Galois theory, Problems 1

To be handed in 11th October 2017

1. Let K be a field; recall that the polynomial ring $K[t]$ is a unique factorisation domain. Recall also that a non-zero polynomial $f \in K[t]$ is monic if its leading coefficient is 1, meaning that $f = t^n + a_{n-1}t^{n-1} + \ldots + a_0$ for some $a_{n-1}, \ldots, a_0 \in K$. Show that $K[t]$ contains infinitely many monic, irreducible polynomials.

   (Suggestion: First show that $K[t]$ contains at least one monic, irreducible polynomial. Then assume that $K[t]$ contains only finitely many monic, irreducible polynomials, and derive a contradiction. You might want to review Euclid's proof that there are infinitely many primes.)

2. For each of the following pairs of polynomials f and g:

   (i) find the quotient and remainder on dividing g by f;

   (ii) use the Euclidean Algorithm to find the highest common factor h of f and g;

   (iii) find polynomials a and b with the property that $h = af + bg$.

   (a) $g = t^3 + 2t^2 - t + 3$, $f = t + 2$ over $\mathbb{F}_5$;

   (b) $g = t^7 - 4t^6 + t^3 - 4t + 6$, $f = 2t^3 - 2$ over $\mathbb{F}_7$.

3. (a) Show that $t^3 + 3t + 1$ is irreducible in $\mathbb{Q}[t]$.

   (b) Suppose that α is a root of $t^3 + 3t + 1$ in $\mathbb{C}$. Express α^{-1} and $(1 + \alpha^2)^{-1}$ as linear combinations, with rational coefficients, of 1, α and α^2.

   (c) Is it possible to express $(1 + \alpha)^{-1}$ as a linear combination, with rational coefficients, of 1 and α? Justify your answer.

4. Let $L : K$ be a field extension with $K \subseteq L$. Let $A \subseteq L$, and let

   $$\mathcal{C} = \{C \subseteq A : C \text{ is a finite set}\}.$$

   Show that $K(A) = \cup_{C \in \mathcal{C}} K(C)$, and further that when $[K(C) : K] < \infty$ for all $C \in \mathcal{C}$, then $K(A) : K$ is an algebraic extension.

5. Let $L : K$ be a field extension, and suppose that $\gamma \in L$ satisfies the property that $\deg m_\gamma(K) = 7$. Suppose that $h \in K[t]$ is a non-zero cubic polynomial. By noting that γ is a root of the cubic polynomial $g(t) = h(t) - h(\gamma) \in K(h(\gamma))[t]$, show that $[K(h(\gamma)) : K] = 7$.

6. Calculate the minimal polynomial of $\sqrt[3]{7 + \sqrt[5]{21}}$ over $\mathbb{Q}$, and hence determine the degree of the field extension $\mathbb{Q}(\sqrt[3]{7 + \sqrt[5]{21}}) : \mathbb{Q}$.