Galois theory, Problems 2

To be handed in 24th October 2017

1. Let $L : K$ be a field extension with $K \subseteq L$. Show that the following are equivalent:

   (i) one has $[L : K] < \infty$;

   (ii) the extension $L : K$ is algebraic, and there exist $\alpha_1, \ldots, \alpha_n \in L$ having the property that $L = K(\alpha_1, \ldots, \alpha_n)$.

2. (a) Show that when p is a prime number, then for every positive integer n the polynomial $X^n - p$ is irreducible over $\mathbb{Q}$.

   (b) By making the substitution $y = X - 1$, or otherwise, show that when p is a prime number, the polynomial $X^{p-1} + X^{p-2} + \cdots + X + 1$ is irreducible over $\mathbb{Q}$.

   (c) Let p be a prime number with $p \equiv 3 \pmod 4$, and consider the polynomial $\pi = t^2 + 1$ in the ring $\mathbb{K} = \mathbb{F}_p[t]$. Show that the polynomial $X^{2016} - \pi X + \pi$ is irreducible over $\mathbb{F}_p(t)$.

3. (a) Show that the polynomial $f(t) = t^7 - 7t^5 + 14t^3 - 7t - 2$ factorises over $\mathbb{Q}[t]$ in the form $f = g_1 g_3^2$, where $g_1, g_3 \in \mathbb{Z}[t]$ have the property that g_1 is linear, and g_3 is cubic and irreducible.

   (b) Using the identity

   $$\cos 7\theta = 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta,$$

   together with the conclusion of part (a), show that the angle $2\pi/7$ is not constructible by ruler and compass. Hence deduce that the regular heptagon is not constructible by ruler and compass.

4. Suppose that $L : K$ is a field extension with $K \subseteq L$, and that $\tau : L \to L$ is a K-homomorphism. Suppose also that $f \in K[t]$ has the property that $\deg f \geq 1$, and additionally that $\alpha \in L$.

   (a) Show that when $f(\alpha) = 0$, then $f(\tau(\alpha)) = 0$.

   (b) Deduce that when τ is a K-automorphism of L, we have that $f(\alpha) = 0$ if and only if $f(\tau(\alpha)) = 0$.

5. Let $L : K$ be a field extension. Show that $\mathrm{Gal}(L : K)$ is a subgroup of $\mathrm{Aut}(L)$.

6. Suppose that L and M are fields with an associated homomorphism $\psi : L \to M$. Show that whenever L is algebraically closed, then $\psi(L)$ is also algebraically closed.