UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc. and M.Sci. (Level C/4)

FOUNDATIONS AND PROOF

MATH 10004

(Paper Code MATH-10004J)

January 2017 1 hour 30 minutes

This paper contains two sections, Section A and Section B. Please use a separate answer booklet for each section.
Section A contains ***five*** *short questions.* ***ALL*** *answers will be used for assessment. This section is worth 40% of the marks for the paper.*
Section B contains ***two*** *longer questions.* ***ALL*** *answers will be used for assessment. This section is worth 60% of the marks for the paper.*

Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

Section A: Short Questions

A1. Define $f = \{(x^2, x) : x \in \mathbb{R}\}$ and $g = \{(x, x^2) : x \in \mathbb{N}\}$.

(i) (3 marks) Is f a function? Justify.

(ii) (3 marks) Is g a function? Justify.

(iii) (3 marks) If f is a function, please list the domain, co-domain and range of f. Similarly, if g is a function, please list the domain, co-domain and range of g.

A2. (i) (2 marks) Negate the following statement:

$$(\forall\, x \in X,\, x \in Y \Rightarrow X \subseteq Y)$$

(ii) (2 marks) State the contrapositive of the following statement:

$$(\text{If } x \text{ is even, then } x = 2n \text{ for some } n \in \mathbb{N})$$

(iii) (5 marks) Let P, Q be two statements. Using a truth table, show that

$$\neg(P \vee Q) \iff ((\neg P) \wedge (\neg Q)).$$

A3. (i) (2 marks) Let $X = \{a, b, c\}$. Let $Y = \emptyset$. Find $X \times Y$.

(ii) Let $A = \{\sqrt{2}, e, \pi, \frac{\sqrt{8}}{2}\}$.

(a) (2 mark) What is the cardinality of A?

(b) (3 marks) List all possible partitions of the set A.

A4. (i) (5 marks) Use Euclid's algorithm to find $\gcd(72, 51)$, and integers s, t such that

$$72s + 51t = \gcd(72, 51).$$

(ii) (3 marks) Find $x \in \mathbb{Z}$ (with $0 \leq x \leq 4$) such that $x \equiv 7^{22} \pmod 5$.

(iii) (2 marks) For natural numbers a and b, when do we have that $a \equiv b \pmod 7$?

A5. (i) (2 marks) Define what it means for a set A to be countable.

(ii) (3 marks) Which of the following sets are countable?

(a) $\mathbb{Z} \times \mathbb{Z}$

(b) $\mathbb{Z} \times \mathbb{Q}^+$

(c) $\mathbb{N}$

Section B: Longer Questions

B1.

(i) (10 marks) Define $f : \mathbb{R}^+ \to \mathbb{R}^+$ by $f(x) = \frac{x^2}{4}$, for all $x \in \mathbb{R}^+$. Prove or disprove that f is bijective.

(ii) (10 marks) Let X, Y be two non-empty sets and suppose that $f : X \to Y$, $g : Y \to X$ and $h : Y \to X$ with both g and h being inverses of f. Show that $g = h$.

(iii) (10 marks) For natural numbers a, b, define a relation $\sim$ by $a \sim b$ if $a \mid b$.

(a) Prove or disprove that $\sim$ is reflexive.

(b) Prove or disprove that $\sim$ is symmetric.

(c) Prove or disprove that $\sim$ is transitive.

(d) Is $\sim$ a equivalence relation? Justify.

B2. (i) (10 marks) Let A, B be sets. Prove that $A \subseteq B$ if and only if $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

(ii) (10 marks) Let X, Y, U, V be sets. Suppose that $f : X \to Y$ and $X = U \cup V$. Show that if f is injective and $U \cap V = \emptyset$, then $f(U) \cap f(V) = \emptyset$.

(iii) (10 marks) Let $a_1 = 1$, $a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$, for all natural numbers $n \geq 3$. Using strong induction, show that for all n, we have that

$$a_n \geq \left(\frac{3}{2}\right)^{n-2}.$$

End of examination.