = h(y)$, for all $y \in Y$. Hence, $g = h$.

(iii) (10 marks) For natural numbers a, b, define a relation $\sim$ by $a \sim b$ if $a \mid b$.

(a) Prove or disprove that $\sim$ is reflexive.

Solution:

We will prove that $\sim$ is reflexive. For any natural number a, we have that $a \mid a$. Hence, $a \sim a$.

(b) Prove or disprove that $\sim$ is symmetric.

Solution:

We will prove that $\sim$ is not symmetric [by giving a counterexample]. Let $a = 3$ and $b = 6$. Then $3 \mid 6$ but $6 \nmid 3$. Then $3 \sim 6$ but $6 \not\sim 3$. Therefore, $\sim$ is not symmetric [since for $\sim$ to be symmetric it must be symmetric for all natural numbers].

(c) Prove or disprove that $\sim$ is transitive.

Solution:

We will prove that $\sim$ is transitive. So suppose we have that $a \sim b$ and $b \sim c$ for natural numbers a, b and c. Then $a \mid b$ and $b \mid c$. Hence, there exist $m, n \in \mathbb{N}$ such that $b = am$ and $c = bn$. Then

$$c = bn = (am)n = a(mn).$$

Since mn is an integer, we have that $a \mid c$. It follows that $a \sim c$. Therefore, $\sim$ is transitive.

(d) Is $\sim$ a equivalence relation? Justify.

Solution:

We have that $\sim$ is not an equivalence relation since $\sim$ is not symmetric.

B2. (i) (10 marks) Let A, B be sets. Prove that $A \subseteq B$ if and only if $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

Solution:

We will first show that if $A \subseteq B$, then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. So suppose that $A \subseteq B$. Then every subset of A is a subset of B, so every element of $\mathscr{P}(A)$ is an element of $\mathscr{P}(B)$ [since $\mathscr{P}(A)$ is the collection of all subsets of A and $\mathscr{P}(B)$ is the collection of all subsets of B]. It follows that $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

Second, we will show that if $\mathscr{P}(A) \subseteq \mathscr{P}(B)$, then $A \subseteq B$. So suppose that $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. We know that $A \in \mathscr{P}(A)$, so $A \in \mathscr{P}(B)$, that is A must be a subset of B. Hence, $A \subseteq B$. Hence, $A \subseteq B$ if and only if $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

(ii) (10 marks) Let X, Y, U, V be sets. Suppose that $f : X \to Y$ and $X = U \cup V$. Show that if f is injective and $U \cap V = \emptyset$, then $f(U) \cap f(V) = \emptyset$.

Solution:

To the contrary, suppose that $f(U) \cap f(V)$ is non-empty. Then there exists some element $y \in f(U) \cap f(V)$. Hence, there exists some $u \in U$ such that $y = f(u)$, and there exists some $v \in V$ such that $y = f(v)$. It follows that $f(u) = y = f(v)$. Since f is injective, we have that $u = v$. Hence, $u \in U \cap V$, which contradicts our initial assumption that $U \cap V = \emptyset$. It follows that $f(U) \cap f(V) = \emptyset$.

(iii) (10 marks) Let $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$, for all natural numbers $n \geq 3$. Using strong induction, show that for all n, we have that

$$a_n \geq \left(\frac{3}{2}\right)^{n-2}.$$

Solution:

For $n \in \mathbb{N}$, let $P(n)$ be the following statement.

$$a_n \geq \left(\frac{3}{2}\right)^{n-2}.$$

First, let us consider $n = 1$. Then

$$a_1 = 1 \geq \frac{2}{3} = \left(\frac{3}{2}\right)^{-1} = \left(\frac{3}{2}\right)^{1-2}.$$

Further, for $n = 2$, we have

$$a_2 = 1 \geq 1 \left(\frac{3}{2}\right)^{0} = \left(\frac{3}{2}\right)^{2-1}.$$

Therefore, $P(1)$ and $P(2)$ are true statements.
Now, let $k \in \mathbb{N}$ such that $k \geq 2$, and suppose that $P(i)$ is a true statement, for all $2 \leq i \leq k$. Hence, we have that a_k and a_{k-1} are true statements. That is, we have

$$a_k \geq \left(\frac{3}{2}\right)^{k-2}$$
$$a_{k-1} \geq \left(\frac{3}{2}\right)^{k-3}$$

Then

$$\begin{aligned}
a_{k+1} &= a_k + a_{k-1} \\
&\geq \left(\frac{3}{2}\right)^{k-2} + \left(\frac{3}{2}\right)^{k-3} \\
&= \left(\frac{3}{2}\right)^{k-2} \left(1 + \left(\frac{3}{2}\right)^{-1}\right) = \left(\frac{3}{2}\right)^{k-2} \left(1 + \frac{2}{3}\right) \\
&= \left(\frac{3}{2}\right)^{k-2} \left(\frac{5}{3}\right) \\
&\geq \left(\frac{3}{2}\right)^{k-2} \left(\frac{3}{2}\right) \\
&= \left(\frac{3}{2}\right)^{k-1},
\end{aligned}$$

showing that if $P(i)$ is true, for $2 \leq i \leq k$ then $P(k+1)$ is true. By the Strong Principle of Mathematical Induction, it follows that $P(n)$ is true for all natural numbers n.