Galois theory, Problems 3

To be handed in 8th November 2017

1. Let M be a field. Show that the following are equivalent:

   (i) the field M is algebraically closed;

   (ii) every non-constant polynomial $f \in M[t]$ factors in $M[t]$ as a product of linear factors;

   (iii) every irreducible polynomial in $M[t]$ has degree 1;

   (iv) the only algebraic extension of M is M itself.

   (Version 1)

2. Let $L : K$ be a field extension with $K \subseteq L$. Let $\gamma \in L$ be transcendental over K, and suppose that $K(\gamma) : K$ is a simple field extension. Show that $K(\gamma)$ is not algebraically closed.

3. For each of the following polynomials, construct a splitting field L over $\mathbb{Q}$ and compute the degree $[L : \mathbb{Q}]$.

   (a) $t^7 - 1$

   (b) $t^4 + t^2 - 6$

4. Construct a splitting field L over $\mathbb{Q}$ for the polynomial $t^8 - 16$, and determine the subgroup of S_4 to which $\mathrm{Gal}(L : \mathbb{Q})$ is isomorphic.

5. Suppose that K is a field and that $L : K$ is a splitting field extension for an irreducible polynomial $f \in K[t]$ of degree n. Assume that $K \subseteq L$.

   (a) Show that whenever α and β are roots of f in L, and σ is a K-automorphism of L, then $\sigma(\alpha) = \sigma(\beta)$ if and only if $\alpha = \beta$;

   (b) Show that the elements of $\mathrm{Gal}(L : K)$ act as permutations on the n roots of f, and hence deduce that $\mathrm{Gal}(L : K)$ has order dividing $n!$;

   (c) Let g be a degree m polynomial in $K[t]$, not necessarily irreducible, and let $M : K$ be a splitting field extension for g. Show that $|\mathrm{Gal}(M : K)|$ divides $m!$.

6. Suppose that $L : K$ is a normal extension, and that $K \subseteq L \subseteq \overline{K}$. Recall that since $L : K$ is algebraic, then any algebraic closure of K is an algebraic closure of L.

   (a) Show that for any K-homomorphism $\tau : L \to \overline{K}$, one has $\tau(L) = L$;

   (b) Suppose that M is a field satisfying $K \subseteq M \subseteq L$. Show that $L : M$ is a normal extension.